Integr. Equ. Oper. Theory (2026) 98:28
https://doi.org/10.1007/s00020-026-02849-7


Integral Equations and Operator Theory

# Fourier Multipliers in Homogeneous Banach Spaces

Ben de Pagter and Werner J. Ricker

*Dedicated to the memory of Rien Kaashoek.*

**Abstract.** Let $G$ be an infinite, compact abelian group, $E$ be a homogeneous Banach space over $G$ and $\mathcal{L}(E)$ be the space of all continuous linear operators from $E$ into itself equipped with the operator norm. Translation operators are isometries in $E$ (by definition) and so the closed subalgebra $\mathfrak{m}(E)$ of $\mathcal{L}(E)$ consisting of those operators which commute with all translations is well defined. It is shown that there exists a contractive projection $\mathcal{Q}$ of $\mathcal{L}(E)$ onto $\mathfrak{m}(E)$ which is positivity preserving. Moreover, every operator $\mathcal{Q}(T) \in \mathfrak{m}(E)$, with $T \in \mathcal{L}(E)$, is induced by a unique Fourier multiplier function $\hat{T} \in \ell^{\infty}(\Gamma)$, where $\Gamma$ is the dual group of $G$. In the setting of the homogeneous Banach spaces $L^p(G)$, for $1 \leq p < \infty$ and $G$ an amenable group, these results are due to W. Arendt and J. Voigt.

**Mathematics Subject Classification.** Primary 43A15, 43A22, 46E30; Secondary 47B38, 47B65, 47B90.

**Keywords.** Homogeneous Banach space, Compact abelian group, Translation operator, Fourier multiplier, Order bounded operator.

## 1. Introduction

The theory of homogeneous Banach spaces over locally compact abelian groups first arose in the seminal paper [21] of G.E. Shilov. Since then, various aspects of the topic have been intensively investigated; see, for example, [7], [10], [17] and the historical comments and references therein.

Infinite, compact abelian groups $G$ form the setting of this article. Our aim is to extend two well known results, for certain Fourier multiplier operators acting in $L^p(G)$, for $1 \leq p < \infty$, due to W. Arendt and J. Voigt whenever $G$ is an amenable group (all compact abelian groups are amenable); see Theorems 1.1 and 3.1 in [1]. Our extensions (see Theorems 5.1 and 5.4 below) pertain to the class of homogeneous Banach spaces over $G$, a class

Published online: 02 July 2026 

which is significantly larger than the spaces $L^p(G)$, $1 \le p < \infty$. In particular, it includes all translation invariant Banach function spaces over $G$ which have order continuous norm and hence, also encompasses all rearrangement invariant spaces over $G$ with order continuous norm.

Section 2 formulates some relevant preliminaries from harmonic analysis, Banach function spaces and the theory of Bochner integrals. Homogeneous Banach spaces $E$ over $G$ are discussed in Sect. 3 , together with certain aspects concerning translation invariant Banach function spaces over $G$. Sect. 4 is devoted to the theory of Fourier multiplier functions defined on the dual group $\Gamma$ of $G$ and the corresponding multiplier operators in $\mathcal{L}(E)$ that they induce, which necessarily commute with all translation operators in $E$ ; see Lemma 4.5. Conversely (cf. Lemma 4.7), if an operator $T \in \mathcal{L}(E)$ commutes with all translations, then it is necessarily induced by a Fourier multiplier function on $\Gamma$. In the final Section 5 we formulate and prove the extension of the results of Arendt and Voigt alluded to above. Some applications and examples are also presented.

## 2. Preliminaries

Throughout this article, $G$ is an infinite, compact abelian group equipped with normalized Haar measure $\mu$. The Borel $\sigma$-algebra of $G$ is denoted by $\mathcal{B}(G)$. The space of all $\mathbb{C}$ -valued Borel measurable functions on $G$, with the usual identification of $\mu$-a.e. functions, is denoted by $L^0(G)$ and, for each $1 \le p \le \infty$, $L^p(G)$ is the usual $L^p$-space on $G$ with respect to $\mu$.

The space of all regular, complex Borel measures on $G$ is denoted by $M(G)$ and is equipped with the variation norm $\|\cdot\|_{M(G)}$, that is, $\|\lambda\|_{M(G)} = |\lambda|(G)$, for $\lambda \in M(G)$, where $|\lambda|$ is the variation measure of $\lambda$. The dual group of $G$ is denoted by $\Gamma$; see, for example, Section 1.2.1 of [18]. For $x \in G$ and $\gamma \in \Gamma$ we write $\gamma(x) = (x, \gamma)$. Since we frequently consider $\Gamma$ as a subset of $C(G)$, we shall write the group operation in $\Gamma$ as *multiplication*, that is, if $\gamma_1, \gamma_2 \in \Gamma$, then $\gamma_1\gamma_2 \in \Gamma$ is given by

$$(x, \gamma_1\gamma_2) = (x, \gamma_1)(x, \gamma_2), \quad x \in G.$$

Here $C(G)$ is the Banach space of all continuous, $\mathbb{C}$ -valued functions on $G$ equipped with the sup-norm. The identity element of $\Gamma$ is, of course, $\mathbf{1} = \chi_G$. Moreover,

$$\left(x, \gamma^{-1}\right) = \overline{(x, \gamma)} = (-x, \gamma), \quad x \in G, \quad \gamma \in \Gamma.$$

The discrete group $\Gamma$ is countable if and only if $G$ is metrizable; see Theorem 2.2.6 in [18].

Given any measure $\lambda \in M(G)$, its Fourier-Stieltjes transform $\hat{\lambda} : \Gamma \to \mathbb{C}$ is defined by

$$\hat{\lambda}(\gamma) = \int_G \overline{(x, \gamma)} d\lambda(x), \quad \gamma \in \Gamma.$$

Then $\hat{\lambda} \in \ell^\infty(\Gamma)$ and $\left\|\hat{\lambda}\right\|_\infty \le \|\lambda\|_{M(G)}$ for every $\lambda \in M(G)$; see Section 1.3.3 of [18]. If $f \in L^1(G)$ and $\lambda = f \cdot \mu$ (that is, $\lambda$ is the measure $\lambda(A) = \int_A f d\mu$, for $A \in \mathcal{B}(G)$), then its Fourier-Stieltjes transform $\hat{\lambda}$ is also denoted by $\hat{f}$ and is called the Fourier transform of $f$, that is,

$$\hat{f}(\gamma) = \int_G f(x)\,\overline{(x,\gamma)}d\mu, \quad \gamma \in \Gamma.$$

By the Riemann-Lebesgue lemma, the Fourier transform $\hat{f} \in c_0(\Gamma)$ for all $f \in L^1(G)$. For each $f \in L^1(G)$, the support of $\hat{f}$ is denoted by

$$\operatorname{supp}\left(\hat{f}\right) = \left\{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\right\} \subseteq \Gamma, \tag{1}$$

which is necessarily a countable subset of $\Gamma$ as $\hat{f} \in c_0(\Gamma)$.

Denote by $\tau(G)$ the linear span of the dual group $\Gamma$ formed in the Banach space $C(G)$, that is, $\tau(G)$ consists of all functions on $G$ which are of the form $f = \sum_{j=1}^{n} \alpha_j \gamma_j$, where $\gamma_j \in \Gamma$ and $\alpha_j \in \mathbb{C}$, for $j = 1, \ldots, n$, and $n \in \mathbb{N}$. The elements of $\tau(G)$ are referred to as *trigonometric polynomials* on $G$. Note that in this case the support of $\hat{f}$ is given by

$$\operatorname{supp}\left(\hat{f}\right) = \{\gamma_j : \alpha_j \neq 0\}. \tag{2}$$

According to Appendix A16 in [18], the Banach space $C(G)$ is separable if and only if $G$ is metrizable.

Let $f \in L^1(G)$ and $\lambda \in M(G)$. The convolution $f * \lambda \in L^1(G)$ of $f$ and $\lambda$ is defined by

$$(f * \lambda)(x) = \int_G f(x-y)\, d\lambda(y), \quad \mu\text{-a.e. } x \in G, \tag{3}$$

(see Section 1.3 of [18]) and satisfies $(f * \lambda)^{\wedge} = \hat{f} \cdot \hat{\lambda}$ pointwise on $\Gamma$.

