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Discrete Triebel-Lizorkin spaces and expansive matrices

Jordy Timo van Velthoven [a,*], Felix Voigtlaender [b]

[a] *Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria*
[b] *Mathematical Institute for Machine Learning and Data Science (MIDS), Catholic University of Eichstätt–Ingolstadt (KU), Auf der Schanz 49, 85049 Ingolstadt, Germany*

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ABSTRACT

We provide a characterization of two expansive dilation matrices yielding equal discrete anisotropic Triebel-Lizorkin spaces. For two such matrices A and B, and arbitrary $\alpha \in \mathbb{R}$ and $p, q \in (0, \infty]$, it is shown that $\dot{\mathbf{f}}^{\alpha}_{p,q}(A) = \dot{\mathbf{f}}^{\alpha}_{p,q}(B)$ if and only if the set $\{A^j B^{-j} : j \in \mathbb{Z}\}$ is finite, or in the trivial case when $|\det(A)|^{\alpha+1/2-1/p} = |\det(B)|^{\alpha+1/2-1/p}$ and $p = q$. This provides an extension of a result by Triebel for diagonal dilations to arbitrary expansive matrices. The obtained classification of dilations is different from corresponding results for anisotropic Triebel-Lizorkin function spaces.

* Corresponding author.
E-mail addresses: jordy.timo.van.velthoven@univie.ac.at (J.T. van Velthoven), felix.voigtlaender@ku.de (F. Voigtlaender).

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1. Introduction

Let $A \in \mathrm{GL}(d,\mathbb{R})$ be an expansive matrix, i.e., all eigenvalues $\lambda \in \mathbb{C}$ of A satisfy $|\lambda| > 1$. The associated discrete (homogeneous) Besov space $\dot{\mathbf{b}}^{\alpha}_{p,q}(A)$ and Triebel-Lizorkin space $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ are defined to consist of those sequences $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ satisfying

$$\|c\|_{\dot{\mathbf{b}}^{\alpha}_{p,q}(A)} := \Bigg(\sum_{j\in\mathbb{Z}} |\det(A)|^{-jq(\alpha+1/2)} \Bigg\| \sum_{k\in\mathbb{Z}^d} |c_{j,k}| \mathbb{1}_{A^j([0,1]^d+k)} \Bigg\|_{L^p}^q \Bigg)^{1/q} < \infty$$

when $\alpha \in \mathbb{R}$ and $p,q \in (0,\infty]$ (with the usual modification for $q = \infty$), respectively

$$\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} := \Bigg\| \Bigg(\sum_{j\in\mathbb{Z}} |\det(A)|^{-jq(\alpha+1/2)} \sum_{k\in\mathbb{Z}^d} |c_{j,k}|^q \mathbb{1}_{A^j([0,1]^d+k)} \Bigg)^{1/q} \Bigg\|_{L^p} < \infty, \tag{1.1}$$

when $\alpha \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$ (with the usual modifications when $q = \infty$); see Section 2 for the more technical definition of the spaces $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ when $p = \infty$. The importance of these sequence spaces is that they characterize the decay of wavelet coefficients of functions/distributions in the associated Besov and Triebel-Lizorkin function spaces. As such, many problems regarding Besov and Triebel-Lizorkin spaces can be reduced to the corresponding sequence spaces, which are often easier to work with; see, e.g., [3,4,6,7]. Moreover, discrete Triebel-Lizorkin spaces (also called discrete Littlewood-Paley spaces) naturally occur in the study of weighted inequalities and Carleson coefficients, see, e.g., [8,9,17] and the references therein.

One question on anisotropic Besov and Triebel-Lizorkin function spaces that has received particular attention over the last years is how they depend on the choice of the dilation matrix; see [1,5,12,16]. The same question for their corresponding sequence spaces appears to have been first considered in [14] and was motivated by the so-called *transference method*, which allows to transfer problems for anisotropic Besov spaces via sequence spaces to *isotropic* Besov spaces. Indeed, note that it follows readily from the definition of discrete Besov spaces that $\dot{\mathbf{b}}^{\alpha}_{p,q}(A) = \dot{\mathbf{b}}^{\alpha}_{p,q}(B)$ for two expansive matrices $A, B \in \mathrm{GL}(d,\mathbb{R})$ if and only if

$$|\det(A)|^{\alpha+1/2-1/p} = |\det(B)|^{\alpha+1/2-1/p}. \tag{1.2}$$

In other words, the scale of discrete anisotropic Besov spaces is *independent* of the choice of the dilation matrix. Similarly, it follows readily from the (quasi-)norms (1.1) that if $p = q$ and the identity (1.2) holds, then $\dot{\mathbf{f}}^{\alpha}_{p,q}(A) = \dot{\mathbf{f}}^{\alpha}_{p,q}(B)$. However, the question whether the full scale of discrete Triebel-Lizorkin spaces is independent of the choice of the dilation matrix is remarkably more subtle. As a matter of fact, it was conjectured in [14, Conjecture 11] that for diagonal matrices $A = \mathrm{diag}(2^{a_1},\dots,2^{a_d})$ and $B = \mathrm{diag}(2^{b_1},\dots,2^{b_d})$ with anisotropies $(a_1,\dots,a_d), (b_1,\dots,b_d) \in (0,\infty)^d$ satisfying $\sum_{i=1}^d a_i = \sum_{i=1}^d b_i = d$ (so that $\det(A) = \det(B)$), the spaces $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ and $\dot{\mathbf{f}}^{\alpha}_{p,q}(B)$ coincide for all $\alpha \in \mathbb{R}$, $p \in (0,\infty)$

and $q \in (0, \infty]$. The fact that the conjecture [14, Conjecture 11] was incorrect as stated was shown by the following theorem; see [15, Proposition 5.26].

Theorem 1.1 *([15]). Let $\alpha \in \mathbb{R}$, $p \in (0, \infty)$ and $q \in (0, \infty]$. Let $A = \operatorname{diag}(2^{a_1}, \dots, 2^{a_d})$ and $B = \operatorname{diag}(2^{b_1}, \dots, 2^{b_d})$ for anisotropies*

$$(a_1, \dots, a_d), (b_1, \dots, b_d) \in (0, \infty)^d \quad \textit{satisfying} \quad \sum_{i=1}^d a_i = \sum_{i=1}^d b_i = d.$$

Suppose that $A \neq B$. Then $\dot{\mathbf{f}}^\alpha_{p,q}(A) = \dot{\mathbf{f}}^\alpha_{p,q}(B)$ if and only if $p = q$.

The aim of the present paper is to provide a characterization of two arbitrary expansive matrices $A, B \in \mathrm{GL}(d, \mathbb{R})$ yielding the same discrete Triebel-Lizorkin spaces $\dot{\mathbf{f}}^\alpha_{p,q}(A) = \dot{\mathbf{f}}^\alpha_{p,q}(B)$. The following theorem provides our sufficient condition for the coincidence of sequence spaces.

Theorem 1.2. *If $A, B \in \mathrm{GL}(d, \mathbb{R})$ are expansive matrices such that $\{A^j B^{-j} : j \in \mathbb{Z}\}$ is a finite set, then $\dot{\mathbf{f}}^\alpha_{p,q}(A) = \dot{\mathbf{f}}^\alpha_{p,q}(B)$ for all $\alpha \in \mathbb{R}$ and $p, q \in (0, \infty]$.*

We mention that the set $\{A^j B^{-j} : j \in \mathbb{Z}\}$ is finite if and only if $A^k = B^k$ for some $k \in \mathbb{N}$, cf. Lemma 3.1. It is of interest to compare this sufficient condition to the characterization of two expansive matrices yielding equal anisotropic (homogeneous) Triebel-Lizorkin function spaces. In [12], it was shown that such spaces coincide if and only if we have $p \in (1, \infty)$, $q = 2$ and $\alpha = 0$, or if

$$\sup_{j \in \mathbb{Z}} \| A^{-j} B^{\lfloor \varepsilon j \rfloor} \| < \infty, \tag{1.3}$$

with $\varepsilon := \ln|\det(A)| / \ln|\det(B)|$, which in turn is equivalent to two quasi-norms associated to A and B being equivalent; see [1, Section 10]. The condition that $\{A^j B^{-j} : j \in \mathbb{Z}\}$ is a finite set (cf. Theorem 1.2) implies, in particular, that $|\det(A)| = |\det(B)|$ and is thus much stronger than the condition (1.3) classifying anisotropic Triebel-Lizorkin function spaces [12].

We make some further comments on the necessity of the sufficient condition of Theorem 1.2. First, we mention that if two expansive matrices $A, B \in \mathrm{GL}(d, \mathbb{R})$ have only positive eigenvalues and satisfy $\det(A) = \det(B)$ and (1.3), then necessarily $A = B$, cf. [5, Theorem 7.9]. In particular, if $A, B \in \mathrm{GL}(d, \mathbb{R})$ are expansive matrices having only positive eigenvalues and satisfy the sufficient condition of Theorem 1.2, then $A = B$. Second, it is not difficult to show that the matrices $A = \operatorname{diag}(2, 2)$ and $B = \operatorname{diag}(2, -2)$ provide examples of matrices $A \neq B$ still yielding $\dot{\mathbf{f}}^\alpha_{p,q}(A) = \dot{\mathbf{f}}^\alpha_{p,q}(B)$. Thus, in contrast to the setting of Theorem 1.1, the coincidence of spaces $\dot{\mathbf{f}}^\alpha_{p,q}(A) = \dot{\mathbf{f}}^\alpha_{p,q}(B)$ is generally *not* equivalent to $A = B$. Lastly, we mention that for the matrices $A = 2 \cdot I$ and $B = 2 \cdot R_\phi$, where

$$R_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}, \quad \phi \in \mathbb{R} \setminus \mathbb{Q},$$

the spaces $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ and $\dot{\mathbf{f}}^{\alpha}_{p,q}(B)$ can be shown to be distinct in case $p \neq q$ (see also Theorem 1.3). Note that for such matrices the set $\{A^j B^{-j} : j \in \mathbb{Z}\}$ is infinite, which follows, for example, from Weyl's equidistribution theorem.

Our main result shows that the sufficient condition provided by Theorem 1.2 is in general also necessary for the coincidence of the scale of discrete Triebel-Lizorkin spaces. More precisely, we show the following general theorem.

Theorem 1.3. *Let $A, B \in \mathrm{GL}(d, \mathbb{R})$ be expansive, $\alpha_1, \alpha_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in (0, \infty]$.*
If $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A) = \dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$, then $p_1 = p_2$ and at least one of the following conditions hold:

(i) The set $\{A^j B^{-j} : j \in \mathbb{Z}\}$ is finite, $\alpha_1 = \alpha_2$ and $q_1 = q_2$;
(ii) $p_1 = p_2 = q_1 = q_2$ and $|\det(A)|^{\alpha_1 + 1/2 - 1/p_1} = |\det(B)|^{\alpha_2 + 1/2 - 1/p_2}$.

