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Closed form representations for the compactly supported radial basis functions of Buhmann, Wendland and Wu

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Abstract
The original compactly supported radial basis functions of Wendland (*Adv. Comput. Math.*, **4**, 389–396, 1995) and Wu (*Adv. Comput. Math.*, **4**, 283–292, 1995) have a polynomial form and are constructed using a two-step dimension walk strategy. Focussing on the Wendland functions, Schaback (*Adv. Comput. Math.*, **34**(1), 67–81, 2011) proposed a one-step dimension walk which is shown to recover the original Wendland functions at every second step but also introduces new examples, the so-called missing Wendland functions at the intermediate steps. In a recent paper (*Science China Mathematics Published online*, 2025), the analogue of Schaback's work is presented for the Wu functions and so delivers the so-called missing Wu functions. The original and missing Wendland functions belong to a much wider class proposed by Buhmann (*Math. Comput.*, **70**(233), 307–318, 2001). The classical Buhmann functions, which are related to thin-plate spline radial basis functions, also belong to this much wider class. The theme uniting the classical Buhmann functions and the missing Wendland/Wu functions is that they are non-polynomial, and closed-form expressions are not known for all of them. In this paper, we revisit these functions and show how closed-form representations can be given using direct techniques. The results for the classical Buhmann and Wu functions are new, and the resulting expressions for the missing Wendland functions improve on those given in Hubbert (*Adv. Comput. Math.*, **36**, 115–136, 2012) and so their implementation should be more straightforward.

Keywords Compactly supported radial basis functions · Kernels · Positive definite functions · Hypergeometric functions · Native Sobolev Spaces

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1 Introduction

To motivate the topic of this paper, we consider the $d-$dimensional scattered data fitting problem where the goal is to approximate an unknown function based only on a set of its values at a finite set $\{\mathbf{x}_i\}_{i=1}^n$ of distinct points in $\mathbb{R}^d$. We choose to approximate via interpolation, and in order to do so, we propose a univariate continuous function ϕ which is used to induce a $d-$dimensional function by composition with the Euclidean norm on $\mathbb{R}^d$. The univariate function $\phi : [0, \infty) \to \mathbb{R}$ is known as the radial basis function (RBF). To ensure our interpolant is location-dependent, we propose that it has the following form

$$s(\mathbf{x}) = \sum_{j=1}^{n} \alpha_i \phi(\mathbf{x} - \mathbf{x}_j\|), \quad \mathbf{x} \in \mathbb{R}^d.$$

Enforcing the interpolation requirements $s(\mathbf{x}_i) = f_i$ $(1 \leq i \leq n)$, we see that the resulting interpolant is unique if and only if the interpolation matrix

$$A \in \mathbb{R}^{n\times n}: \quad A_{ij} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|), \quad (1 \leq i, j \leq n), \tag{1.1}$$

is non-singular. One way to achieve uniqueness is to choose ϕ to be strictly positive definite in $\mathbb{R}^d$; such functions, by definition, are guaranteed to give rise to positive definite (and hence non-singular) interpolation matrices (1.1). The following result provides a characterisation (see [19, Theorem 6.18]).

Theorem 1.1 *A radial basis function $\phi : [0, \infty) \to \mathbb{R}$ such that $r \mapsto r^{d-1}\phi(r) \in L_1[0, \infty)$ is strictly positive definite in $\mathbb{R}^d$ if and only if the $d-$dimensional Fourier transform*

$$\widehat{\phi}(z) = z^{1-\frac{d}{2}} \int_0^\infty \phi(y)\, y^{\frac{d}{2}}\, J_{\frac{d}{2}-1}(yz)\, dy, \tag{1.2}$$

where $J_\nu(\cdot)$ denotes the Bessel function of the first kind with order ν, is non-negative and not identically equal to zero.

In the RBF literature it is common to work with both the underlying RBF ϕ and also radial kernel it induces via $\Phi(\mathbf{x}, \mathbf{y}) = \phi(\|\boldsymbol{x} - \boldsymbol{y}\|)$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$. Indeed, if we assume that $\widehat{\phi}(z)$ (1.2) is positive for all $z \geq 0$, then we can appeal to the theory of RBFs (see [19]) for the following result.

Theorem 1.2 *Let $d \geq 1$ denote a fixed spatial dimension and $\Phi(\mathbf{x}, \mathbf{y}) = \phi(\|\boldsymbol{x} - \boldsymbol{y}\|)$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ be a radial kernel on $\mathbb{R}^d$ induced by an RBF ϕ whose $d-$dimensional Fourier transform $\widehat{\phi}(z)$ is positive for all $z \geq 0$. Define*

$$N_\phi := \left\{ f \in L_2(\mathbb{R}^d) : \|f\|_\phi^2 = \int_{\mathbb{R}^d} \frac{|\widehat{f}(\boldsymbol{\omega})|^2}{\widehat{\phi}(\|\boldsymbol{\omega}\|)} d\boldsymbol{\omega} < \infty \right\}, \tag{1.3}$$

where $\|\cdot\|_\phi$ is a norm induced by the inner product

$$(f, g)_\phi := \int_{\mathbb{R}^d} \frac{\widehat{f}(\boldsymbol{\omega})\widehat{g}(\boldsymbol{\omega})}{\widehat{\phi}(\|\boldsymbol{\omega}\|)} d\boldsymbol{\omega}. \tag{1.4}$$

Then, N_ϕ is a real Hilbert space with inner product $(\cdot, \cdot)_\phi$ and reproducing kernel Φ.