We will have the occasion to use Bochner integrals over $G$. For the definition and basic properties of the Bochner integral we refer the reader to Chapter II in [4]. Let $(X, \|\cdot\|_X)$ be a Banach space with dual Banach space $X^*$. The value of a functional $x^* \in X^*$ at an element $x \in X$ is denoted by $\langle x, x^* \rangle$. Let $\lambda \in M(G)^+$ be a positive measure and $F : G \to X$ be Bochner $\lambda$-integrable. The Bochner integral of $F$ with respect to $\lambda$ is denoted by $\int_G^{(B)} F d\lambda \in X$. Recall, in particular, that

$$\left\| \int_G^{(B)} F d\lambda \right\|_X \leq \int_G \|F\|_X \, d\lambda \tag{4}$$

and that

$$\left\langle \int_G^{(B)} F d\lambda, x^* \right\rangle = \int_G \langle F, x^* \rangle \, d\lambda, \quad x^* \in X^*. \tag{5}$$

Let $\lambda \in M(G)$, not necessarily positive. By the Radon-Nikodym theorem, there exists a $\mathcal{B}(G)$ -measurable function $h : G \to \mathbb{C}$ such that $|h(x)| = 1$, for $x \in G$, and $\lambda = h|\lambda|$ (see, for example, [19], Theorem 6.12). Let $X$ be a Banach space and $F : G \to X$ be Bochner $|\lambda|$-integrable. Then the function $hF : G \to X$ is also Bochner $|\lambda|$-integrable. Therefore, we can define the Bochner $\lambda$-integral of $F$ by setting

$$\int_G^{(B)} Fd\lambda = \int_G^{(B)} hFd\,|\lambda|\,.$$

Using (4) and (5), it is readily verified that

$$\left\| \int_G^{(B)} Fd\lambda \right\|_X \leq \int_G \|F\|_X \, d\,|\lambda| \tag{6}$$

and that

$$\left\langle \int_G^{(B)} Fd\lambda, x^* \right\rangle = \int_G \langle F, x^* \rangle \, d\lambda, \quad x^* \in X^*. \tag{7}$$

The following fact will be used.

**Lemma 2.1.** *Let $X$ be a Banach space, $\lambda \in M(G)$ and $F : G \to X$ be continuous. Then $F$ is Bochner $|\lambda|$-integrable over $G$ and so $\int_G^{(B)} Fd\lambda \in X$ exists.*

*Proof.* For each $x^* \in X^*$ the scalar-valued function $\langle F, x^* \rangle$ defined on $G$ is continuous and hence, Borel measurable. Therefore, the function $F$ is weakly $|\lambda|$-measurable; see p. 41 of [4]. Furthermore, the range $F(G)$ of $F$ is a compact subset of $X$ and so $F(G)$ is separable. Accordingly, it follows from Pettis's measurability theorem (see Theorem II.1.2 in [4]) that $F$ is strongly $|\lambda|$-measurable. Furthermore, since $F(G)$ is a bounded subset of $X$, it is clear that $\int_G \|F\|_X \, d\,|\lambda| < \infty$. Via Theorem II.2.2 of [4] we can conclude that $F$ is Bochner $|\lambda|$ -integrable. This completes the proof of the lemma. $\square$

We end this section by recalling the definition of Banach function spaces (briefly, B.f.s.') over $G$.

**Definition 2.2.** A *Banach function space* over $G$ is an ideal $E \subseteq L^0(G)$ (that is, $E$ is a linear subspace of $L^0(G)$ with the property that $f \in E$ whenever $f \in L^0(G)$ satisfies $|f| \leq |g|$ for some $g \in E$) equipped with a norm $\|\cdot\|_E$ such that:

(i) it follows from $f, g \in E$ with $|f| \leq |g|$ that $\|f\|_E \leq \|g\|_E$;
(ii) the normed space $(E, \|\cdot\|_E)$ is complete.

For the general theory of B.f.s.' we refer to Chapter 15 in [22]. A special class consists of the rearrangement invariant B.f.s.' Recall, for any function $f \in L^0(G)$, that the decreasing rearrangement $f^* : [0,1] \to [0,\infty]$ of $|f|$ is defined by

$$f^*(t) = \inf\{s \geq 0 : \mu(\{x \in G : |f(x)| > s\}) \leq t\}, \quad t \in [0,1]\,.$$

The function $f^*$ is decreasing, right-continuous and is equimeasurable with $|f|$. For properties of decreasing rearrangements we refer to Section II.1 in [2] and Section II.2 in [11].

A B.f.s. $(E, \|\cdot\|_E)$ over $G$ is called *rearrangement invariant* if it follows from $f \in E$, $g \in L^0(G)$ and $f^* = g^*$ that $g \in E$ and $\|g\|_E = \|f\|_E$. Examples of rearrangement invariant B.f.s.' include the $L^p$-spaces, for $1 \leq p \leq \infty$, Orlicz spaces, Marcinkiewicz spaces and Lorentz spaces; see, for example, [11], Chapter II and [2], Chapter 2, as well as [3].

## 3. Homogeneous Banach Spaces

As before, $G$ is an infinite, compact abelian group with normalized Haar measure $\mu$. For each $y \in G$ the translation operator $\tau_y : L^0(G) \to L^0(G)$ is the linear map defined by

$$(\tau_y f)(x) = f(x-y), \quad x \in G, \quad f \in L^0(G).$$

The following definition is motivated by Chapter I, Sect. 2 in [10].

**Definition 3.1.** A *homogeneous Banach space* over $G$ is a Banach space $(E, \|\cdot\|_E)$ which is a linear subspace of $L^1(G)$ satisfying the following three conditions.

(H-1) The natural inclusion $E \subseteq L^1(G)$ is continuous.
(H-2) If $f \in E$ and $y \in G$, then $\tau_y f \in E$ with $\|\tau_y f\|_E = \|f\|_E$.
(H-3) For each $f \in E$ we have $\lim_{y\to 0} \|f - \tau_y f\|_E = 0$.

Note that condition (H-2) implies that each translation operator $\tau_y : E \to E$ is a linear isometry. Furthermore, condition (H-3) implies, for each $f \in E$, that the map $y \mapsto \tau_y f$ is continuous from $G$ into $E$.

The classical examples of homogeneous Banach spaces are the B.f.s.' $L^p(G)$, for $1 \leq p < \infty$. It can be shown that the space $L^\infty(G)$ is *not* homogeneous (since we assume that $G$ is infinite). For the circle group $G = \mathbb{T}$ this is easily verified.

Let $E$ be a B.f.s. over $G$. If $E$ satisfies condition (H-2), then $E$ is called a *translation invariant B.f.s.* over $G$; see [14], [15], [16] for various aspects of such spaces. Any translation invariant B.f.s. $E$ over $G$ satisfies $E \subseteq L^1(G)$ with a continuous embedding (see Proposition 3.4 in [15]) and so $E$ satisfies condition (H-1). However, such a B.f.s. need not be a homogeneous B.f.s. (consider, for example, the space $L^\infty(G)$). Actually, a translation invariant B.f.s. $E$ over $G$ satisfies (H-3) if and only if $E$ has order continuous norm (briefly, o.c.-norm), that is, $\|f_n\|_E \downarrow_n 0$ for every sequence $(f_n)_{n=1}^\infty$ in $E$ satisfying $f_n(x) \downarrow_n 0$ for $\mu$-a.e. $x \in G$; see Proposition 3.8 in [15] and Theorem 7.11 in [16]).

Every rearrangement invariant B.f.s. over $G$ is, in particular, translation invariant (but, not conversely; see Example 3.2 in [15]). Indeed, $(\tau_y f)^* = f^*$ for all $f \in L^0(G)$ and $y \in G$. Consequently, a rearrangement invariant B.f.s. $E$ over $G$ is a homogeneous Banach space if and only if $E$ has o.c.-norm.