Theorem 1.2 and Theorem 1.3 provide a full classification of the expansive dilation matrices yielding equal discrete Triebel-Lizorkin spaces, and form a nontrivial extension of Theorem 1.1 to arbitrary expansive dilations. In addition, the necessary conditions for possibly different integrability and smoothness exponents provided by Theorem 1.3 appear to be new even for diagonal dilation matrices. Similarly, a classification of dilations for the case $p = \infty$ seems new for diagonal dilations.

As already mentioned above, the classification provided by Theorem 1.2 and Theorem 1.3 is different from the one for anisotropic Triebel-Lizorkin function spaces (1.3). To illustrate this, we recall that an expansive matrix $A \in \mathrm{GL}(d, \mathbb{R})$ is equivalent to the isotropic dilation matrix $2 \cdot I_d$ in the sense of (1.3) if and only if A is diagonalizable over $\mathbb{C}$ with all eigenvalues being equal in absolute value, see, e.g., [1, Example, p. 7]. Combined with the classification of Triebel-Lizorkin function spaces [12], except in the trivial case where $p \in (1, \infty)$, $q = 2$, and $\alpha = 0$, an expansive $A \in \mathrm{GL}(d, \mathbb{R})$ therefore generates the isotropic Triebel-Lizorkin function space $\dot{\mathbf{F}}^{\alpha}_{p,q}(A) = \dot{\mathbf{F}}^{\alpha}_{p,q}(2 \cdot I_d)$ if and only if A is diagonalizable over $\mathbb{C}$ with all eigenvalues being equal in absolute value. In contrast, an expansive matrix $A \in \mathrm{GL}(d, \mathbb{R})$ generates the classical Triebel-Lizorkin sequence space $\dot{\mathbf{f}}^{\alpha}_{p,q}(A) = \dot{\mathbf{f}}^{\alpha}_{p,q}(2 \cdot I_d)$ for $p \neq q$ if and only if $A^k = 2^k \cdot I_d$ for some $k \in \mathbb{N}$. In turn, this is equivalent to A being diagonalizable over $\mathbb{C}$ and such that each eigenvalue is of the form $2z$ for some $z \in \mathbb{C}$ satisfying $z^k = 1$ for some $k \in \mathbb{N}$. See Section 5 for further details.

Our proofs of Theorem 1.2 and Theorem 1.3 are elementary and essentially self-contained. The condition that the set $\{A^j B^{-j} : j \in \mathbb{Z}\}$ is finite is equivalent to the set $\{B^j A^{-j} : j \in \mathbb{Z}\}$ being finite. Using this, the central idea in our proof of Theorem 1.2 is to partition the integers $\mathbb{Z} = \biguplus_{1 \leq t \leq N} J_t$ into sets $J_t := \{j \in \mathbb{Z} : B^j A^{-j} = M_t\}$ for matrices M_t, $1 \leq t \leq N$, where $N := \#\{B^j A^{-j} : j \in \mathbb{Z}\}$. This allows us to rewrite the (quasi-)norms (1.1) in such a way that by means of a change of variable the (quasi-)norm of $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ can be compared to that of $\dot{\mathbf{f}}^{\alpha}_{p,q}(B)$. For the case $p = \infty$, we use a characterization of the usual (quasi-)norm via a local q-power function as shown in [3] (see Lemma 2.1).

The necessary condition provided by Theorem 1.3 requires significantly more work than the proof of Theorem 1.2. For proving Theorem 1.3, we construct sequences $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ that allow us to compare the (quasi-)norms of $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)$ and $\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$ to that of some (weighted) ℓ^r-spaces for suitable $r \in \{p_1, p_2, q_1, q_2\}$. In combination with the equivalence of the (quasi-)norms of $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)$ and $\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$ this allows us then to show the coincidence of the integrability exponents. Among these different cases, the proof of $p_1 = p_2 = q_1 = q_2$ (see Theorem 1.3(ii)) is most difficult as it requires the construction of a sequence whose $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)$-norm is comparable to some (weighted) ℓ^{q_1}-norm, whereas its $\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$-norm should be comparable to some (weighted) ℓ^{p_2}-norm. The construction of such sequences are based on some ideas underlying the proof of Theorem 1.1 as given in [15] and form nontrivial adaptations of those sequences to general expansive matrices.

The organization of the paper is as follows: Section 2 provides basic notation and properties for discrete Triebel-Lizorkin spaces that will be used throughout the paper. In Section 3, we provide a proof of Theorem 1.2. The proof of Theorem 1.3 is given in Section 4 and split into various subresults. Finally, Section 5 provides a characterization of expansive matrices A for which $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ coincides with the isotropic spaces $\dot{\mathbf{f}}^{\alpha}_{p,q}(2 \cdot I_d)$.

Notation. Unless otherwise noted, $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^d$. For a matrix $A \in \mathbb{R}^{k\times d}$, $\|A\|$ denotes the operator norm of A. The open and closed Euclidean balls with radius $r > 0$ and center $x \in \mathbb{R}^d$ are denoted by $\mathcal{B}_r(x)$ and $\overline{\mathcal{B}}_r(x)$, respectively. The r-neighborhood (resp. diameter) of a set $X \subseteq \mathbb{R}^d$ with respect to the Euclidean distance is denoted by $\mathcal{B}_r(X) = \bigcup_{x\in X} \mathcal{B}_r(x)$ (resp. $\operatorname{diam}(X)$). The standard basis of $\mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ is denoted by $(e_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ and the Kronecker delta function δ is as usual defined by $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$.

The cardinality of a set M is denoted by $\#M$, with $\#M \in \mathbb{N}_0$ for a finite set and $\#M = \infty$ for an infinite set. The Lebesgue measure of a measurable set $X \subseteq \mathbb{R}^d$ is denoted by $|X|$, and integration of a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ over X is written as $\int_X f(x)\, dx$. For a set X of finite positive measure, we write $⨍_X f(x)\, dx := |X|^{-1} \int_X f(x)\, dx$.

Given two functions $f_1, f_2 : X \to [0, \infty)$ on a set X, we write $f_1 \lesssim f_2$ if there exists $C > 0$ such that $f_1(x) \leq C f_2(x)$ for all $x \in X$. We use the notation $f_1 \asymp f_2$ whenever $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$. Subscripted variants such as $f_1 \lesssim_{a,b} f_2$ indicate that the implicit constant depends only on quantities a, b.

2. Discrete Triebel-Lizorkin spaces

For an expansive matrix $A \in \mathrm{GL}(d, \mathbb{R})$, we define associated *dilated cubes* by

$$Q^A_{j,k} := A^j([0,1]^d + k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^d.$$

The *scale* of a dyadic cube $Q^A_{j,k}$ is defined by $\operatorname{scale}(Q^A_{j,k}) = \operatorname{scale}_A(Q^A_{j,k}) = \log_{|\det(A)|}(|Q^A_{j,k}|)$. We denote the family of all dyadic cubes associated to A by $\mathcal{Q}^A = \{Q^A_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$.

For $\alpha \in \mathbb{R}$ and $p, q \in (0, \infty]$, the (homogeneous) anisotropic discrete Triebel-Lizorkin space $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ is defined as the space of all sequences $c \in \mathbb{C}^{\mathbb{Z} \times \mathbb{Z}^d}$ satisfying $\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} < \infty$, where

$$\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} := \left\| \left(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left(|\det(A)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{Q^A_{j,k}} \right)^q \right)^{\frac{1}{q}} \right\|_{L^p}$$

if $p < \infty$ (with the usual modification for $q = \infty$), and

$$\|c\|_{\dot{\mathbf{f}}^{\alpha}_{\infty,q}(A)} := \sup_{P \in \mathcal{Q}^A} \left(\fint_P \sum_{\substack{j \in \mathbb{Z} \\ j \le \mathrm{scale}(P)}} \sum_{k \in \mathbb{Z}^d} \left(|\det(A)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{Q^A_{j,k}}(x) \right)^q \, dx \right)^{\frac{1}{q}}, \quad (2.1)$$

where the case $q = \infty$ in (2.1) has to be interpreted as

$$\|c\|_{\dot{\mathbf{f}}^{\alpha}_{\infty,\infty}(A)} := \sup_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} |\det(A)|^{-j(\alpha+1/2)} |c_{j,k}|;$$

see [2–4] for various basic properties.

In order to give similar proofs for the cases $p < \infty$ and $p = \infty$, we will often use the following equivalent (quasi-)norms. The lemma is a direct consequence of a characterization of $\dot{\mathbf{f}}^{\alpha}_{\infty,q}(A)$ in terms of a so-called local q-power function. See [3, Corollary 3.4] for a proof.

Lemma 2.1 *([3]). Let $\alpha \in \mathbb{R}$ and $p, q \in (0, \infty]$. Fix $0 < \varepsilon < 1$. Then $\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} \asymp \|c\|^*_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}$ for any sequence $c \in \mathbb{C}^{\mathbb{Z} \times \mathbb{Z}^d}$, where*

$$\|c\|^*_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} := \inf \left\{ \left\| \left(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left(|\det(A)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{E_{j,k}} \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} : E_{j,k} \subseteq Q^A_{j,k}, \ \frac{|E_{j,k}|}{|Q^A_{j,k}|} > \varepsilon \right\}$$

where $E_{j,k} \subseteq Q^A_{j,k}$ are Borel sets, with the usual modification for $q = \infty$. The implicit constant is independent of the sequence c.

Remark 2.2. Strictly speaking, the statement of [3, Corollary 3.4] provides a (quasi-)norm characterization in terms of the L^2-normalized indicators $|E_{j,k}|^{-1/2} \mathbb{1}_{E_{j,k}}$ rather than the functions $|Q^A_{j,k}|^{-1/2} \mathbb{1}_{E_{j,k}}$ appearing in the statement of Lemma 2.1. However, by using that $\varepsilon < |E_{j,k}|/|Q^A_{j,k}| \le 1$, the (quasi-)norm characterization provided by Lemma 2.1 is easily seen to be equivalent to [3, Corollary 3.4].

The following basic properties of discrete Triebel-Lizorkin spaces appear to be well-known. However, as we will use both properties, but could not locate a proof, we provide short arguments in the appendix.

Lemma 2.3. *Let $A \in \mathrm{GL}(d,\mathbb{R})$ be expansive, $\alpha \in \mathbb{R}$ and $p,q \in (0,\infty]$. The following assertions hold:*

(i) The space $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ is complete with respect to the quasi-norm $\|\cdot\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}$, and continuously embedded into $\mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ equipped with the topology of pointwise, i.e., componentwise, convergence.

(ii) If $r := \min\{1,p,q\}$, then for any $\varepsilon \in (0,1)$, the quasi-norm $\|\cdot\|^*_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}$ introduced in Lemma 2.1 satisfies

$$\left(\|a+b\|^*_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}\right)^r \le \left(\|a\|^*_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}\right)^r + \left(\|b\|^*_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}\right)^r$$

for all $a,b \in \dot{\mathbf{f}}^{\alpha}_{p,q}(A)$.

