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ORIGINAL RESEARCH PAPER

Bases of *k*-modified harmonic polynomials

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Abstract
In this article we establish bases of the various spaces of homogeneous k-modified harmonic polynomials on $\mathbb{R}^d$ for every degree of homogeneity and for all but finite values of $k \in \mathbb{R}$. In particular, we prove a conjecture of the first author concerning the case $d = 3$.

Keywords k-modified harmonic function · k-modified harmonic polynomial

Mathematics Subject Classification 35C11 · 33E30

1 Introduction

A function u on a domain in $\mathbb{R}^d$ $(d \geq 2)$ is called *k-modified harmonic* if and only if it satisfies the so-called Weinstein equation,

$$x_d \cdot \Delta u(x_1, \ldots, x_d) + k \cdot \frac{\partial u(x_1, \ldots, x_d)}{\partial x_d} = 0,$$

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where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$ and $k \in \mathbb{R}$. If such a function is a polynomial on $\mathbb{R}^d$, it is called a *k*-modified harmonic polynomial. For historical information and the justification of the name the reader is referred to [1] or [2].

Those *k*-modified harmonic polynomials on $\mathbb{R}^d$ that are homogeneous of the same degree *n* form a real vector space, which we shall denote by $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$, whereas $\mathcal{P}_n(\mathbb{R}^d)$ will stand for the real vector space of all homogeneous polynomials of degree *n* on $\mathbb{R}^d$ ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$). Since the monomials

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_d^{\alpha_d}$$

for $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{N}_0$, $\alpha_1 + \alpha_2 + \ldots + \alpha_d = n$ form a basis of $\mathcal{P}_n(\mathbb{R}^d)$,

$$\dim \mathcal{P}_n(\mathbb{R}^d) = \binom{d+n-1}{n}$$

(the monomials can be regarded as *n*-combinations with repetitions of *d* elements).

In [2] we proved that

$$\dim \mathcal{H}_n^{(k)}(\mathbb{R}^d) = \binom{d+n-2}{n}$$

if and only if $-k \notin 2\mathbb{N}_0 \cap [0, n-1]$. In the latter, exceptional cases, the dimension is bigger:

$$\dim \mathcal{H}_n^{(-2m)}(\mathbb{R}^d) = \binom{d+n-2}{n} + \binom{d+n-3-2m}{d-2} \tag{1}$$

for $m \in \mathbb{N}_0$, $m \leq \frac{n-1}{2}$ (see [3]).

The question of bases appears to be more difficult. In [2] we showed that for $\alpha_1, \ldots, \alpha_{d-1} \in \mathbb{N}_0$, $\alpha_1 + \ldots + \alpha_{d-1} = n$, the functions

$$v^{(k)}_{\alpha_1 \ldots \alpha_{d-1}}(x_1, \ldots, x_d) := r^{2n+d+k-2} \cdot \frac{\partial^n r^{2-k-d}}{\partial x_1^{\alpha_1} \ldots \partial x_{d-1}^{\alpha_{d-1}}}, \tag{2}$$

where $r := \sqrt{x_1^2 + \ldots + x_d^2}$, which were shown to belong to $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$, are linearly independent if $-k \notin d - 2 + 2\,(\mathbb{N}_0 \cap [0, n-1])$. Combining this with the above result on the dimension, we conclude that if additionally $-k \notin 2\mathbb{N}_0 \cap [0, n-1]$, these polynomials form a basis of $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$.

In the case of classical harmonic polynomials ($k = 0$), a basis of $\mathcal{H}_n^{(0)}(\mathbb{R}^d)$ is given in [4]. Moreover, orthonormal bases in the cases $d = 3$ and $d = 4$ are there determined.

Finally, we mention that for $k \in \mathbb{N}$, all major results from the theory of classical spherical harmonics remain valid in the *k*-modified setting. The interested reader is referred to [5] and [6].

2 The odd-dimensional case

The problem of finding bases in the spaces $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$ for exceptional values of *k* can be settled in a unified manner in the odd-dimensional case (*d* odd), when $-k \in 2\mathbb{N}_0 \cap [0, n-1]$. This is the goal of this section. We start with a lemma, which establishes a connection between spaces $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$ for different values of *k*.

Lemma 1 *For every function u on a domain in $\mathbb{R}^d$ and for every $k \in \mathbb{R}$, u is k-modified harmonic if and only if the function v defined by*

$$v(x_1, \ldots, x_d) := x_d^{k-1} \cdot u(x_1, \ldots, x_d)$$

is $(2-k)$-modified harmonic.

The proof follows by a straightforward verification.

Now we are ready to prove our result.