Another well known example of a homogeneous Banach space is

$$A(G) = \left\{ f \in C(G) : \|f\|_{A(G)} = \sum\nolimits_{\gamma\in\Gamma} \left| \hat{f}(\gamma) \right| < \infty \right\};$$

see, for example, Ch. 9 in [9] and, for $G$ the circle group $\mathbb{T}$, Section I.6 in [10]. The space $A(G)$ is in fact a Banach algebra (with respect to the pointwise operations) which is isometrically isomorphic to the convolution algebra $\ell^1(\Gamma)$. For the case $G = \mathbb{T}$ the space $A(\mathbb{T})$ is sometimes called the Wiener algebra. It is known (cf. Lemma 37.3 and Theorem 37.4 in [9]) that

$$A(G) \subsetneqq C(G) \subsetneqq L^\infty(G).$$

Evidently, $A(G)$ is *not* a B.f.s. over $G$. As noted before, $\Gamma$ is countable if and only if $G$ is metrizable.

To verify that $A(G)$ is a homogeneous Banach space, let $f \in A(G)$. Then $f = \sum_{\gamma\in\Gamma} \hat{f}(\gamma)\gamma$ as an absolutely convergent series in $C(G)$ and so

$$\|f\|_1 \leq \|f\|_\infty \leq \sum\nolimits_{\gamma\in\Gamma} \left\|\hat{f}(\gamma)\gamma\right\|_\infty = \sum\nolimits_{\gamma\in\Gamma} \left|\hat{f}(\gamma)\right| = \|f\|_{A(G)}.$$

Hence, $A(G)$ is continuously included in $L^1(G)$. Furthermore, for every $f \in L^1(G)$, in particular, for every $f \in A(G)$, we have that

$$(\tau_y f)^\wedge(\gamma) = \overline{(y,\gamma)}\hat{f}(\gamma), \quad \gamma \in \Gamma,\ y \in G, \tag{8}$$

from which it is clear that condition (H-2) is also satisfied. Finally, using (8) once again, a routine argument shows that $A(G)$ satisfies (H-3) as well. Hence, $A(G)$ is a homogeneous Banach space over $G$.

## 4. Fourier Multipliers

Let $E$ be a homogeneous Banach space over $G$. The space of all bounded linear operators from $E$ into itself is denoted by $\mathcal{L}(E)$ and is equipped with the operator norm. An operator $T \in \mathcal{L}(E)$ is called an *$E$-multiplier operator* if $T\tau_y = \tau_y T$ for all $y \in G$. The space of all such operators is denoted by $\mathfrak{m}(E)$, which is a closed linear subspace of $\mathcal{L}(E)$. The following lemma exhibits a large class of $E$-multiplier operators.

**Lemma 4.1.** *Let $E$ be a homogeneous Banach space over $G$ and $\lambda \in M(G)$. Then $f * \lambda \in E$, for all $f \in E$, and the linear map $T_\lambda : E \to E$, defined by $T_\lambda f = f * \lambda$, for $f \in E$, satisfies $T_\lambda \in \mathfrak{m}(E)$ with $\|T_\lambda\| \leq \|\lambda\|_{M(G)}$.*

*Proof.* Fix $f \in E$. As observed before, the convolution $f * \lambda \in L^1(G)$, as defined in (3), exists. Define the function $F : G \to E$ by $F(y) = \tau_y f$, for $y \in G$. It follows from (H-2) and (H-3) that $F$ is well defined and continuous on $G$. Lemma 2.1 implies that the Bochner integral $\int_G^{(B)} \tau_y f d\lambda = \int_G^{(B)} F d\lambda \in E$ exists.

For each $g \in L^\infty(G)$, define the linear functional $\varphi_g \in L^1(G)^*$ by $\langle h, \varphi_g\rangle = \int_G hg\, d\mu$, for $h \in L^1(G)$. Since the embedding of $E$ into $L^1(G)$ is continuous, it is clear that the restriction of $\varphi_g$ to $E$ belongs to $E^*$. Noting that

$$\int_G \int_G |(\tau_y f)(x)|\,|g(x)|\, d\mu(x)\, d|\lambda|(y) \leq \|g\|_\infty \|f\|_1 |\lambda|(G) < \infty,$$

we can use (7) and apply Fubini's theorem to compute

$$\begin{aligned}\left\langle \int_G^{(B)} F(y)\, d\lambda(y), \varphi_g \right\rangle &= \int_G \langle F(y), \varphi_g\rangle\, d\lambda \\ &= \int_G \left( \int_G (F(y))(x)\, g(x)\, d\mu(x) \right) d\lambda(y) \\ &= \int_G \left( \int_G (\tau_y f)(x)\, d\lambda(y) \right) g(x)\, d\mu(x) \\ &= \int_G (f * \lambda)(x)\, g(x)\, d\mu(x) = \langle f * \lambda, \varphi_g\rangle.\end{aligned}$$

Since $\int_G^{(B)} F d\lambda \in E \subseteq L^1(G)$ and $L^\infty(G)$ separates the points of $L^1(G)$, we can conclude that $f * \lambda = \int_G^{(B)} F d\lambda \in E$.

Define the linear map $T_\lambda : E \to E$ by $T_\lambda f = f * \lambda$, for $f \in E$. It follows from (6) that

$$\begin{aligned}\|T_\lambda f\|_E &= \left\| \int_G^{(B)} F(y)\, d\lambda(y) \right\|_E \leq \int_G \|F(y)\|_E\, d|\lambda|(y) \\ &= \int_G \|\tau_y f\|_E\, d|\lambda|(y) = \int_G \|f\|_E\, d|\lambda|(y) = \|f\|_E \|\lambda\|_{M(G)}\end{aligned}$$

and hence, $T_\lambda \in \mathcal{L}(E)$ with $\|T_\lambda\| \leq \|\lambda\|_{M(G)}$. Finally, it is now routine to check that $T_\lambda \in \mathfrak{m}(E)$. □

In particular, it follows from Lemma 4.1 that $f * g \in E$ for all $f \in E$ and $g \in L^1(G)$. Let $\gamma \in \Gamma$, the dual group of $G$. Then $\gamma \in C(G) \subseteq L^1(G)$ and so $f * \gamma \in E$ for all $f \in E$. It is readily verified that $f * \gamma = \hat{f}(\gamma)\gamma$. Hence,

$$\hat{f}(\gamma)\gamma = f * \gamma \in E, \quad f \in E, \quad \gamma \in \Gamma. \tag{9}$$

It follows from (9) that $\gamma \in E$ whenever there exists a function $f \in E$ satisfying $\hat{f}(\gamma) \neq 0$. Defining

$$\Gamma_E = \{\gamma \in \Gamma : \gamma \in E\},$$

it is now clear that

$$\Gamma_E = \bigcup_{f \in E} \operatorname{supp}\left(\hat{f}\right). \tag{10}$$

Note that $\Gamma_E \neq \emptyset$ whenever $E \neq \{0\}$.

The set of all trigonometric polynomials on $G$ which belong to $E$ is denoted by $\tau_E(G)$, that is,

$$\tau_E(G) = \tau(G) \cap E.$$

Using (2) and (10) it is readily verified that $\tau_E(G)$ is equal to the linear span of $\Gamma_E$ in $C(G)$. Furthermore, it follows from (9) that $f * h \in \tau_E(G)$ for all $f \in E$ and $h \in \tau(G)$. For later reference we formulate the following observation as a lemma.