3. Sufficient condition

The aim of this section is to prove the sufficient condition (Theorem 1.2) for the coincidence of discrete Triebel-Lizorkin spaces. Before doing so, we show the following simple lemma that provides different equivalent formulations of this sufficient condition.

Lemma 3.1. *Let $A,B \in \mathrm{GL}(d,\mathbb{R})$. Then the following are equivalent:*

(i) The set $\{B^jA^{-j} : j \in \mathbb{Z}\}$ is finite;
(ii) The set $\{A^jB^{-j} \ : \ j \in \mathbb{Z}\}$ is finite;
(iii) There exists $k \in \mathbb{N}$ such that $A^k = B^k$.

Proof. The equivalence of (i) and (ii) is immediate since $(A^jB^{-j})^{-1} = B^jA^{-j}$, so it remains to show the equivalence of (ii) and (iii). For this, let $F := \{A^jB^{-j} \ : \ j \in \mathbb{Z}\}$ and suppose that F is finite. Then the map $j \mapsto A^jB^{-j}$ cannot be injective from $\mathbb{N}$ to F, and hence there exist $j,\ell \in \mathbb{N}$ with $j \neq \ell$ and $A^jB^{-j} = A^\ell B^{-\ell}$. Without loss of generality, we may assume that $j > \ell$. Then we also get that $A^j = A^\ell B^{j-\ell}$, and thus $A^{j-\ell} = B^{j-\ell}$, so that setting $k := j-\ell \in \mathbb{N}$ shows (iii).

Conversely, suppose there exists $k \in \mathbb{N}$ with $A^k = B^k$. Then induction shows that $A^{k\ell} = B^{k\ell}$ for all $\ell \in \mathbb{N}$, and hence also $A^{-k\ell} = B^{-k\ell}$ for $\ell \in \mathbb{N}$, which shows that $A^{k\ell} = B^{k\ell}$ for all $\ell \in \mathbb{Z}$. Let $j \in \mathbb{Z}$ be arbitrary. Then $j = \ell k + r$ for suitable $\ell \in \mathbb{Z}$ and $r \in \{0,\dots,k-1\}$, and thus

$$A^jB^{-j} = A^rA^{\ell k}B^{-\ell k}B^{-r} = A^rB^{-r}.$$

This shows that $\left\{A^jB^{-j} \ : \ j \in \mathbb{Z}\right\} \subseteq \left\{A^rB^{-r} \ : \ r \in \{0,\dots,k-1\}\right\}$ is finite. □

The following theorem corresponds to Theorem 1.2.

Theorem 3.2. *If $A, B \in \mathrm{GL}(d,\mathbb{R})$ are two expansive matrices such that $\{A^j B^{-j} : j \in \mathbb{Z}\}$ is a finite set, then $\dot{\mathbf{f}}^\alpha_{p,q}(A) = \dot{\mathbf{f}}^\alpha_{p,q}(B)$ for all $\alpha \in \mathbb{R}$ and $p, q \in (0,\infty]$.*

Proof. We will use that $\{A^j B^{-j} : j \in \mathbb{Z}\}$ is finite if and only if $\{B^j A^{-j} : j \in \mathbb{Z}\}$ is finite; see Lemma 3.1. Again by Lemma 3.1, there exists $k \in \mathbb{N}$ with $A^k = B^k$, so that $(\det A)^k = (\det B)^k$ and hence $|\det A| = |\det B|$. Let $N := \#\{B^j A^{-j} : j \in \mathbb{Z}\}$ and write $\{B^j A^{-j} : j \in \mathbb{Z}\} = \{M_1, \dots, M_N\}$ for necessarily pairwise distinct invertible matrices $M_1, \dots, M_N$. For $t \in \mathbb{N}$ with $t \leq N$, let $J_t := \{j \in \mathbb{Z} : B^j A^{-j} = M_t\}$, and note that $\mathbb{Z} = \biguplus_{1 \leq t \leq N} J_t$.

We split the proof into the cases $p < \infty$ and $p = \infty$.

Case 1. Let $p < \infty$. If $q < \infty$, then for arbitrary $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$,

$$\begin{aligned}
\|c\|_{\dot{\mathbf{f}}^\alpha_{p,q}(B)} &= \left\| \Big(\sum_{j\in\mathbb{Z}} |\det(B)|^{-jq(\alpha+1/2)} \sum_{k\in\mathbb{Z}^d} |c_{j,k}|^q \mathbb{1}_{B^j([0,1]^d+k)} \Big)^{\frac{1}{q}} \right\|_{L^p} \\
&\asymp_{p,q,N} \sum_{t=1}^N \left\| \Big(\sum_{j\in J_t} |\det(A)|^{-jq(\alpha+1/2)} \sum_{k\in\mathbb{Z}^d} |c_{j,k}|^q \mathbb{1}_{M_t A^j([0,1]^d+k)} \Big)^{\frac{1}{q}} \right\|_{L^p}.
\end{aligned}$$

Using that $\mathbb{1}_{M_t A^j([0,1]^d+k)} = \mathbb{1}_{A^j([0,1]^d+k)}(M_t^{-1}\cdot)$, a change of variable gives

$$\begin{aligned}
&\|c\|_{\dot{\mathbf{f}}^\alpha_{p,q}(B)} \\
&\asymp_{p,q,N} \sum_{t=1}^N |\det(M_t)|^{1/p} \left\| \Big(\sum_{j\in J_t} |\det(A)|^{-jq(\alpha+1/2)} \sum_{k\in\mathbb{Z}^d} |c_{j,k}|^q \mathbb{1}_{A^j([0,1]^d+k)} \Big)^{\frac{1}{q}} \right\|_{L^p} \\
&\asymp_{p,A,B} \left\| \Big(\sum_{j\in\mathbb{Z}} |\det(A)|^{-jq(\alpha+1/2)} \sum_{k\in\mathbb{Z}^d} |c_{j,k}|^q \mathbb{1}_{A^j([0,1]^d+k)} \Big)^{\frac{1}{q}} \right\|_{L^p} \\
&= \|c\|_{\dot{\mathbf{f}}^\alpha_{p,q}(A)},
\end{aligned}$$

where the second step used that $|\det(M_t)| \asymp 1$ for an implicit constant independent of t.

The case $q = \infty$ follows by similar arguments: For arbitrary $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\sup_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}^d} |\det(B)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{B^j([0,1]^d+k)}(x) \\
&\qquad \asymp_N \sum_{t=1}^N \sup_{j\in J_t} \sup_{k\in\mathbb{Z}^d} |\det(B)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{B^j([0,1]^d+k)}(x),
\end{aligned}$$

so that a change of variable yields

$$
\begin{aligned}
\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,\infty}(B)} &= \Big\| \sup_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}^d} |\det(B)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{B^j([0,1]^d+k)} \Big\|_{L^p} \\
&\asymp_{p,N} \sum_{t=1}^{N} \Big\| \sup_{j\in J_t} \sup_{k\in\mathbb{Z}^d} |\det(B)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{A^j([0,1]^d+k)}(M_t^{-1}\cdot) \Big\|_{L^p} \\
&\asymp_{p,A,B} \Big\| \sup_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}^d} |\det(A)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{A^j([0,1]^d+k)} \Big\|_{L^p} \\
&= \|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,\infty}(A)},
\end{aligned}
$$

as required.

Case 2. Let $p = \infty$. If $q < \infty$ and $0 < \varepsilon < 1$, then Lemma 2.1 yields, for $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$,

$$
\|c\|_{\dot{\mathbf{f}}^{\alpha}_{\infty,q}(A)} \asymp \sum_{t=1}^{N} \inf \Bigg\{ \Big\| \Big(\sum_{j\in J_t} \sum_{k\in\mathbb{Z}^d} \big(|\det(A)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{E_{j,k}}\big)^q \Big)^{\frac{1}{q}} \Big\|_{L^\infty} : E_{j,k} \subseteq Q^A_{j,k},\ \frac{|E_{j,k}|}{|Q^A_{j,k}|} > \varepsilon \Bigg\}.
$$

Given a Borel set $E_{j,k} \subseteq Q^A_{j,k}$ with $|E_{j,k}|/|Q^A_{j,k}| > \varepsilon$, define $E^*_{j,k} := A^{-j}E_{j,k} - k \subseteq [0,1]^d$, so that $E_{j,k} = A^j(E^*_{j,k} + k) = M_t^{-1}B^j(E^*_{j,k} + k)$ for $j \in J_t$, and $|E^*_{j,k}| > \varepsilon$. Then, using that

$$
\mathbb{1}_{E_{j,k}} = \mathbb{1}_{M_t^{-1}B^j(E^*_{j,k}+k)} = \mathbb{1}_{B^j(E^*_{j,k}+k)}(M_t \cdot) \quad \text{for all} \quad j \in J_t, k \in \mathbb{Z}^d,
$$

a change of variable yields

$$
\begin{aligned}
&\Big\| \Big(\sum_{j\in J_t} \sum_{k\in\mathbb{Z}^d} \big(|\det(A)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{E_{j,k}}\big)^q \Big)^{\frac{1}{q}} \Big\|_{L^\infty} \\
&\quad = \Big\| \Big(\sum_{j\in J_t} \sum_{k\in\mathbb{Z}^d} \big(|\det(A)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{B^j(E^*_{j,k}+k)}\big)^q \Big)^{\frac{1}{q}} \Big\|_{L^\infty}.
\end{aligned}
$$

As $E_{j,k}$ runs through all subsets $E_{j,k} \subseteq Q^A_{j,k}$ with $|E_{j,k}|/|Q^A_{j,k}| > \varepsilon$, the set $E'_{j,k} := B^j(E^*_{j,k} + k)$ runs through all subsets $E'_{j,k} \subseteq Q^B_{j,k}$ with $|E'_{j,k}|/|Q^B_{j,k}| > \varepsilon$. Thus, a combination of the above equivalences, together with $|\det(A)| = |\det(B)|$ and another application of Lemma 2.1, yields

$$
\|c\|_{\dot{\mathbf{f}}^{\alpha}_{\infty,q}(A)} \asymp \sum_{t=1}^{N} \inf \Bigg\{ \Big\| \Big(\sum_{j\in J_t} \sum_{k\in\mathbb{Z}^d} \big(|\det(B)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{F_{j,k}}\big)^q \Big)^{\frac{1}{q}} \Big\|_{L^\infty} : F_{j,k} \subseteq Q^B_{j,k},\ \frac{|F_{j,k}|}{|Q^B_{j,k}|} > \varepsilon \Bigg\}
$$

$$\asymp \|c\|_{\dot{\mathbf{f}}^{\alpha}_{\infty,q}(B)},$$

which shows the claim for $q < \infty$.