Theorem 1 *Let $d, n \in \mathbb{N}$, $d \geq 3$ odd, $m \in \mathbb{N}_0$, $m \leq \frac{n-1}{2}$. Then, the polynomials $v^{(-2m)}_{\beta_1 \ldots \beta_{d-1}}$, $\beta_1 + \ldots + \beta_{d-1} = n$ (as defined in (2)), together with the polynomials $u^{(-2m)}_{\alpha_1 \ldots \alpha_{d-1}}$, defined by*

$$u^{(-2m)}_{\alpha_1 \ldots \alpha_{d-1}}(x_1, \ldots, x_d) := x_d^{2m+1} \cdot v^{(2m+2)}_{\alpha_1 \ldots \alpha_{d-1}}(x_1, \ldots, x_d),$$

$\alpha_1 + \ldots + \alpha_{d-1} = n - 1 - 2m$, **form a basis of $\mathcal{H}_n^{(-2m)}(\mathbb{R}^d)$.**

Proof Since the polynomials $v^{(2m+2)}_{\alpha_1 \ldots \alpha_{d-1}}(x_1, \ldots, x_d)$ for $\alpha_1, \ldots, \alpha_{d-1} \in \mathbb{N}_0$, $\alpha_1 + \ldots + \alpha_{d-1} = n - 1 - 2\,m$ form a basis of $\mathcal{H}^{(2m+2)}_{n-1-2m}(\mathbb{R}^d)$ (see the last statement in the introduction), they are linearly independent. From the above lemma we deduce that the polynomials $u^{(-2\,m)}_{\alpha_1 \ldots \alpha_{d-1}}$ in the theorem are linearly independent in $\mathcal{H}_n^{(-2m)}(\mathbb{R}^d)$. Obviously, their number is

$$\binom{d-1+n-1-2m-1}{n-1-2m} = \binom{d+n-3-2m}{d-2}.$$

Moreover, since *d* is odd, the penultimate statement in the introduction supplies the linearly independent polynomials $v^{(-2\,m)}_{\beta_1 \ldots \beta_{d-1}}(x_1, \ldots, x_d)$, $\beta_1 + \ldots + \beta_{d-1} = n$ in $\mathcal{H}_n^{(-2m)}(\mathbb{R}^d)$. Their number is $\binom{d+n-2}{n}$.

Since the polynomials $v^{(-2m)}_{\beta_1 \ldots \beta_{d-1}}$ are even in x_d, whereas the $u^{(-2m)}_{\alpha_1 \ldots \alpha_{d-1}}$ are odd, the union of these two systems of polynomials remains linearly independent. Since their total number is equal to the dimension of $\mathcal{H}_n^{(-2m)}(\mathbb{R}^d)$ (see (1)), they constitute a basis of $\mathcal{H}_n^{(-2m)}(\mathbb{R}^d)$. $\square$

In [1], where the case $d = 3$ is considered, we have

$$\dim \mathcal{H}_n^{(-2m)}(\mathbb{R}^3) = \binom{n+1}{n} + \binom{n-2m}{1} = 2n - 2m + 1.$$

There, the $n+1$ linearly independent polynomials u_n^i $(0 \le i \le n)$ together with the $n - 2m$ linearly independent polynomials ν_n^i $(0 \le i \le n - 2m - 1)$ form a linearly independent system, because the former are even, the latter are odd in the last variable. Since their total number equals $2n - 2m + 1$, they clearly constitute a basis of $\mathcal{H}_n^{(-2m)}(\mathbb{R}^3)$, a fact that proves the conjecture formulated in that article (following Theorem 3.1).

Declarations

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this article.

References

1. Leutwiler, H. 2024. Further results on modified harmonic functions in three dimensions. *Mathematical Methods in the Applied Sciences* 47:7879–7887. https://doi.org/10.1002/mma.7277.
2. Symeonidis, E. 2024. Spaces of k-modified harmonic polynomials. *Revue Roumaine de Mathématiques Pures et Appliquées* 69 (3–4): 575–583. https://doi.org/10.59277/RRMPA.2024.575.583.
3. Symeonidis, E. On k-Modified Harmonic Polynomials and Their Dimensions, to appear in Bull. Transilv. Univ. Brașov.
4. Leutwiler, H., and E. Symeonidis. 2023. On orthonormal harmonic polynomials. *Journal of Analysis* 31 (4): 2755–2775.
5. Leutwiler, H. 2019. More on modified spherical harmonics. *Advances in Applied Clifford Algebras* 29 (4): 15. https://doi.org/10.1007/s00006-019-0990-z.
6. Leutwiler, H. 2019. Modified spherical harmonics in several dimensions. *Advances in Applied Clifford Algebras* 29 (5): 17. https://doi.org/10.1007/s00006-019-1021-9.