**Lemma 4.2.** *Let $E$ be a homogeneous Banach space over $G$. Then $\tau_E(G)$ is dense in $E$.*

*Proof.* According to Theorem (28.53) of [9] there exists an approximate identity $\{h_\alpha\} \subseteq \tau(G)$ for $L^1(G)$ such that $h_\alpha \geq 0$ with $\|h_\alpha\|_1 = \int_G h_\alpha d\mu = 1$ for all $\alpha$ and $\lim_\alpha \left\|h_\alpha \chi_{G \setminus U}\right\|_\infty = 0$ for each neighbourhood $U$ of $0 \in G$. As observed prior to the lemma, $h_\alpha * f \in \tau_E(G)$ for each $f \in E$ and each $\alpha$. Using the stated properties of $\{h_\alpha\}$ it is routine to check that $f * h_\alpha \to_\alpha f$ in $E$ for each $f \in E$; cf. the proof of Lemma I.2.2 in [10]. Hence, we can conclude that $\tau_E(G)$ is dense in $E$ . □

Since $|\gamma| = \chi_G$, for every $\gamma \in E$, given a translation invariant B.f.s. $E \neq \{0\}$ over $G$, it follows that $\Gamma_E \neq \emptyset$ if and only if $\chi_G \in E$, if and only if $\tau_E(G) = \tau(G)$, if and only if $L^\infty(G) \subseteq E$. There exist translation invariant B.f.s.' over (every) $G$ such that $\Gamma_E = \emptyset$; see Example 3.6 in [15]. In view of the observation after (10), such a space $E$ is not a homogeneous Banach space over $G$.

Let $\{\gamma_j\}_{j=1}^n$ be any finite subset of $\Gamma$ and define $E = \operatorname{span}\{\gamma_j : 1 \leq j \leq n\}$ in $C(G)$. Then $E$ is a (finite dimensional) homogeneous Banach space over $G$ with respect to any of the norms $\|\cdot\|_p$, for $1 \leq p \leq \infty$. The fact that $\tau_y f \in E$ , for all $y \in G$ and $f \in E$, follows from the identity

$$\tau_y \gamma = \overline{(y,\gamma)}\gamma, \quad \gamma \in \Gamma. \tag{11}$$

In particular, this example shows that homogeneous Banach spaces need not be dense in $L^1(G)$. It follows from (2), for such spaces $E$, that $\Gamma_E = \{\gamma_j\}_{j=1}^n$, which shows that $\Gamma_E \subsetneqq \Gamma$ (hence, also $C(G) \nsubseteq E$) in general. If $f \in E$, then it need not follow that $\bar{f}$ or $|f|$ belongs to $E$ (consider $G = \mathbb{T}$ and $E = \operatorname{span}\{z\}$ in $C(\mathbb{T})$, for example).

The homogeneous Banach spaces $C^{(n)}(\mathbb{T})$ , for $n \in \mathbb{N}$, and $\operatorname{lip}_\alpha(\mathbb{T})$, for $0 < \alpha < 1$ (see pp. 14-16 in [10]) show that $\Gamma_E$ need not be a norm bounded subset of $E$.

Let $E \neq \{0\}$ be a homogeneous Banach space over $G$. A function $\varphi : \Gamma \to \mathbb{C}$ is called a *Fourier $E$-multiplier function* if $\operatorname{supp}(\varphi) \subseteq \Gamma_E$ and $\varphi$ has the property that, for every $f \in E$, there exists $g \in E$ satisfying $\hat{g} = \varphi\hat{f}$. By property (H-1) in Definition 3.1 and injectivity of the Fourier transform on $L^1(G)$, it follows that such a function $g$ is unique; we denote it by $M_\varphi^E f$. The so defined linear map $M_\varphi^E : E \to E$ is also continuous, as follows via the Closed Graph Theorem (using the continuity of the Fourier transform from $E$ into $c_0(\Gamma)$; see property (H-1)). Hence, $M_\varphi^E \in \mathcal{L}(E)$. A routine calculation, using the identity $\hat{\gamma} = \chi_{\{\gamma\}}$, shows that $M_\varphi^E \gamma = \varphi(\gamma)\gamma$ for every $\gamma \in \Gamma_E$. Since $\gamma \neq 0$, this implies that $|\varphi(\gamma)| \leq \left\|M_\varphi^E \gamma\right\|_E / \|\gamma\|_E$ for all $\gamma \in \Gamma_E$. But, $\operatorname{supp}(\varphi) \subseteq \Gamma_E$ and so we can conclude that

$$\|\varphi\|_\infty \leq \left\|M_\varphi^E\right\|. \tag{12}$$

The collection of all Fourier $E$-multiplier functions is denoted by $\mathcal{M}_E(\Gamma)$, which is clearly a linear subspace of $\ell^\infty(\Gamma)$. Actually, $\mathcal{M}_E(\Gamma)$ is a *subalgebra* of $\ell^\infty(\Gamma)$. Indeed, let $\varphi, \psi \in \mathcal{M}_E(\Gamma)$. The identity $\operatorname{supp}(\varphi\psi) = \operatorname{supp}(\varphi) \cap \operatorname{supp}(\psi)$ implies that $\operatorname{supp}(\varphi\psi) \subseteq \Gamma_E$. Moreover, given $f \in E$, there exist functions $g, h \in E$ satisfying $\hat{g} = \varphi\hat{f}$ and $\hat{h} = \psi\hat{g}$. Then $\hat{h} = \varphi\psi\hat{f}$. Accordingly, $\varphi\psi \in \mathcal{M}_E(\Gamma)$. It follows from (10) that $\chi_{\Gamma_E}$ is the unit element of $\mathcal{M}_E(\Gamma)$, corresponding to the multiplier operator $M_{\chi_{\Gamma_E}}^E = I$.

For $E = \operatorname{span}\{\gamma_j : 1 \leq j \leq n\}$ with $\{\gamma_j\}_{j=1}^\infty \subseteq \Gamma$ a finite set, it is clear that

$$\mathcal{M}_E(\Gamma) = \left\{\sum\nolimits_{j=1}^n \alpha_j \chi_{\{\gamma_j\}} : \alpha_j \in \mathbb{C},\ 1 \leq j \leq n\right\}.$$

The homogeneous Banach space $A(G)$ given prior to (8) satisfies $\mathcal{M}_{A(G)}(\Gamma) = \ell^\infty(\Gamma)$. Indeed, fix $\varphi \in \ell^\infty(\Gamma)$. Given $f \in A(G)$, it is clear that $\varphi\hat{f} \in \ell^1(\Gamma)$.

Recalling that $A(G)$ is isometrically isomorphic with $\ell^1(\Gamma)$ via the Fourier transform, it follows that there exists a unique $g \in A(G)$ satisfying $\hat{g} = \varphi \hat{f}$. Note that actually $g = \sum_{\gamma \in \Gamma} \varphi(\gamma) \hat{f}(\gamma) \gamma$ as an absolutely convergent series in $A(G)$. Hence, $\varphi \in \mathcal{M}_{A(G)}(\Gamma)$. Note also that $\left\| M_{\varphi}^{A(G)} \right\| = \|\varphi\|_{\infty}$.

*Remark 4.3.* Let $E$ be a homogeneous Banach space over $G$ and $\varphi : \Gamma \to \mathbb{C}$ be a function with the property that, for every $f \in E$, there exists $g \in E$ satisfying $\hat{g} = \varphi \hat{f}$. Then $\operatorname{supp}(\varphi \chi_{\Gamma_E}) \subseteq \Gamma_E$ and so $\varphi \chi_{\Gamma_E} \in \mathcal{M}_E(\Gamma)$. Note also that Lemma 4.1 implies that

$$\left\{ \hat{\lambda} \chi_{\Gamma_E} : \lambda \in M(G) \right\} \subseteq \mathcal{M}_E(\Gamma),$$

since $(f * \lambda)^{\wedge} = \hat{\lambda} \cdot \hat{f}$ for all $\lambda \in M(G)$ and all $f \in E$.