The remaining case $q = \infty$ follows immediately from the fact that $|\det(A)| = |\det(B)|$. $\square$

4. Necessary condition

This section is devoted to proving Theorem 1.3. We start by proving some simple lemmas that will be used in various parts of the proof.

4.1. Two lemmas

The first lemma provides two simple consequences of the coincidence of spaces.

Lemma 4.1. *Let $A, B \in \mathrm{GL}(d, \mathbb{R})$ be expansive matrices, $\alpha_1, \alpha_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in (0, \infty]$.*

If $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A) = \dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$, then the following assertions hold:

(i) There exists $C \geq 1$ such that

$$\frac{1}{C}\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)} \leq \|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)} \leq C\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)}$$

for all $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$;

(ii) $|\det(A)|^{\alpha_1+\frac{1}{2}-\frac{1}{p_1}} = |\det(B)|^{\alpha_2+\frac{1}{2}-\frac{1}{p_2}}$.

Proof. (i) The proof is analogous that of function spaces [12, Lemma 5.2].

If $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A) = \dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$, then the identity map $\iota : \dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A) \to \dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$ given by $c \mapsto c$ is well-defined, and it follows from the continuous embeddings $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A), \dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B) \hookrightarrow \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ (cf. Lemma 2.3) that the graph of ι is closed. Let $r := \min\{1, p_1, q_1, p_2, q_2\}$. Since $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)$ and $\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$ are complete with respect to their "natural" (quasi-)norms, they are also complete, respectively, with respect to the equivalent r-norms $\|\cdot\|^*_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)}$ and $\|\cdot\|^*_{\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)}$ introduced in Lemma 2.1; cf. Lemma 2.3. Therefore, an application of the closed graph theorem (see, e.g., [13, Theorem 2.15]) implies that $\|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)} \asymp \|c\|^*_{\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)} \lesssim \|c\|^*_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)} \asymp \|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)}$ for all $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$. The reverse inequality follows by symmetry.

(ii) Define $c := e_{j_0,0}$ for $j_0 \in \mathbb{Z}$, where $(e_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ denotes the standard basis for $\mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$. If $p_1 < \infty$, then

$$\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)} = \left\| |\det(A)|^{-j_0(\alpha_1+\frac{1}{2})} \mathbb{1}_{A^{j_0}[0,1]^d} \right\|_{L^{p_1}} = |\det(A)|^{-j_0(\alpha_1+\frac{1}{2}-\frac{1}{p_1})},$$

while if $p_1 = \infty$, then using that

$$\Big\| |\det(A)|^{-j_0(\alpha_1+\frac{1}{2})} \mathbb{1}_{E_{j_0,0}} \Big\|_{L^\infty} = |\det(A)|^{-j_0(\alpha_1+\frac{1}{2})} = |\det(A)|^{-j_0(\alpha_1+\frac{1}{2}-\frac{1}{p_1})}$$

for Borel sets $E_{j_0,0}$ of positive measure, it follows from an application of Lemma 2.1 that $\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{\infty,q_1}(A)} \asymp |\det(A)|^{-j_0(\alpha_1+\frac{1}{2}-\frac{1}{p_1})}$. Similarly, $\|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)} \asymp |\det(B)|^{-j_0(\alpha_2+\frac{1}{2}-\frac{1}{p_2})}$.

Since $\|\cdot\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)} \asymp \|\cdot\|_{\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)}$ by assertion (i), it follows that

$$|\det(A)|^{-j_0(\alpha_1+\frac{1}{2}-\frac{1}{p_1})} \asymp |\det(B)|^{-j_0(\alpha_2+\frac{1}{2}-\frac{1}{p_2})}, \quad j_0 \in \mathbb{Z},$$

which easily implies the claim. □

Lemma 4.2. *Suppose $A, B \in \mathrm{GL}(d, \mathbb{R})$ are expansive matrices such that $\{B^j A^{-j} : j \in \mathbb{Z}\}$ is infinite. Then, given any $N \in \mathbb{N}$, there exist $j_1, \dots, j_N \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$ such that*

$$B^{j_t} A^{-j_t} x_0, \quad 1 \le t \le N,$$

are pairwise distinct. Moreover, there exists $\varepsilon > 0$ such that

$$B^{j_t} A^{-j_t} (x_0 + [-\varepsilon, \varepsilon]^d), \quad 1 \le t \le N,$$

are pairwise disjoint. In addition, for any $R > 0$, the sets $B^{j_t} A^{-j_t} \mathcal{B}_{R\varepsilon}(R x_0)$, $1 \le t \le N$, are pairwise disjoint.

Proof. For proving the first claim, assume towards a contradiction that there do not exist $j_1, \dots, j_N \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$ such that the points $B^{j_t} A^{-j_t} x_0$, $1 \le t \le N$, are pairwise distinct. Then, for every $x_0 \in \mathbb{R}^d$, it follows that $\#\{B^j A^{-j} x_0 : j \in \mathbb{Z}\} < N$. In particular, this implies that the set $\{B^j A^{-j} e_i : j \in \mathbb{Z}\}$ is finite for every standard basis vector e_i with $1 \le i \le d$. Setting $\mathcal{C}_i := \{B^j A^{-j} e_i : j \in \mathbb{Z}\}$, it follows that $\#\{B^j A^{-j} : j \in \mathbb{Z}\} \le \Pi_{i=1}^d \#\mathcal{C}_i < \infty$, which is a contradiction.

For the remaining claims, let $j_1, \dots, j_N \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$ be such that $B^{j_t} A^{-j_t} x_0$, $1 \le t \le N$, are pairwise distinct. Choose $R' > 1$ such that $\max_{1 \le t \le N} \|B^{j_t} A^{-j_t}\| \le R'$. Moreover, choose some $\delta > 0$ satisfying

$$\delta < \frac{1}{2} \min_{t \ne t'} \|B^{j_t} A^{-j_t} x_0 - B^{j_{t'}} A^{-j_{t'}} x_0\|,$$

so that $\overline{\mathcal{B}}_\delta(B^{j_t} A^{-j_t} x_0)$, $1 \le t \le N$, are pairwise disjoint. Then, choosing $0 < \varepsilon < \delta/(R'\sqrt{d})$ yields

$$\begin{aligned} B^{j_t} A^{-j_t} (x_0 + [-\varepsilon, \varepsilon]^d) &\subseteq B^{j_t} A^{-j_t} \overline{\mathcal{B}}_{\varepsilon\sqrt{d}}(x_0) = B^{j_t} A^{-j_t} \overline{\mathcal{B}}_{\varepsilon\sqrt{d}}(0) + B^{j_t} A^{-j_t} x_0 \\ &\subseteq R' \overline{\mathcal{B}}_{\varepsilon\sqrt{d}}(0) + B^{j_t} A^{-j_t} x_0 \subseteq \mathcal{B}_\delta(B^{j_t} A^{-j_t} x_0). \end{aligned}$$

Lastly, note that

$$B^{j_t} A^{-j_t} \mathcal{B}_{R\varepsilon}(Rx_0) = R \cdot (B^{j_t} A^{-j_t} \mathcal{B}_{\varepsilon}(x_0)) \subseteq R \cdot \big(B^{j_t} A^{-j_t} (x_0 + [-\varepsilon, \varepsilon]^d)\big),$$

which proves the final claim. □

4.2. Key results

In this section, we prove the various necessary conditions for the coincidence of discrete Triebel-Lizorkin spaces associated to possibly different exponents and dilation matrices. For clarity, we prove these necessary conditions by establishing various subresults.

We start by showing that $p_1 = p_2$ whenever $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A) = \dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$.

Proposition 4.3. *Let $A, B \in \mathrm{GL}(d, \mathbb{R})$ be expansive, $\alpha_1, \alpha_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in (0, \infty]$. If $\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A) = \dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)$, then $p := p_1 = p_2$ and $|\det(A)|^{\alpha_1 + \frac{1}{2} - \frac{1}{p}} = |\det(B)|^{\alpha_2 + \frac{1}{2} - \frac{1}{p}}$.*

Proof. Let $(a_k)_{k\in\mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d}$ be arbitrary and define $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ by $c_{j,k} = \delta_{0,j} a_k$. We will show that $\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)} \asymp \|a\|_{\ell^{p_1}}$. Once this is shown, it follows by symmetry and an application of Lemma 4.1(i) that $\|a\|_{\ell^{p_1}} \asymp \|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)} \asymp \|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{p_2,q_2}(B)} \asymp \|a\|_{\ell^{p_2}}$, hence $p := p_1 = p_2$, and $|\det(A)|^{\alpha_1 + \frac{1}{2} - \frac{1}{p}} = |\det(B)|^{\alpha_2 + \frac{1}{2} - \frac{1}{p}}$ by Lemma 4.1(ii).

For showing that $\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)} \asymp \|a\|_{\ell^{p_1}}$, we will consider the cases $p_1 < \infty$ and $p_1 = \infty$.

Case 1. Let $p_1 \in (0, \infty)$. If $q_1 < \infty$, then

$$\begin{aligned}
\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,q_1}(A)} &= \bigg\| \bigg(\sum_{j\in\mathbb{Z}} |\det(A)|^{-jq_1(\alpha_1 + \frac{1}{2})} \sum_{k\in\mathbb{Z}^d} |c_{j,k}|^{q_1} \mathbb{1}_{Q^A_{j,k}} \bigg)^{1/q_1} \bigg\|_{L^{p_1}} \\
&= \bigg\| \bigg(\sum_{k\in\mathbb{Z}^d} |a_k|^{q_1} \mathbb{1}_{[0,1]^d + k} \bigg)^{1/q_1} \bigg\|_{L^{p_1}} \\
&= \bigg\| \sum_{k\in\mathbb{Z}^d} |a_k|^{p_1} \mathbb{1}_{[0,1]^d + k} \bigg\|_{L^1}^{1/p_1} = \bigg(\sum_{k\in\mathbb{Z}^d} |a_k|^{p_1} \bigg)^{1/p_1}.
\end{aligned}$$

Similarly, if $q_1 = \infty$, then

$$\begin{aligned}
\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p_1,\infty}(A)} &= \bigg\| \sup_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}^d} |\det(A)|^{-j(\alpha_1 + \frac{1}{2})} |c_{j,k}| \mathbb{1}_{Q^A_{j,k}} \bigg\|_{L^{p_1}} \\
&= \bigg\| \sup_{k\in\mathbb{Z}^d} |a_k| \mathbb{1}_{[0,1]^d + k} \bigg\|_{L^{p_1}} = \bigg\| \sum_{k\in\mathbb{Z}^d} |a_k| \mathbb{1}_{[0,1]^d + k} \bigg\|_{L^{p_1}} \\
&= \bigg(\sum_{k\in\mathbb{Z}^d} |a_k|^{p_1} \bigg)^{1/p_1},
\end{aligned}$$

where the penultimate step used that the sets $[0,1]^d + k$, $k \in \mathbb{Z}^d$ are pairwise disjoint up to null-sets.