*Remark 4.4.* Let $\varphi \in \mathcal{M}_E(\Gamma)$ and let $S \in \mathcal{L}(E)$ be the operator $S = M_{\varphi}^E$. For each $\gamma \in \Gamma_E$ we have that $(S\gamma)^{\wedge} = \varphi \hat{\gamma} = \varphi \chi_{\{\gamma\}}$. Consequently, the function $\varphi$ can be recovered from the operator $S$ via the formula

$$\varphi(\gamma) = \begin{cases} (S\gamma)^{\wedge}(\gamma) & \text{if } \gamma \in \Gamma_E \\ 0 & \text{if } \gamma \in \Gamma \backslash \Gamma_E \end{cases}. \tag{13}$$

Let $\varphi \in \mathcal{M}_E(\Gamma)$. If $y \in G$ and $f \in L^1(G)$, then $(\tau_y f)^{\wedge}(\gamma) = \overline{\gamma(y)} \hat{f}(\gamma)$ for all $\gamma \in \Gamma$. Consequently, using (10), we find that

$$\left(\tau_y M_{\varphi}^E f\right)^{\wedge}(\gamma) = \left(M_{\varphi}^E \tau_y f\right)^{\wedge}(\gamma) = \overline{\gamma(y)} \varphi(\gamma) \hat{f}(\gamma), \quad \gamma \in \Gamma, \quad f \in E,$$

which implies that $\tau_y M_{\varphi}^E = M_{\varphi}^E \tau_y$ for all $y \in G$, that is, $M_{\varphi}^E \in \mathfrak{m}(E)$. Note that the estimate (12) implies that the linear, multiplicative map $\varphi \longmapsto M_{\varphi}^E$ from $\mathcal{M}_E(\Gamma)$ into $\mathfrak{m}(E)$ is injective. In Lemma 4.7 below we will see that this map is also surjective.

Summarizing the above discussion yields the following result.

**Lemma 4.5.** *Let $E \neq \{0\}$ be a homogeneous Banach space over $G$. Every $\varphi \in \mathcal{M}_E(\Gamma)$ belongs to $\ell^{\infty}(\Gamma)$ and there exists an operator $M_{\varphi}^E \in \mathfrak{m}(E)$ satisfying the inequality ( 12) and*

$$\left(M_{\varphi}^E f\right)^{\wedge} = \varphi \hat{f}, \quad f \in E.$$

*Moreover, $\mathcal{M}_E(\Gamma)$ is a subalgebra of $\ell^{\infty}(\Gamma)$ with unit element $\chi_{\Gamma_E}$.*

*Remark 4.6.* Let $\varphi \in \mathcal{M}_E(\Gamma)$. If $f \in \tau_E(G)$ is given by $f = \sum_{j=1}^n c_j \gamma_j$, with $c_j \in \mathbb{C}$ and $\gamma_j \in \Gamma_E$, for $j = 1, \ldots, n$, then

$$M_{\varphi}^E f = \sum_{j=1}^{n} \varphi(\gamma_j) c_j \gamma_j.$$

Indeed, apply the identity $M_{\varphi}^E \gamma = \varphi(\gamma) \gamma$ for all $\gamma \in \Gamma_E$ and use the linearity of $M_{\varphi}^E$.

The converse of Lemma 4.5 is also valid.

**Lemma 4.7.** *Let $E \neq \{0\}$ be a homogeneous Banach space over $G$ and $T \in \mathfrak{m}(E)$. Then there exists a unique function $\varphi \in \mathcal{M}_E(\Gamma)$ such that $T = M_{\varphi}^E$.*

*Proof.* Fix $\gamma \in \Gamma_E$. It follows from (11) that $\tau_y T\gamma = T\tau_y\gamma = \overline{\gamma(y)}T\gamma$ for all $y \in G$. Taking Fourier transforms yields, via (8), that

$$\overline{\alpha(y)}\,(T\gamma)^\wedge(\alpha) = \overline{\gamma(y)}\,(T\gamma)^\wedge(\alpha), \quad \alpha \in \Gamma,\ y \in G.$$

For each $\alpha \in \Gamma\backslash\{\gamma\}$ there exists $y \in G$ with $\alpha(y) \neq \gamma(y)$, from which it follows that $\operatorname{supp}\left((T\gamma)^\wedge\right) \subseteq \{\gamma\}$. Hence, for every $\gamma \in \Gamma_E$, there is $\varphi(\gamma) \in \mathbb{C}$ satisfying $(T\gamma)^\wedge = \varphi(\gamma)\chi_{\{\gamma\}} = \varphi(\gamma)\hat{\gamma}$. For $\gamma \in \Gamma\backslash\Gamma_E$ we set $\varphi(\gamma) = 0$. Then $\operatorname{supp}(\varphi) \subseteq \Gamma_E$ and

$$(Tf)^\wedge = \varphi\hat{f}, \quad f \in \tau_E(G). \tag{14}$$

The density of $\tau_E(G)$ in $E$ (see Lemma 4.2) and the continuity of the Fourier transform from $E$ into $c_0(\Gamma)$ can then be used, together with (14), to deduce that $(Tf)^\wedge = \varphi\hat{f}$ for all $f \in E$. Hence, $\varphi \in \mathcal{M}_E(\Gamma)$ and $T = M_\varphi^E$.

The uniqueness of the function $\varphi$ has already been observed in the discussion prior to Lemma 4.5. The proof is thereby complete. □

Note that Lemma 4.7 implies that the operator algebra

$$\mathfrak{m}(E) = \left\{M_\varphi^E : \varphi \in \mathcal{M}_E(\Gamma)\right\}$$

and so $\mathfrak{m}(E)$ is commutative for every homogeneous Banach space $E$ over $G$. Indeed, it is evident that $M_\varphi^E M_\psi^E = M_{\varphi\psi}^E = M_\psi^E M_\varphi^E$ for all $\varphi, \psi \in \mathcal{M}_E(\Gamma)$.

*Remark 4.8.* Let $G$ be any infinite, compact abelian group. If $E$ is a Banach space of functions on $G$ which only satisfies (H1) and (H2), then there may exist operators $T \in \mathfrak{m}(E)$ which are *not* of the form $M_\psi^E$ for any $\psi \in \mathcal{M}_E(\Gamma)$. Indeed, consider $E = L^\infty(G)$ and let $\varphi \in L^\infty(G)$ be an *invariant mean* with the properties stated in Theorem 3.4 of [20]. In particular, $\varphi$ is *not* given by integration over $G$ with respect to normalized Haar measure $\mu$ on $G$. Define $T \in \mathcal{L}(L^\infty(G))$ to be the rank-1 projection given by $Tf = \langle f, \varphi\rangle\chi_G$, for $f \in L^\infty(G)$, in which case $T \in \mathfrak{m}(L^\infty(G))$. Also, the function $\psi = \chi_{\{\mathbf{1}\}}$ on $\Gamma$ belongs to $\mathcal{M}_{L^\infty(G)}(\Gamma)$ and the corresponding multiplier operator $M_\psi^{L^\infty(G)} \in \mathfrak{m}(L^\infty(G))$ is given by

$$M_\psi^{L^\infty(G)} f = \left(\int_G f\,d\mu\right)\chi_G = f * \chi_G, \quad f \in L^\infty(G).$$

A direct calculation shows that the condition $TM_\psi^{L^\infty(G)} = M_\psi^{L^\infty(G)}T$ would imply that $\langle f, \varphi\rangle = \int_G f\,d\mu$ for all $f \in L^\infty(G)$, which is a contradiction. This shows that the algebra $\mathfrak{m}(L^\infty(G))$ is not commutative. Consequently, the result of Lemma 4.7 is not valid for $E = L^\infty(G)$.

## 5. The Projection of $\mathcal{L}(E)$ onto $\mathfrak{m}(E)$

As before, $G$ is an infinite, compact abelian group with normalized Haar measure $\mu$. Given a homogeneous Banach space $E$ over $G$, let $E^+$ denote the cone of all $f \in E$ such that $f(x) \geq 0$ for $\mu$-a.e. $x \in E$. Considering $E = \operatorname{span}\{z\}$ in $C(\mathbb{T})$ shows that $E^+ = \{0\}$ is possible. An operator $T \in \mathcal{L}(E)$ is called *positive* if $T(E^+) \subseteq E^+$. Observe that each translation operator $\tau_y$, for $y \in G$, is positive.

Since compact abelian groups are amenable, for such compact groups $G$ the following result is an extension of Theorem 1.1 of [1] to a significantly larger class of spaces than $L^p(G)$ for $1 \le p < \infty$.