Case 2. Let $p_1 = \infty$ and $0 < \varepsilon < 1$. If $q_1 < \infty$, then it follows by Lemma 2.1 that

$$\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{\infty,q_1}(A)} \asymp \inf\left\{ \left\| \left(\sum_{k\in\mathbb{Z}^d} \left(|a_k| \mathbb{1}_{E_{0,k}}\right)^{q_1} \right)^{\frac{1}{q_1}} \right\|_{L^\infty} : E_{0,k} \subseteq Q^A_{0,k},\ \frac{|E_{0,k}|}{|Q^A_{0,k}|} > \varepsilon \right\}$$

where $E_{0,k} \subseteq Q^A_{0,k} = [0,1]^d + k$ are Borel sets. Since the sets $E_{0,k}$, $k \in \mathbb{Z}^d$, are pairwise disjoint up to null-sets, a direct calculation gives

$$\left\| \left(\sum_{k\in\mathbb{Z}^d} \left(|a_k| \mathbb{1}_{E_{0,k}}\right)^{q_1} \right)^{\frac{1}{q_1}} \right\|_{L^\infty} = \|a\|_{\ell^\infty},$$

which shows that $\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{\infty,q_1}(A)} \asymp \|a\|_{\ell^\infty}$ whenever $q_1 < \infty$. The remaining case $p_1 = q_1 = \infty$ is immediate. □

We next show that necessarily $p = q_1 = q_2$ whenever $\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A) = \dot{\mathbf{f}}^{\alpha_2}_{p,q_2}(B)$ and the set $\{B^j A^{-j} : j \in \mathbb{Z}\}$ is infinite. This is the most difficult part of the proof of Theorem 1.3.

Theorem 4.4. *Let $A, B \in \mathrm{GL}(d,\mathbb{R})$ be expansive, $\alpha_1, \alpha_2 \in \mathbb{R}$, $p \in (0,\infty]$ and $q_1, q_2 \in (0,\infty]$.*

If $\{B^j A^{-j} : j \in \mathbb{Z}\}$ is infinite and $\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A) = \dot{\mathbf{f}}^{\alpha_2}_{p,q_2}(B)$, then $p = q_1 = q_2$.

Our proof for the case $p < \infty$ of Theorem 4.4 is based on some ideas used for the construction of sequences in the proof of Theorem 1.1 (see [15, Proposition 5.26]).

Proof of Theorem 4.4. Since $\{B^j A^{-j} : j \in \mathbb{Z}\}$ is infinite, Lemma 4.2 shows for any given $N \in \mathbb{N}$ that there exist $\varepsilon > 0$, $j_1, \ldots, j_N \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$ such that, for any $R > 0$, the sets $B^{j_t} A^{-j_t} \mathcal{B}_{R\varepsilon}(Rx_0)$, $1 \le t \le N$, are pairwise disjoint. In particular, $j_t \neq j_{t'}$ for $t \neq t'$. We will choose the value of $R > 0$ depending on the cases $p < \infty$ and $p = \infty$, which we treat separately.

Case 1. We first consider the case $p < \infty$. In this case, we let $R' := \max_{1\le t\le N} \sqrt{d}\|A^{j_t}\|$, and fix some $R \ge \frac{2}{\varepsilon} R'$. Define $P_R := \mathcal{B}_{\frac{R\varepsilon}{2}}(Rx_0)$ and set

$$I_{t,R} := \left\{ k \in \mathbb{Z}^d : A^{j_t}([0,1]^d + k) \cap P_R \neq \varnothing \right\}, \quad 1 \le t \le N.$$

Let $(\tau_t)_{t=1}^N \in \mathbb{R}^N$ be arbitrary and define $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ by

$$c_{j,k} := \begin{cases} |\tau_t| \cdot |\det(A)|^{j_t(\alpha_1 + \frac{1}{2} - \frac{1}{p})}, & \text{if } j = j_t \text{ for a (unique) } 1 \le t \le N \text{ and } k \in I_{t,R} \\ 0, & \text{otherwise.} \end{cases}$$

We will show that $p = q_1 = q_2$ by comparing the norm of c for $\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A)$ and $\dot{\mathbf{f}}^{\alpha_2}_{p,q_2}(B)$.

We start by estimating the norm of c for the space $\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A)$. For this, consider the set $\Omega_{t,R} := \bigcup_{k\in I_{t,R}} A^{j_t}([0,1]^d + k)$ and note that $P_R \subseteq \Omega_{t,R}$. Second, note that

$$\operatorname{diam}(A^{j_t}([0,1]^d + k)) \leq \|A^{j_t}\|\sqrt{d} \leq R'$$

and that if two sets $\Omega, \Omega' \subseteq \mathbb{R}^d$ satisfy $\Omega \cap \Omega' \neq \varnothing$ and $\rho = \operatorname{diam}(\Omega)$, then $\Omega \subseteq \overline{\mathcal{B}}_\rho(\Omega')$. Therefore, if $k \in I_{t,R}$, then

$$A^{j_t}([0,1]^d + k) \subseteq \overline{\mathcal{B}}_{R'}(P_R) \subseteq \mathcal{B}_{R\frac{\varepsilon}{2}+R'}(Rx_0) \subseteq \mathcal{B}_{R\varepsilon}(Rx_0),$$

where the last inclusion uses that $R' \leq R\frac{\varepsilon}{2}$. In combination, this shows $P_R \subseteq \Omega_{t,R} \subseteq \mathcal{B}_{R\varepsilon}(Rx_0)$, whence $|\Omega_{t,R}| \asymp_d (R\varepsilon)^d$. On the one hand, if $q_1 < \infty$, then a direct calculation gives

$$\begin{aligned}
\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A)} &= \bigg\| \bigg(\sum_{j\in\mathbb{Z}} |\det(A)|^{-jq_1(\alpha_1+\frac{1}{2})} \sum_{k\in\mathbb{Z}^d} |c_{j,k}|^{q_1} \mathbb{1}_{Q^A_{j,k}} \bigg)^{1/q_1} \bigg\|_{L^p} \\
&= \bigg\| \bigg(\sum_{t=1}^N |\det(A)|^{-j_t \frac{q_1}{p}} |\tau_t|^{q_1} \sum_{k\in I_{t,R}} \mathbb{1}_{A^{j_t}([0,1]^d+k)} \bigg)^{1/q_1} \bigg\|_{L^p} \\
&= \bigg\| \bigg(\sum_{t=1}^N |\det(A)|^{-j_t \frac{q_1}{p}} |\tau_t|^{q_1} \mathbb{1}_{\Omega_{t,R}} \bigg)^{1/q_1} \bigg\|_{L^p} \\
&\geq \|\mathbb{1}_{P_R}\|_{L^p} \big\| \big(|\det(A)|^{-j_t/p} \tau_t \big)_{t=1}^N \big\|_{\ell^{q_1}} \\
&\gtrsim_{d,p} (R\varepsilon)^{d/p} \big\| \big(|\det(A)|^{-j_t/p} \tau_t \big)_{t=1}^N \big\|_{\ell^{q_1}},
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A)} &= \bigg\| \bigg(\sum_{t=1}^N |\det(A)|^{-j_t \frac{q_1}{p}} |\tau_t|^{q_1} \mathbb{1}_{\Omega_{t,R}} \bigg)^{1/q_1} \bigg\|_{L^p} \\
&\leq \|\mathbb{1}_{\mathcal{B}_{R\varepsilon}(Rx_0)}\|_{L^p} \big\| \big(|\det(A)|^{-j_t/p} \tau_t \big)_{t=1}^N \big\|_{\ell^{q_1}} \\
&\lesssim_{d,p} (R\varepsilon)^{d/p} \big\| \big(|\det(A)|^{-j_t/p} \tau_t \big)_{t=1}^N \big\|_{\ell^{q_1}}.
\end{aligned}$$

On the other hand, if $q_1 = \infty$, then using that $\sup_{k\in I_{t,R}} \mathbb{1}_{A^{j_t}([0,1]^d+k)} = \mathbb{1}_{\Omega_{t,R}}$ almost everywhere for $1 \leq t \leq N$ and that $\mathbb{1}_{P_R} \leq \mathbb{1}_{\Omega_{t,R}} \leq \mathbb{1}_{\mathcal{B}_{R\varepsilon}(Rx_0)}$, it follows that

$$\begin{aligned}
\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A)} &= \bigg\| \sup_{1\leq t\leq N} |\det(A)|^{-\frac{j_t}{p}} |\tau_t| \sup_{k\in I_{t,R}} \mathbb{1}_{A^{j_t}([0,1]^d+k)} \bigg\|_{L^p} \\
&= \bigg\| \sup_{1\leq t\leq N} |\det(A)|^{-\frac{j_t}{p}} |\tau_t| \mathbb{1}_{\Omega_{t,R}} \bigg\|_{L^p}
\end{aligned}$$

$$\asymp_{d,p} (R\varepsilon)^{d/p} \big\| \big(|\det(A)|^{-j_t/p} \tau_t \big)_{t=1}^N \big\|_{\ell^{q_1}}.$$

Thus, $\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A)} \asymp_{d,p} (R\varepsilon)^{d/p} \big\| \big(|\det(A)|^{-j_t/p} \tau_t \big)_{t=1}^N \big\|_{\ell^{q_1}}$ for any possible $q_1 \in (0,\infty]$.

For estimating the norm of c for the space $\dot{\mathbf{f}}^{\alpha_2}_{p,q_2}(B)$, define $\Lambda_{t,R} := \biguplus_{k \in I_{t,R}} B^{j_t}([0,1]^d + k)$. Note that $\Lambda_{t,R} = B^{j_t} A^{-j_t} \Omega_{t,R}$ and that the sets $\Lambda_{t,R}$, $1 \le t \le N$, are pairwise disjoint as $\Omega_{t,R} \subseteq \mathcal{B}_{R\varepsilon}(Rx_0)$ and $B^{j_t} A^{-j_t} \mathcal{B}_{R\varepsilon}(Rx_0)$ are pairwise disjoint for $1 \le t \le N$. Using that $|\det(A)|^{\alpha_1 + \frac{1}{2} - \frac{1}{p}} = |\det(B)|^{\alpha_2 + \frac{1}{2} - \frac{1}{p}}$ (cf. Proposition 4.3), a direct calculation yields for the case $q_2 < \infty$ that

$$\begin{aligned}
\|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{p,q_2}(B)} &= \Bigg\| \Bigg(\sum_{t=1}^N |\det(B)|^{-j_t q_2 (\alpha_2 + \frac{1}{2} - \frac{1}{p})} |\det(B)|^{-j_t \frac{q_2}{p}} \sum_{k \in \mathbb{Z}^d} |c_{j_t,k}|^{q_2} \mathbb{1}_{Q^B_{j_t,k}} \Bigg)^{1/q_2} \Bigg\|_{L^p} \\
&= \Bigg\| \Bigg(\sum_{t=1}^N |\det(B)|^{-j_t \frac{q_2}{p}} |\tau_t|^{q_2} \sum_{k \in I_{t,R}} \mathbb{1}_{B^{j_t}([0,1]^d + k)} \Bigg)^{1/q_2} \Bigg\|_{L^p} \\
&= \Bigg\| \sum_{t=1}^N |\det(B)|^{-j_t} |\tau_t|^p \mathbb{1}_{\Lambda_{t,R}} \Bigg\|_{L^1}^{1/p} = \Bigg(\sum_{t=1}^N |\det(B)|^{-j_t} |\tau_t|^p |\Lambda_{t,R}| \Bigg)^{1/p} \\
&\asymp_{d,p} (R\varepsilon)^{d/p} \big\| \big(|\det(A)|^{-\frac{j_t}{p}} \tau_t \big)_{t=1}^N \big\|_{\ell^p},
\end{aligned}$$

where the last step used that $|\Lambda_{t,R}| \asymp_d \big(|\det(B)| / |\det(A)| \big)^{j_t} (R\varepsilon)^d$ since $\Lambda_{t,R} = B^{j_t} A^{-j_t} \Omega_{t,R}$. The estimate

$$\|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{p,\infty}(B)} \asymp (R\varepsilon)^{d/p} \big\| \big(|\det(A)|^{-\frac{j_t}{p}} \tau_t \big)_{t=1}^N \big\|_{\ell^p}$$

for the case $q_2 = \infty$ is shown using similar arguments.

A combination of the above obtained estimates with Lemma 4.1(i) thus yields

$$\begin{aligned}
(R\varepsilon)^{d/p} \big\| \big(|\det(A)|^{-\frac{j_t}{p}} \tau_t \big)_{t=1}^N \big\|_{\ell^p} &\asymp \|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{p,q_2}(B)} \\
&\asymp \|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A)} \asymp (R\varepsilon)^{d/p} \big\| \big(|\det(A)|^{-\frac{j_t}{p}} \tau_t \big)_{t=1}^N \big\|_{\ell^{q_1}},
\end{aligned}$$

which implies that $q_1 = p$ since $N \in \mathbb{N}$ and $\tau = (\tau_t)_{t=1}^N \in \mathbb{R}^N$ were chosen arbitrary and the implied constants do not depend on N, R, ε or τ. Since the condition that $\{B^j A^{-j} : j \in \mathbb{Z}\}$ is finite is symmetric in A, B, it follows by symmetry that also $q_2 = p$.

Case 2. Suppose that $p = \infty$. Throughout, we fix some $\delta \in (0, 1/6)$ and choose $\ell_0 \in \mathbb{N}$ such that $A^{-\ell}[0,1]^d \subseteq [-\delta, \delta]^d$ for all $\ell \ge \ell_0$, which is possible since A is expansive and hence $\|A^{-j}\| \to 0$ as $j \to \infty$; this follows from the spectral radius formula, since the spectral radius of A^{-1} satisfies $\rho(A^{-1}) < 1$.

Now, given $N \in \mathbb{N}$, choose $\varepsilon > 0$ and $j_1, \ldots, j_N \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$ such that the sets $B^{j_t} A^{-j_t} \mathcal{B}_{R\varepsilon}(Rx_0)$, $1 \le t \le N$, are pairwise disjoint for all $R > 0$; this is possible by

Lemma 4.2. We then choose $j_0 \in \mathbb{Z}$ such that $j_0 \geq \ell_0 + \max_{1 \leq t \leq N} j_t$. With this choice, we set $R := 10\sqrt{d}\|A^{j_0}\|/\varepsilon$ and $R_0 := R/10$.

Choose next $k_0 \in \mathbb{Z}^d$ such that $Rx_0 \in A^{j_0}([0,1]^d + k_0)$. Then, since

$$\operatorname{diam}(A^{j_0}[0,1]^d + k_0) \leq \|A^{j_0}\|\sqrt{d} \leq R_0\varepsilon,$$

it follows that $A^{j_0}([0,1]^d + k_0) \subseteq \overline{\mathcal{B}}_{R_0\varepsilon}(Rx_0) \subseteq \mathcal{B}_{R\varepsilon}(Rx_0)$, and hence it follows that also the sets $B^{j_t}A^{-j_t}A^{j_0}([0,1]^d + k_0)$ are pairwise disjoint. Finally, we set $P_\delta := A^{j_0}([\frac{1}{2} - \delta, \frac{1}{2} + \delta)^d + k_0)$ and define

$$I_{t,\delta} := \{k \in \mathbb{Z}^d : A^{j_t}([0,1]^d + k) \cap P_\delta \neq \varnothing\}, \quad 1 \leq t \leq N.$$

Similar to Case 1, we define a sequence $c \in \mathbb{C}^{\mathbb{Z} \times \mathbb{Z}^d}$ by

$$c_{j,k} := \begin{cases} |\det(A)|^{j_t(\alpha_1 + \frac{1}{2})} \cdot |\tau_t|, & \text{if } j = j_t \text{ for a (unique) } 1 \leq t \leq N \text{ and } k \in I_{t,\delta} \\ 0, & \text{otherwise,} \end{cases}$$

where $\tau = (\tau_t)_{t=1}^N$ is an arbitrary given sequence in $\mathbb{R}^N$.

For showing that $q_1 = q_2 = p$, we will estimate the norms $\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A)}$ and $\|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{p,q_2}(B)}$. For estimating $\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A)}$, we define $\Omega_{t,\delta} := \bigcup_{k \in I_{t,\delta}} A^{j_t}([0,1]^d + k)$. Then clearly $P_\delta \subseteq \Omega_{t,\delta}$, and we claim that $\Omega_{t,\delta} \subseteq Q^A_{j_0,k_0} = A^{j_0}([0,1]^d + k_0)$. Indeed, note that if $k \in I_{t,\delta}$, then

$$\begin{aligned} k &\in A^{j_0 - j_t}k_0 + A^{j_0 - j_t}\big([1/2 - \delta, 1/2 + \delta)^d - A^{j_t - j_0}[0,1]^d\big) \\ &\subseteq A^{j_0 - j_t}k_0 + A^{j_0 - j_t}\big([1/2 - \delta, 1/2 + \delta)^d - [-\delta, \delta]^d\big) \\ &\subseteq A^{j_0 - j_t}k_0 + A^{j_0 - j_t}[1/2 - 2\delta, 1/2 + 2\delta)^d, \end{aligned}$$

where we used that $j_t - j_0 \leq -\ell_0$ and $A^{-\ell}[0,1]^d \subseteq [-\delta, \delta]^d$ for all $\ell \geq \ell_0$. Using again that $A^{j_t - j_0}[0,1]^d \subseteq [-\delta, \delta]^d$ and that $\delta < 1/6$, we finally see that

$$\begin{aligned} k + [0,1]^d &\subseteq A^{j_0 - j_t}k_0 + A^{j_0 - j_t}\big([1/2 - 2\delta, 1/2 + 2\delta)^d + A^{j_t - j_0}[0,1]^d\big) \\ &\subseteq A^{j_0 - j_t}k_0 + A^{j_0 - j_t}[1/2 - 3\delta, 1/2 + 3\delta]^d \\ &\subseteq A^{j_0 - j_t}k_0 + A^{j_0 - j_t}[0,1]^d, \end{aligned}$$

whence $A^{j_t}\big([0,1]^d + k\big) \subseteq A^{j_0}\big([0,1]^d + k_0\big) = Q^A_{j_0,k_0}$ for any $k \in I_{t,\delta}$, as claimed. As $j_0 \geq \ell_0 + \max_{1 \leq t \leq N} j_t$, we have $j_0 \geq j_t$ for each $1 \leq t \leq N$. Hence, using the definition of $\dot{\mathbf{f}}^{\alpha_1}_{\infty,q_1}(A)$ for $q_1 < \infty$, we estimate

$$\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{\infty,q_1}(A)} \geq \bigg(\fint_{Q^A_{j_0,k_0}} \sum_{j \in \mathbb{Z}, j \leq j_0} |\det(A)|^{-jq_1(\alpha_1 + \frac{1}{2})} \sum_{k \in \mathbb{Z}^d} |c_{j,k}|^{q_1} \mathbb{1}_{Q^A_{j,k}}(x)\, dx \bigg)^{\frac{1}{q_1}}$$

$$= \Bigg(\fint_{Q^A_{j_0,k_0}} \sum_{t=1}^N |\tau_t|^{q_1} \sum_{k \in I_{t,\delta}} \mathbb{1}_{Q^A_{j_t,k}}(x)\, dx \Bigg)^{\frac{1}{q_1}}$$

$$= \Bigg(\fint_{Q^A_{j_0,k_0}} \sum_{t=1}^N |\tau_t|^{q_1} \mathbb{1}_{\Omega_{t,\delta}}(x)\, dx \Bigg)^{\frac{1}{q_1}} \geq \Bigg(\frac{|P_\delta|}{|Q^A_{j_0,k_0}|} \sum_{t=1}^N |\tau_t|^{q_1} \Bigg)^{1/q_1}$$

$$\gtrsim_{\delta,q_1} \|\tau\|_{\ell^{q_1}}.$$

Clearly, $\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{\infty,\infty}(A)} = \|\tau\|_{\ell^\infty}$, so that $\|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{\infty,q_1}(A)} \gtrsim \|\tau\|_{\ell^{q_1}}$ for arbitrary $q_1 \in (0,\infty]$. Here, we crucially used that $\frac{|P_\delta|}{|Q^A_{j_0,k_0}|} = (2\delta)^d$ is independent of the choice of $N \in \mathbb{N}$ and of $\tau \in \mathbb{R}^N$.