**Theorem 5.1.** *Let $E \neq \{0\}$ be a homogeneous Banach space over $G$. For each $T \in \mathcal{L}(E)$ define the function $\hat{T} : \Gamma \to \mathbb{C}$ by*

$$\hat{T}(\gamma) = \begin{cases} (T\gamma)^{\wedge}(\gamma) & \text{if } \gamma \in \Gamma_E \\ 0 & \text{if } \gamma \in \Gamma \backslash \Gamma_E \end{cases}. \tag{15}$$

*Then $\hat{T} \in \mathcal{M}_E(\Gamma)$ and the linear map $\mathcal{Q} : \mathcal{L}(E) \to \mathcal{L}(E)$, given by*

$$\mathcal{Q}(T) = M^E_{\hat{T}}, \quad T \in \mathcal{L}(E),$$

*is a contractive projection of $\mathcal{L}(E)$ onto $\mathfrak{m}(E)$, preserving positivity.*

*Proof.* Fix $T \in \mathcal{L}(E)$. For each $f \in E$, the $E$-valued function $\Phi_f : x \mapsto \tau_{-x} T \tau_x f$ is continuous on $G$ and hence, by Lemma 2.1, it is Bochner $\mu$-integrable. Define the linear map $S_T : E \to E$ by

$$S_T f = \int_G^{(B)} \Phi_f(x)\, d\mu(x), \quad f \in E. \tag{16}$$

Using estimate (4), it follows that $S_T \in \mathcal{L}(E)$ as

$$\|S_T f\|_E \le \int_G \|\Phi_f(x)\|_E\, d\mu(x) \le \int_G \|T\| . \|f\|_E\, d\mu(x) = \|T\| . \|f\|_E,$$

for each $f \in E$. In particular, $\|S_T\| \le \|T\|$.

To see that $S_T \in \mathfrak{m}(E)$, fix $y \in G$. Then, for $f \in E$, Theorem II.2.6 of [4] and the translation invariance of $\mu$ yields

$$\begin{aligned} \tau_y S_T f &= \tau_y \int_G^{(B)} \tau_{-x} T \tau_x f d\mu(x) = \int_G^{(B)} \tau_{y-x} T \tau_x f d\mu(x) \\ &= \int_G^{(B)} \tau_{-x} T \tau_{x+y} f d\mu(x) = S_T \tau_y f, \end{aligned}$$

which shows that $\tau_y S_T = S_T \tau_y$. We can conclude that $S_T \in \mathfrak{m}(E)$. Furthermore, whenever $T \in \mathfrak{m}(E) \subseteq \mathcal{L}(E)$, we note that $\Phi_f(x) = Tf$ for all $f \in E$ and all $x \in G$ and hence, (16) implies that $S_T = T$. Defining $\mathcal{Q}(T) = S_T$, for $T \in \mathcal{L}(E)$, it is now clear that $\mathcal{Q} : \mathcal{L}(E) \to \mathcal{L}(E)$ is a contractive projection in $\mathcal{L}(E)$ with range $\mathfrak{m}(E)$.

It follows from Lemma 4.7, for each $T \in \mathcal{L}(E)$, that there exists a unique Fourier multiplier function $\hat{T} \in \mathcal{M}_E(\Gamma)$ such that $S_T = M^E_{\hat{T}}$. Next we establish formula (15). To this end recall, for each $g \in L^\infty(G)$, that the linear functional $\varphi_g \in E^*$ is defined by $\langle f, \varphi_g \rangle = \int_G fg\, d\mu$, for $f \in E$ (see the proof of Lemma 4.1). A direct computation shows that $\tau_y^* \varphi_g = \varphi_{\tau_{-y} g}$ for all $g \in L^\infty(G)$ and $y \in G$, where $\tau_y^* : E^* \to E^*$ is the Banach space adjoint operator of $\tau_y$.

Let $T \in \mathcal{L}(E)$ be fixed. It follows from formula ( 13), applied to the operator $S = S_T$, that the function $\hat{T}$ is given by

$$\hat{T}(\gamma) = (S_T\gamma)^{\wedge}(\gamma) = \int_G (S_T\gamma)(x)\,\bar{\gamma}(x)\,d\mu(x) = \langle S_T\gamma, \varphi_{\bar{\gamma}}\rangle \tag{17}$$

for all $\gamma \in \Gamma_E$ (and $\hat{T}(\gamma) = 0$ for $\gamma \in \Gamma\backslash\Gamma_E$). Using (16) in combination with (5), we find that

$$\langle S_T\gamma, \varphi_{\bar{\gamma}}\rangle = \int_G \langle \tau_{-x}T\tau_x\gamma, \varphi_{\bar{\gamma}}\rangle\,d\mu(x), \quad \gamma \in \Gamma_E. \tag{18}$$

It follows from (11), for all $x \in G$ and $\gamma \in \Gamma_E$, that

$$\begin{aligned}\langle \tau_{-x}T\tau_x\gamma, \varphi_{\bar{\gamma}}\rangle &= \left\langle T\tau_x\gamma, \tau^*_{-x}\varphi_{\bar{\gamma}}\right\rangle = \langle T\tau_x\gamma, \varphi_{\tau_x\bar{\gamma}}\rangle \\ &= \overline{\gamma(x)}\gamma(x)\,\langle T\gamma, \varphi_{\bar{\gamma}}\rangle = \langle T\gamma, \varphi_{\bar{\gamma}}\rangle = (T\gamma)^{\wedge}(\gamma).\end{aligned} \tag{19}$$

A combination of (17), (18) and (19) yields the formula (15).

Finally, let $T \in \mathcal{L}(E)$ be a positive operator. If $g \in L^\infty(G)^+$ and $h \in E^+$, then $\langle h, \varphi_g\rangle \geq 0$. So, fix $f \in E^+$. Then $\tau_{-x}T\tau_x f \geq 0$ for all $x \in G$. Hence, for each $g \in L^\infty(G)^+$, we have that

$$\langle \mathcal{Q}(T)f, \varphi_g\rangle = \int_G \langle \tau_{-x}T\tau_x f, \varphi_g\rangle\,d\mu(x) \geq 0,$$

which implies that $\mathcal{Q}(T)f \geq 0$. This shows that $\mathcal{Q}(T)$ is a positive operator. The proof is thereby complete. □

*Remark 5.2.* Consider $E = L^2(\mathbb{T})$, where $G = \mathbb{T}$ is the circle group, identified with the interval $[0, 2\pi)$ equipped with normalized Haar measure $(2\pi)^{-1}dx$. For each $n \in \mathbb{Z}$, define the function $e_n(x) = e^{inx}$, for $x \in [0, 2\pi)$. Then $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis in the Hilbert space $E$ and so every operator $T \in \mathcal{L}(E)$ can be represented by an infinite matrix with respect to $\{e_n\}_{n\in\mathbb{Z}}$. Let $T \in \mathcal{L}(E)$. Then the decomposition of $T$ with respect to the projection $\mathcal{Q}$ of Theorem 5.1 corresponds to the decomposition of the matrix of $T$ into its diagonal and off diagonal parts.

As an application, recall *Bochner's theorem* which states, for a function $\varphi : \Gamma \to \mathbb{C}$, that $\varphi = \hat{\lambda}$ for some measure $\lambda \in M(G)^+$ if and only if $\varphi$ is positive definite; see, for example, Section 1.4 of [18]. Since $C(G)$ is a homogeneous Banach space over $G$, by Theorem 5.1 an equivalent statement is that $\varphi = \hat{T}$ for some positive operator $T \in \mathcal{L}(C(G))$. Indeed, by Theorem 5.1 we have, for each $R \in \mathcal{L}(C(G))$, that $\mathcal{Q}(R) \in \mathfrak{m}(C(G))$ and hence, $\hat{R} = \hat{\lambda}$ for some $\lambda \in M(G)$; see, for example, Theorem 3.3.2 in [12]. The positivity of $T$ ensures that $\mathcal{Q}(T)$ is also positive (cf. Theorem 5.1), that is, $f * \lambda = \mathcal{Q}(T)f \geq 0$ for all $f \in C(G)^+$. Accordingly, $\int_G f(-x)\,d\lambda(x) = (f * \lambda)(0) \geq 0$ for all $f \in C(G)^+$. Via the Riesz representation theorem this implies that $\lambda \in M(G)^+$.

For our final result, let $E \neq \{0\}$ be a translation invariant B.f.s. over $G$ with o.c.-norm. Then $E$ is a homogeneous Banach space over $G$; see the discussion after Definition 3.1. Furthermore, $L^{\infty}(G) \subseteq E$; see Proposition 3.8 in [15]. In particular, $\Gamma_E = \Gamma$ and $\tau(G) \subseteq E$.

A linear operator $T : E \to E$ is called *order bounded* if $T$ maps order bounded subsets of $E$ onto order bounded sets, that is, for every $u \in E^+$ there exists $v \in E^+$ such that $|Tf| \leq v$ whenever $f \in E$ satisfies $|f| \leq u$. Evidently, every positive operator is order bounded. The collection of all order bounded linear operators is denoted by $\mathcal{L}_b(E)$ and it has the structure of a Dedekind complete, complex Riesz space (see Section 92 of [23]), as $E$ itself is a Dedekind complete complex Banach lattice. In particular, each $T \in \mathcal{L}_b(E)$ is a linear combination of at most four positive operators (in other words, every $T \in \mathcal{L}_b(E)$ is a *regular operator* on $E$). Since every positive, linear operator on a Banach lattice is necessarily continuous (see, for example, Theorem 83.12 in [23]), it follows that $\mathcal{L}_b(E) \subseteq \mathcal{L}(E)$.

Recall from Lemma 4.1 that, for each $\lambda \in M(G)$ , the operator $T_\lambda \in \mathcal{L}(E)$ of convolution with $\lambda$ exists. The following fact will be used.

**Lemma 5.3.** *Let $E \neq \{0\}$ be a translation invariant B.f.s. over $G$ with o.c.-norm. Then*

$$\mathcal{L}_b(E) \cap \mathfrak{m}(E) = \{T_\lambda : \lambda \in M(G)\}.$$

*Proof.* Let $\lambda \in M(G)$. Then $\lambda$ is a linear combination of at most four positive measures from $M(G)^+$. Consequently, $T_\lambda$ is a linear combination of at most four positive operators from $\mathfrak{m}(E)$. Hence, $T_\lambda \in \mathcal{L}_b(E) \cap \mathfrak{m}(E)$. This shows that $\{T_\lambda : \lambda \in M(G)\} \subseteq \mathcal{L}_b(E) \cap \mathfrak{m}(E)$.

For the proof of the converse implication, first observe that $\mathfrak{m}_b(E) = \mathcal{L}_b(E) \cap \mathfrak{m}(E)$ is a (complex) vector sublattice of $\mathcal{L}_b(E)$; see Proposition 5.3 of [16]. Therefore, each operator $T \in \mathfrak{m}_b(E)$ is a linear combination of at most four positive operators in $\mathfrak{m}_b(E)$.

Fix $T \in \mathfrak{m}_b(E)^+$. It follows from Lemma 4.7 that there exists a unique $\varphi \in \mathcal{M}_E(\Gamma)$ such that $T = M_\varphi^E$. Let $f \in \tau(G) = \tau_E(G)$ be given by $f = \sum_{j=1}^n c_j\gamma_j$, with $c_j \in \mathbb{C}$ and $\gamma_j \in \Gamma$ for $1 \leq j \leq n$. Since

$$\begin{aligned} |f|^2 = f\overline{f} &= \left(\sum\nolimits_{j=1}^n c_j\gamma_j\right)\left(\sum\nolimits_{k=1}^n \overline{c_k}\gamma_k^{-1}\right) \\ &= \sum\nolimits_{j,k=1}^n c_j\overline{c_k}\gamma_j\gamma_k^{-1} \end{aligned}$$

belongs to $\tau(G)$, Remark 4.6 implies that

$$T\left(|f|^2\right) = \sum\nolimits_{j,k=1}^n \varphi\left(\gamma_j\gamma_k^{-1}\right) c_j\overline{c_k}\gamma_j\gamma_k^{-1}. \tag{20}$$

Using that $T$ is positive and $|f|^2 \geq 0$ in $E$, it follows that $T\left(|f|^2\right) \geq 0$ in $E$ (that is, $\mu$-a.e. on $G$). Since the function (20) is continuous on $G$, this implies that

$$\sum\nolimits_{j,k=1}^n \varphi\left(\gamma_j\gamma_k^{-1}\right) c_j\overline{c_k}\left(x, \gamma_j\gamma_k^{-1}\right) \geq 0, \quad x \in G.$$

Selecting $x = 0$ yields

$$\sum\nolimits_{j,k=1}^n \varphi\left(\gamma_j\gamma_k^{-1}\right) c_j\overline{c_k} \geq 0.$$

This holds for all choices of $c_1, \dots, c_n \in \mathbb{C}$, all $\gamma_1, \dots, \gamma_n \in \Gamma$ and all $n \in \mathbb{N}$. Hence, the function $\varphi : \Gamma \to \mathbb{C}$ is positive definite. It follows from Bochner's Theorem (see, for example, Theorem 1.4.3 in [18] or Theorem 4.18 in [8]) that there exists $\lambda \in M(G)^+$ such that $\varphi = \hat{\lambda}$. This suffices for the proof of the lemma. □

Let $B(\Gamma) = \left\{ \hat{\lambda} : \lambda \in M(G) \right\}$. Its closure in $\ell^\infty(\Gamma)$ is denoted by $\overline{B(\Gamma)}$. For compact abelian groups $G$ the next fact is an extension of Theorem 3.1 in [1].

**Theorem 5.4.** *Let $E \neq \{0\}$ be a translation invariant B.f.s. over $G$ with o.c.-norm and $\overline{\mathcal{L}_b(E)}$ denote the operator norm closure of $\mathcal{L}_b(E)$ in $\mathcal{L}(E)$. Whenever $T \in \overline{\mathcal{L}_b(E)}$ the Fourier $E$-multiplier function $\hat{T}$, as defined in Theorem 5.1, satisfies $\hat{T} \in \overline{B(\Gamma)}$.*

*Proof.* Let $T \in \overline{\mathcal{L}_b(E)}$. Then there exists a sequence $(T_n)_{n=1}^\infty$ in $\mathcal{L}_b(E)$ such that $\|T - T_n\| \to 0$ as $n \to \infty$. Let $\mathcal{Q} : \mathcal{L}(E) \to \mathcal{L}(E)$ be the contractive projection onto $\mathfrak{m}(E)$ given in Theorem 5.1. Since $\mathcal{Q}$ preserves positivity, it follows, for each $n \in \mathbb{N}$, that $\mathcal{Q}(T_n) \in \mathcal{L}_b(E) \cap \mathfrak{m}(E)$ and so, by Lemma 5.3, there exists $\lambda_n \in M(G)$ satisfying $\hat{T}_n = \hat{\lambda}_n$ . Furthermore, the estimate (12) implies that

$$\left\| \hat{T} - \hat{\lambda}_n \right\|_\infty \leq \left\| M^E_{\hat{T} - \hat{T}_n} \right\| = \|\mathcal{Q}(T - T_n)\| \leq \|T - T_n\|, \quad n \in \mathbb{N},$$

and hence, $\left\| \hat{T} - \hat{\lambda}_n \right\|_\infty \to 0$ as $n \to \infty$. The proof is thereby complete. □

We conclude with some examples. Let $E$ be any rearrangement invariant B.f.s. with o.c.-norm over the circle group $\mathbb{T}$. Then $E$ is a homogeneous Banach space over $\mathbb{T}$; see the discussion at the end of Section 3. The *lower and upper Boyd indices* of $E$ are denoted by $\overline{\alpha}_E$ and $\underline{\alpha}_E$, respectively, and always satisfy $0 \leq \underline{\alpha}_E \leq \overline{\alpha}_E \leq 1$; see Sections III.5 and III.6 of [2] for the definition and properties of these indices. According to a theorem of D.W. Boyd, the *Hilbert transform* $H$ is a continuous linear operator on $E$ (and hence, $H \in \mathfrak{m}(E)$) if and only if $0 < \underline{\alpha}_E \leq \overline{\alpha}_E < 1$; see Theorem III.6.10 and Corollary III.6.11 in [2]. Recall that $H$ is the multiplier operator corresponding to the Fourier multiplier function

$$\varphi(n) = -i \operatorname{sgn}(n), \quad n \in \mathbb{Z},$$

with $\operatorname{sgn}(0) = 0$; see [2], Lemma III.6.7. Since $\lim_{n\to\infty} \varphi(n) \neq \lim_{n\to-\infty} \varphi(n)$ and both limits exist, it follows from Corollary 3.8 of [1] that $\varphi \notin \overline{B(\mathbb{Z})}$. Consequently, Theorem 5.4 implies that $H \notin \overline{\mathcal{L}_b(E)}$.