We next provide an upper bound for $\|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{\infty,q_2}(B)}$. Let $\Lambda_{t,\delta} := \bigcup_{k \in I_{t,\delta}} B^{j_t}([0,1]^d + k)$, and observe that $\Lambda_{t,\delta} = B^{j_t} A^{-j_t} \Omega_{t,\delta} \subseteq B^{j_t} A^{-j_t} Q^A_{j_0,k_0}$, where the inclusion $\Omega_{t,\delta} \subseteq Q^A_{j_0,k_0}$ was shown already above. In particular, since we showed towards the beginning of Part 2 of the proof that the sets $B^{j_t} A^{-j_t} A^{j_0}([0,1]^d + k_0)$, $1 \leq t \leq N$, are pairwise disjoint, this implies that the sets $\Lambda_{t,\delta}$, $1 \leq t \leq N$, are pairwise disjoint as well. Using this, together with the fact that $|\det(A)|^{\alpha_1 + \frac{1}{2}} = |\det(B)|^{\alpha_2+\frac{1}{2}}$ (cf. Proposition 4.3), a direct calculation entails for $q_2 < \infty$ that

$$\|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{\infty,q_2}(B)}$$

$$= \sup_{Q \in \mathcal{Q}^B} \Bigg(\fint_Q \Bigg[\Bigg(\sum_{\substack{j \in \mathbb{Z} \\ j \leq \mathrm{scale}_B(Q)}} |\det(B)|^{-jq_2(\alpha_2 + \frac{1}{2})} \sum_{k \in \mathbb{Z}^d} |c_{j,k}|^{q_2} \mathbb{1}_{Q^B_{j,k}}(x) \Bigg)^{1/q_2} \Bigg]^{q_2} dx \Bigg)^{1/q_2}$$

$$\leq \Bigg\| \Bigg(\sum_{j \in \mathbb{Z}} |\det(B)|^{-jq_2(\alpha_2+\frac{1}{2})} \sum_{k \in \mathbb{Z}^d} |c_{j,k}|^{q_2} \mathbb{1}_{Q^B_{j,k}} \Bigg)^{\frac{1}{q_2}} \Bigg\|_{L^\infty}$$

$$= \Bigg\| \Bigg(\sum_{t=1}^N |\tau_t|^{q_2} \sum_{k \in I_{t,\delta}} \mathbb{1}_{Q^B_{j_t,k}} \Bigg)^{\frac{1}{q_2}} \Bigg\|_{L^\infty} = \Bigg\| \Bigg(\sum_{t=1}^N |\tau_t|^{q_2} \mathbb{1}_{\Lambda_{t,\delta}} \Bigg)^{\frac{1}{q_2}} \Bigg\|_{L^\infty}$$

$$= \|\tau\|_{\ell^\infty}.$$

Clearly, also $\|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{\infty,\infty}(B)} = \|\tau\|_{\ell^\infty}$.

A combination of the estimates obtained above with Lemma 4.1(i) gives

$$\|\tau\|_{\ell^\infty} \leq \|\tau\|_{\ell^{q_1}} \lesssim \|c\|_{\dot{\mathbf{f}}^{\alpha_1}_{\infty,q_1}(A)} \asymp \|c\|_{\dot{\mathbf{f}}^{\alpha_2}_{\infty,q_2}(B)} \leq \|\tau\|_{\ell^\infty},$$

which implies that $\|\tau\|_{\ell^{q_1}} \asymp \|\tau\|_{\ell^\infty}$ for arbitrary $N \in \mathbb{N}$ and $\tau \in \mathbb{R}^N$ with an implicit constant independent of N, τ. Thus, $q_1 = \infty$. Since the condition of $\{B^j A^{-j} : j \in \mathbb{Z}\}$ being infinite is symmetric in A, B, it follows by symmetry that also $q_2 = \infty$, so that $p = q_1 = q_2$. This completes the proof. □

Lastly, we treat the case when $\{B^j A^{-j} : j \in \mathbb{Z}\}$ is finite.

Proposition 4.5. *Let $A, B \in \mathrm{GL}(d, \mathbb{R})$ be expansive matrices, $\alpha_1, \alpha_2 \in \mathbb{R}$, $p \in (0, \infty]$ and $q_1, q_2 \in (0, \infty]$.*

If $\{B^j A^{-j} : j \in \mathbb{Z}\}$ is finite and $\dot{\mathbf{f}}^{\alpha_1}_{p,q_1}(A) = \dot{\mathbf{f}}^{\alpha_2}_{p,q_2}(B)$, then $\alpha_1 = \alpha_2$ and $q_1 = q_2$.

Proof. If $\{B^j A^{-j} : j \in \mathbb{Z}\}$ is finite, then necessarily $|\det(A)| = |\det(B)|$ (cf. the proof of Theorem 3.2), and hence it follows by Lemma 4.1 that $|\det(A)|^{\alpha_1 + \frac{1}{2} - \frac{1}{p}} = |\det(A)|^{\alpha_2 + \frac{1}{2} - \frac{1}{p}}$. Therefore, since $|\det(A)| > 1$, we get $\alpha := \alpha_1 = \alpha_2$. Moreover, an application of Theorem 3.2 (which is applicable by Lemma 3.1) yields that $\dot{\mathbf{f}}^{\alpha}_{p,q_1}(A) = \dot{\mathbf{f}}^{\alpha}_{p,q_2}(B) = \dot{\mathbf{f}}^{\alpha}_{p,q_2}(A)$.

Given an arbitrary sequence $\tau = (\tau_t)_{t \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$, define $c \in \mathbb{C}^{\mathbb{Z} \times \mathbb{Z}^d}$ by

$$
c_{j,k} := \begin{cases} |\det(A)|^{j(\alpha + \frac{1}{2})} |\tau_{-j}|, & \text{if } j \leq 0 \text{ and } k \in I_j \\ 0, & \text{otherwise,} \end{cases}
$$

where $I_j := \{k \in \mathbb{Z}^d : Q^A_{j,k} \cap [0,1]^d \neq \varnothing\}$ for $j \in \mathbb{Z}$. We let $\Omega_j := \bigcup_{k \in I_j} Q^A_{j,k}$ for $j \leq 0$. Clearly, $[0,1]^d \subseteq \Omega_j$. On the other hand, if $j \leq 0$ and $k \in I_j$, then $Q^A_{j,k} = A^j([0,1]^d + k) \subseteq \overline{\mathcal{B}}_R([0,1]^d)$ for $R := \sqrt{d} \sup_{j \leq 0} \|A^j\|$, which is finite since A is expansive, and thus $Q^A_{j,k} \subseteq \mathcal{B}_{R'}(0)$ for some $R' > 0$ (only depending on d, A). In combination, this yields $[0,1]^d \subseteq \Omega_j \subseteq \mathcal{B}_{R'}(0)$.

We now show that $\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} \asymp \|\tau\|_{\ell^q}$ for any $q \in (0, \infty]$, with implicit constant independent of τ. For this, we distinguish the cases $p < \infty$ and $p = \infty$.

Case 1. Suppose $p < \infty$. For $q \in (0, \infty)$, the fact that $\mathbb{1}_{[0,1]^d} \leq \mathbb{1}_{\Omega_j} \leq \mathbb{1}_{\mathcal{B}_{R'}(0)}$ for all $j \leq 0$ yields

$$
\begin{aligned}
\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} &= \left\| \left(\sum_{j \leq 0} |\tau_{-j}|^q \sum_{k \in I_j} \mathbb{1}_{Q^A_{j,k}} \right)^{1/q} \right\|_{L^p} \\
&= \left\| \left(\sum_{j \leq 0} |\tau_{-j}|^q \mathbb{1}_{\Omega_j} \right)^{1/q} \right\|_{L^p} \\
&\leq \left(\sum_{j \in \mathbb{N}_0} |\tau_j|^q \right)^{1/q} \|\mathbb{1}_{\mathcal{B}_{R'}(0)}\|_{L^p}
\end{aligned}
$$

and $\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} \geq \|\tau\|_{\ell^q} \|\mathbb{1}_{[0,1]^d}\|_{L^p}$. Similar arguments also give $\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,\infty}(A)} \asymp \|\tau\|_{\ell^\infty}$.

Case 2. Suppose $p = \infty$. For $q < \infty$, it follows from the definition of $\|\cdot\|_{\dot{\mathbf{f}}^{\alpha}_{\infty,q}(A)}$ that

$$
\|c\|_{\dot{\mathbf{f}}^{\alpha}_{\infty,q}(A)} \geq \left(\fint_{Q^A_{0,0}} \sum_{j \in \mathbb{Z}, j \leq 0} |\det(A)|^{-jq(\alpha + \frac{1}{2})} \sum_{k \in \mathbb{Z}^d} |c_{j,k}|^q \mathbb{1}_{Q^A_{j,k}}(x) \, dx \right)^{\frac{1}{q}}
$$

$$
\begin{aligned}
&= \Big(\int_{[0,1]^d} \sum_{j \in \mathbb{Z}, j \le 0} |\tau_{-j}|^q \sum_{k \in I_j} \mathbb{1}_{Q^A_{j,k}}(x) \, dx \Big)^{1/q} \\
&= \Big(\int_{[0,1]^d} \sum_{j \in \mathbb{Z}, j \le 0} |\tau_{-j}|^q \mathbb{1}_{\Omega_j}(x) \, dx \Big)^{1/q} = \Big(\int_{[0,1]^d} \sum_{j \in \mathbb{Z}, j \le 0} |\tau_{-j}|^q \, dx \Big)^{1/q} \\
&= \|\tau\|_{\ell^q},
\end{aligned}
$$

where the penultimate step used that $[0,1]^d \subseteq \Omega_j$ for $j \le 0$. On the other hand, using that $\mathbb{1}_{\Omega_j} \le \mathbb{1}_{\mathcal{B}_{R'}(0)}$ for $j \le 0$, yields

$$
\begin{aligned}
&\|c\|_{\dot{\mathbf{f}}^\alpha_{\infty,q}(A)} \\
&\quad = \sup_{Q \in \mathcal{Q}^A} \Big(\fint_Q \Big[\Big(\sum_{j \in \mathbb{Z}, j \le \mathrm{scale}_A(Q)} |\det(A)|^{-jq(\alpha + \frac{1}{2})} \sum_{k \in \mathbb{Z}^d} |c_{j,k}|^q \mathbb{1}_{Q^A_{j,k}}(x) \Big)^{1/q} \Big]^q dx \Big)^{1/q} \\
&\quad \le \Big\| \Big(\sum_{j \in \mathbb{Z}} |\det(A)|^{-jq(\alpha + \frac{1}{2})} \sum_{k \in \mathbb{Z}^d} |c_{j,k}|^q \mathbb{1}_{Q^A_{j,k}} \Big)^{\frac{1}{q}} \Big\|_{L^\infty} \\
&\quad = \Big\| \Big(\sum_{j \in \mathbb{Z}, j \le 0} |\tau_{-j}|^q \mathbb{1}_{\Omega_j} \Big)^{1/q} \Big\|_{L^\infty} \\
&\quad \le \|\tau\|_{\ell^q} \|\mathbb{1}_{\mathcal{B}_{R'}(0)}\|_{L^\infty},
\end{aligned}
$$

whenever $q < \infty$. It is immediate that $\|c\|_{\dot{\mathbf{f}}^\alpha_{\infty,\infty}(A)} = \|\tau\|_{\ell^\infty}$. Thus, also $\|c\|_{\dot{\mathbf{f}}^\alpha_{\infty,q}(A)} \asymp \|\tau\|_{\ell^q}$ for all $q \in (0, \infty]$.