Further examples of a similar nature to the Hilbert transform occur for certain elements from the space $\operatorname{bv}(\mathbb{Z})$ of all functions $\varphi : \mathbb{Z} \to \mathbb{C}$ of bounded variation. Indeed, functions $\varphi \in \operatorname{bv}(\mathbb{Z})$ have the property that both limits $\lim_{n\to\infty} \varphi(n)$ and $\lim_{n\to-\infty} \varphi(n)$ always exist; see Proposition 8 (i) in [13]. Moreover, Ste čkin's theorem (cf. Theorem 6.3.5 in [6]) ensures that $\operatorname{bv}(\mathbb{Z}) \subseteq \bigcap_{1<p<\infty} \mathcal{M}_{L^p(\mathbb{T})}(\mathbb{Z})$. Again Corollary 3.8 of [1] implies, whenever

$\lim_{n\to\infty}\varphi(n) \neq \lim_{n\to-\infty}\varphi(n)$, that $\varphi \notin \overline{B(\mathbb{Z})}$ and hence, Theorem 3.1 of [1] (see also Theorem 5.4 above) ensures that the corresponding multiplier operator $M_\varphi^{L^p(\mathbb{T})} \notin \overline{\mathcal{L}_b(L^p(\mathbb{T}))}$, for $1 < p < \infty$.

The finite dimensional, homogeneous Banach spaces presented after Lemma 4.2 are surely not dense in $L^1(G)$. Infinite dimensional examples also exist. Indeed, for each $1 \le p < \infty$, define

$$\mathcal{H}^p = \left\{ f \in L^p(\mathbb{T}) : \hat{f}(n) = 0 \text{ for all } n \in \mathbb{Z} \text{ with } n < 0 \right\}$$

equipped with the norm from $L^p(\mathbb{T})$; cf. Section 3.2 of [5]. The space $\mathcal{H}^p$ consists of all the boundary values of functions in the space $H^p$ of analytic functions on the open unit disc in $\mathbb{C}$; see Theorem 3.4 in [5]. As is well known, the space $\mathcal{H}^p$ is a closed subspace of $L^p(\mathbb{T})$ and so it is a Banach space which is continuously included in $L^1(\mathbb{T})$. Using (8), it is clear that property (H-1) in Definition 3.1 is satisfied. Conditions (H-2) and (H-3) are also satisfied since $L^p(\mathbb{T})$ has these properties. So, we can conclude that $\mathcal{H}^p$ is a homogeneous Banach space over $\mathbb{T}$. Note that $\mathbb{Z}_{\mathcal{H}^p} = \mathbb{Z}^+$, where $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. Evidently, $\mathcal{H}^p$ is infinite dimensional and not dense in $L^1(\mathbb{T})$.

We note, for every $1 < p < \infty$, that $\mathcal{H}^p$ is actually a proper, complemented subspace of $L^p(\mathbb{T})$. Indeed, as is well known, the *Riesz projection* $R : L^p(\mathbb{T}) \to L^p(\mathbb{T})$, which is the multiplier operator corresponding to the Fourier multiplier function $\chi_{\mathbb{Z}^+}$, is a continuous projection in $L^p(\mathbb{T})$ onto $\mathcal{H}^p$. Let $P \in \mathcal{L}(L^p(\mathbb{T}))$ denote the rank-1 projection given by $Pf = \hat{f}(0)\chi_{\mathbb{T}}$, for $f \in L^p(\mathbb{T})$. Note that $P \in \mathfrak{m}(E) \cap \mathcal{L}_b(E)$. Since $R = \frac{1}{2}(I + P + iH)$, where $H$ is the Hilbert transform as in the previous example, it is also clear that $R \notin \overline{\mathcal{L}_b(L^p(\mathbb{T}))}$.

**Data Availability** There are no data related to this publication.

## Declarations

**Ethical statement** The authors did not receive support from any organization for the submitted work.

**Competing interests** The authors have no competing interests to declare that are relevant to the content of this article.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

## References

[1] Arendt, W., Voigt, J.: Approximation of multipliers by regular operators. Indag. Math. N.S **2**, 159–170 (1991)

[2] Bennett, C., Sharpley, R.: Interpolation of Operators. Academic Press, London-New York (1988)

[3] Dodds, P.G., de Pagter, B., Sukochev, F.A.: Noncommutative Integration and Operator Theory, Progress in Mathematics 349. Birkhäuser, Cham (2023)

[4] Diestel, J., Uhl, J.J., J.: Vector Measures, Amer. Math. Soc., Providence R.I (1977)

[5] Duren, P.L.: Theory of $H^p$ Spaces, Pure and Applied Math, vol. 38. Academic Press, New York-San Francisco-London (1970)

[6] Edwards, R.E., Gaudry, G.I.: Littlewood-Paley and Multiplier Theory. Springer-Verlag, Berlin-Heidelberg-New York (1977)

[7] Feichtinger, H.G.: Homogeneous Banach spaces as Banach convolution modules over $M(G)$. MDPI, Mathematics **10**(3), 1–22 (2022)

[8] Folland, G.B.: A Course in Abstract Harmonic Analysis. CRC Press, Boca Raton (1995)

[9] Hewitt, E., Ross, K.A.: Abstract Harmonic Analysis II. Springer Verlag, Berlin (1970)

[10] Katznelson, Y.: An Introduction to Harmonic Analysis, Dover Publ, New York (1976). (2nd corrected ed.)

[11] Krein, S.G., Petunin, J.I., Semenov, E.M.: Interpolation of Linear Operators, Amer. Math. Soc., Providence R.I (1982)

[12] Larsen, R.: An Introduction to the Theory of Multipliers. Springer-Verlag, Berlin-Heidelberg (1971)

[13] Mockenhaupt, G., Ricker, W.J.: Approximation of $p$-multiplier operators via their spectral projections. Positivity **12**, 133–150 (2008)

[14] de Pagter, B., Ricker, W.J.: Fourier multipliers and spectral measures in Banach function spaces. Positivity **13**, 225–241 (2009)

[15] de Pagter, B., Ricker, W.J.: A Fuglede type theorem for Fourier multiplier operators. Indag. Math. **34**, 259–273 (2023)

[16] B. de Pagter and W.J. Ricker, Translation invariant Banach function spaces and Fourier multiplier operators, *Geometry of Banach spaces and Related Fields*, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence R.I., 106 (2024), 67-96

[17] Pedersen, T.V.: Homogeneous Banach spaces on the unit circle. Publ. Mat. **44**, 135–155 (2000)

[18] Rudin, W.: Fourier Analysis on Groups. Wiley-Interscience, New York (1967)

[19] Rudin, W.: Real and Complex Analysis. McGraw-Hill, London (1970)

[20] Rudin, W.: Invariant means on $L^\infty$. Studia Math. **44**, 219–227 (1972)

[21] Shilov, G.E.: Homogeneous rings of functions. Amer. Math. Soc. Transl (1) **8**, 392–455 (1962)

[22] Zaanen, A.C.: Integration, 2nd ed. North-Holland, Amsterdam (1967)

[23] Zaanen, A.C.: Riesz Spaces II. North-Holland, Amsterdam (1983)

Ben de Pagter (✉)
Delft Institute of Applied Mathematics, Faculty EEMCS
Delft University of Technology
P.O. Box 5031 2600 GADelft
The Netherlands
e-mail: b.depagter@tudelft.nl

Werner Ricker
Math.-Geogr. Fakultät
Katholische Universität Eichstätt-Ingolstadt
D-85072 Eichstätt
Germany
e-mail: werner.ricker@ku.de

Received: November 24, 2025.
Accepted: May 28, 2026.