To complete the proof, note that Lemma 4.1 implies because of $\dot{\mathbf{f}}^\alpha_{p,q_1}(A) = \dot{\mathbf{f}}^\alpha_{p,q_2}(A)$ (cf. the beginning of the proof) that

$$
\|\tau\|_{\ell^{q_1}} \asymp \|c\|_{\dot{\mathbf{f}}^\alpha_{p,q_1}(A)} \asymp \|c\|_{\dot{\mathbf{f}}^\alpha_{p,q_2}(A)} \asymp \|\tau\|_{\ell^{q_2}},
$$

with implicit constant independent of $\tau \in \mathbb{C}^{N_0}$, and hence $q_1 = q_2$. □

Proof of Theorem 1.3. Theorem 1.3 follows from a combination of Proposition 4.3, Theorem 4.4 and Proposition 4.5. □

5. Application: a spectral characterization of isotropic Triebel-Lizorkin sequence spaces

This section provides a spectral characterization of those expansive matrices $A \in \mathrm{GL}(d, \mathbb{R})$ such that $\dot{\mathbf{f}}^\alpha_{p,q}(A) = \dot{\mathbf{f}}^\alpha_{p,q}(2 \cdot I_d)$ for all $p, q \in (0, \infty]$ and $\alpha \in \mathbb{R}$, that is, those matrices generating the classical isotropic Triebel-Lizorkin sequence space $\dot{\mathbf{f}}^\alpha_{p,q}(2 \cdot I_d)$.

We will use the following lemma on periodic matrices. Recall that a matrix $A \in \mathrm{GL}(d, \mathbb{R})$ is called periodic whenever $A^k = I_d$ for some $k \in \mathbb{N}$. Although we expect this lemma to be part of the folklore, we provide a proof for the sake of completeness.

Lemma 5.1. *Let $A \in \mathbb{R}^{d\times d}$. Then the following assertions are equivalent:*

(i) *A is periodic;*
(ii) *A is diagonalizable over $\mathbb{C}$ and all eigenvalues of A belong to the set*

$$\{z \in \mathbb{C} \ : \ \exists\, k \in \mathbb{N} : z^k = 1\};$$

(iii) *There exists an invertible matrix $S \in \mathbb{R}^{d\times d}$ such that*

$$S^{-1}AS = \operatorname{diag}(B_1, \ldots, B_b)$$

is a block-diagonal matrix, where each block B_j is either a 1×1 matrix of the form $B_j = (\pm 1)$, or a 2×2 rotation matrix with a "rational angle", i.e.,

$$B_j = R_{\phi_j} = \begin{pmatrix} \cos(\phi_j) & -\sin(\phi_j) \\ \sin(\phi_j) & \cos(\phi_j) \end{pmatrix} \qquad \text{with} \qquad \phi_j \in 2\pi\mathbb{Q}.$$

Proof. (i) ⇒ (ii): Let $k \in \mathbb{N}$ with $A^k = I_d$. Then, for the polynomial $p(X) = X^k - 1$, we have $p(A) = 0$, meaning the minimal polynomial of A divides p. But p has k distinct zeros, namely $e^{2\pi i j/k}, j = 0, \ldots, k-1$. Hence, the minimal polynomial of A factors into distinct linear factors over $\mathbb{C}$. By [10, Chapter 6, Theorem 6], this means that A is diagonalizable over $\mathbb{C}$. Moreover, each eigenvalue $z \in \mathbb{C}$ of A is a zero of the minimal polynomial, and hence of p, and thus satisfies $z^k = 1$.

(ii) ⇒ (iii): Write the eigenvalues of A (repeated according to their multiplicity) as

$$\mu_1, \ldots, \mu_{b_r}, \mu_{b_r+1}, \overline{\mu_{b_r+1}}, \ldots, \mu_b, \overline{\mu_b},$$

where $\mu_1, \ldots, \mu_{b_r} \in \mathbb{R}$ and where $\operatorname{Im}(\mu_j) > 0$ for $b_r < j \leq b$. This is possible, since for a real matrix, the complex eigenvalues come in "conjugate pairs". Since all eigenvalues of A belong to $\{z \in \mathbb{C} \ : \ \exists\, k \in \mathbb{N} : z^k = 1\}$, there exists $k \in \mathbb{N}$ such that all of the eigenvalues μ_j are of the form $\mu_j = e^{2\pi i t_j/k}$ for some $t_j \in \mathbb{N}_0$. This in particular implies $\mu_j \in \{\pm 1\}$ for $1 \leq j \leq b_r$. Since A is diagonalizable over $\mathbb{C}$, we can invoke [11, Corollary 3.4.1.10] about the "real Jordan normal form" of diagonalizable matrices to conclude that there exists an invertible matrix $S \in \mathbb{R}^{d\times d}$ such that

$$S^{-1}AS = \operatorname{diag}(B_1, \ldots, B_b)$$

is a block-diagonal matrix, where

$$B_j = \begin{cases} (\mu_j) = (\pm 1), & \text{for } 1 \leq j \leq b_r, \\ \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}, & \text{for } b_r < j \leq m \text{ and } \mu_j = a_j + i b_j. \end{cases}$$

Finally, note for $b_r < j \leq m$ that

$$\begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix} = \begin{pmatrix} \cos(2\pi t_j/k) & \sin(2\pi t_j/k) \\ -\sin(2\pi t_j/k) & \cos(2\pi t_j/k) \end{pmatrix} = R_{-2\pi t_j/k}$$

is a rotation matrix with a "rational rotation angle".

(iii) ⇒ (i): It is well-known that the rotation matrices satisfy $R_\phi R_\theta = R_{\phi+\theta}$. Hence, for the case where $B_j = R_{\phi_j}$ with $\phi_j = 2\pi \frac{t_j}{\ell_j}$ with $t_j \in \mathbb{Z}$, $\ell_j \in \mathbb{N}$, we have

$$B_j^{\ell_j} = R_{\phi_j}^{\ell_j} = R_{\ell_j \phi_j} = R_{2\pi t_j} = I_2.$$

Similarly, if $B_j = (\pm 1)$, then for $\ell_j := 2$ we have $B_j^{\ell_j} = I_1$. Overall, this shows for $k := \ell_1 \cdots \ell_b$ that

$$\begin{aligned} A^k &= \big(S \operatorname{diag}(B_1, \dots, B_b) S^{-1}\big)^k \\ &= S \operatorname{diag}(B_1^k, \dots, B_b^k) S^{-1} \\ &= S I_d S^{-1} = I_d, \end{aligned}$$

which shows that A is periodic. □

The following theorem provides a spectral characterization of the matrices inducing isotropic Triebel-Lizorkin sequence spaces.

Theorem 5.2. *Let $A \in \mathrm{GL}(d, \mathbb{R})$. Then the following assertions are equivalent:*

(i) $\dot{\mathbf{f}}^\alpha_{p,q}(A) = \dot{\mathbf{f}}^\alpha_{p,q}(2 \cdot I_d)$ *for all* $\alpha \in \mathbb{R}$ *and* $p, q \in (0, \infty]$*;*
(ii) There exists an invertible matrix $S \in \mathbb{R}^{d \times d}$ *such that*

$$\frac{A}{2} = S \operatorname{diag}(B_1, \dots, B_b) S^{-1}$$

is a block-diagonal matrix, where each B_j *is either a* 1×1 *matrix of the form* (± 1)*, or a* 2×2 *rotation matrix with a rotation angle in* $2\pi\mathbb{Q}$*;*
(iii) A *is diagonalizable over* $\mathbb{C}$ *and all eigenvalues of* A *belong to the set*

$$\big\{ 2z \in \mathbb{C} \ : \ z \in \mathbb{C} \text{ and } \exists\, k \in \mathbb{N} : z^k = 1 \big\}.$$

Proof. A combination of Theorem 1.2, Theorem 1.3 and Lemma 3.1 shows that (i) is equivalent to $A^k = 2^k \cdot I_d$ for some $k \in \mathbb{N}$, i.e., $A/2$ is a periodic matrix. The equivalences follow therefore directly from Lemma 5.1. □

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Appendix A. Postponed proof

Proof of Lemma 2.3. (i) If $(c^{(n)})_{n\in\mathbb{N}}$ is a sequence consisting of elements $c^{(n)} \in \dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ with $\liminf_{n\to\infty} \|c^{(n)}\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} < \infty$ and $c \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ satisfies $|c_{j,k}| \leq \liminf_{n\to\infty} |c^{(n)}_{j,k}|$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, then it is easily verified via an application of Fatou's lemma that $c \in \dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ with $\|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} \leq \liminf_{n\to\infty} \|c^{(n)}\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}$. This means that the (quasi-)normed space $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ satisfies the Fatou property, and hence it is complete by [19, Section 65, Theorem 1]; see also [18, Lemma 2.2.15] for the case of quasi-norms. Moreover, it follows that $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ is solid, meaning that if $c \in \dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ and $c' \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}^d}$ are such that $|c'_{j,k}| \leq |c_{j,k}|$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, then $c' \in \dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ and $\|c'\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} \leq \|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}$. In turn, this implies the pointwise convergence of sequences converging in $\dot{\mathbf{f}}^{\alpha}_{p,q}(A)$. More precisely, $|c_{j,k}| \cdot \|e_{j,k}\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} = \|c_{j,k} e_{j,k}\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} \leq \|c\|_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}$.

(ii) Fix $\varepsilon \in (0,1)$ and consider the (quasi-)norm $\|\cdot\|^{*}_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)}$ defined in Lemma 2.1. Given $c \in \dot{\mathbf{f}}^{\alpha}_{p,q}(A)$ and Borel sets $E_{j,k} \subseteq Q^{A}_{j,k}$ with $|E_{j,k}|/|Q^{A}_{j,k}| > \varepsilon$, we define

$$c^{E_{j,k}}_{j,k}(x) := |\det(A)|^{-j(\alpha+1/2)} |c_{j,k}| \mathbb{1}_{E_{j,k}}(x), \quad x \in \mathbb{R}^d.$$

Then Lemma 2.1 yields

$$\|c\|^{*}_{\dot{\mathbf{f}}^{\alpha}_{p,q}(A)} = \inf\left\{ \left\| \left\| \left(c^{E_{j,k}}_{j,k}(\cdot) \right)_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \right\|_{\ell^q} \right\|_{L^p} : E_{j,k} \subseteq Q^{A}_{j,k},\ \frac{|E_{j,k}|}{|Q^{A}_{j,k}|} > \varepsilon \right\}.$$

Using that

$$\begin{aligned} \left\| \left\| \left(c^{E_{j,k}}_{j,k}(\cdot) \right)_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \right\|_{\ell^q} \right\|_{L^p} &= \left\| \left\| \left(c^{E_{j,k}}_{j,k}(\cdot) \right)^{r}_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \right\|^{1/r}_{\ell^{q/r}} \right\|_{L^p} \\ &= \left\| \left\| \left(c^{E_{j,k}}_{j,k}(\cdot) \right)^{r}_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \right\|_{\ell^{q/r}} \right\|^{1/r}_{L^{p/r}} \end{aligned}$$

and

$$\left((a+b)_{j,k}^{E_{j,k}}\right)^r \leq \left(a_{j,k}^{E_{j,k}}\right)^r + \left(b_{j,k}^{E_{j,k}}\right)^r$$

since $r \leq 1$, the claim follows easily from applications of the triangle inequality, which is applicable since $q/r, p/r \geq 1$. $\square$

Data availability

No data was used for the research described in the article.

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