The above space is known as the native space associated with the RBF ϕ, and it serves as the natural space of target functions upon which accuracy results can be formulated, see [19]. We remark that if there exist positive constants $\kappa_1 < \kappa_2$ such that

$$\frac{\kappa_1}{(1+z^2)^\lambda} \leq \widehat{\phi}(z) \leq \frac{\kappa_2}{(1+z^2)^\lambda}, \quad z \geq 0 \tag{1.5}$$

where $\lambda > \frac{d}{2}$, then the space N_ϕ in (1.3) is a reproducing kernel Hilbert space which is isomorphic to the Sobolev space $H^\lambda(\mathbb{R}^d) :=$

$$\left\{ f \in L_2(\mathbb{R}^d) \cap C(\mathbb{R}^d) : \|f\|^2_{H^\lambda(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\widehat{f}(\boldsymbol{\omega})|^2 (1 + \|\boldsymbol{\omega}\|^2)^\lambda d\boldsymbol{\omega} < \infty \right\}.$$

Further, under the same assumption, it can be shown that N_ϕ and $H^\lambda(\mathbb{R}^d)$ are norm equivalent; see [20].

In this paper, we will investigate three families of compactly supported RBFs, the Buhmann class, the (generalised) Wendland class and the (generalised) Wu class. Members from each class give rise to positive definite interpolation matrices (1.1) on the prescribed Euclidean space; due to the positivity of their Fourier transforms. In real-world cases, the property of compact support is beneficial as it ensures that the interpolation matrices are sparse, which can be useful when the set of data points is large. The implementation of compactly supported functions has also found popularity in the field of spatial statistics, where these families have been shown to be successful as covariance models for studies of estimation and prediction of Gaussian random fields, see [2] for a study based on generalised Wendland covariance functions. Further favourable results, for both the generalised Wendland and the Buhmann classes of functions, have been reported in [13] for their handling of the so-called screening effect; the situation whereby the value of an interpolant at a given location depends mainly on the observations closest to that location. The screening effect has a long history, see [17], with a non-trivial theoretical analysis, thus the positive results attributed to compactly supported families are likely to raise their profile further.

The most commonly used compactly supported classes are typically characterised via an integral representation and thus, at present, a potential user is required to derive a particular example by computing the integral using the desired parameter choices. The results in this paper help to remove this burden and provide the potential user with a master formula that can be easily programmed and modified if needed. Specifically, for each family under consideration, we will prove a theorem which establishes closed-form representations for certain parameter ranges; the proofs of these results are technical and involved, so they are deferred to a long Appendix. We will also highlight, where appropriate, when the native spaces of these functions are norm equivalent to the more familiar integer order Sobolev spaces.

2 The Buhmann functions

We begin with the contribution from Buhmann [3], who presented a large parameterised family of compactly supported radial basis functions, defined by

$$\phi_{d,\lambda,\ell,\alpha,\delta}(r) = \begin{cases} \int_{r^2}^{1} \left(1 - \frac{r^2}{\beta}\right)^{\lambda} \beta^{\alpha}(1-\beta^{\delta})^{\ell} d\beta & \text{for } 0 \le r \le 1, \\ 0 & \text{for } r > 1, \end{cases} \tag{2.6}$$

where the parameters are real valued with $0 < \delta \le \frac{1}{2}$, $\ell \ge 1$ and where d denotes the space dimension in which one wishes to employ the functions. We note that while d does not directly appear in the above expression, it can be shown, see [3], that the parameter choices required for positive definite matrices satisfy dimension-dependent constraints. We observe that by making the substitution $\beta = t^2$, we can re-express Buhmann's class as

$$\phi_{d,\lambda,\ell,\alpha,\delta}(r) = \begin{cases} 2\int_{r}^{1} t^{2\alpha-2\lambda+1}(1-t^{2\delta})^{\ell}(t^2-r^2)^{\lambda} dt & \text{for } 0 \le r \le 1, \\ 0 & \text{for } r > 1. \end{cases} \tag{2.7}$$

It is worthwhile to use (2.7) as our working definition for Buhmann's class due to an important extension/investigation of Buhmann's work by Zastavnyi [22], who considered functions of the form

$$\mathcal{Z}_{d,\nu,\mu,\gamma,\beta}(r) = \begin{cases} \int_{r}^{1} t^{\gamma-2\nu+1}(1-t^{\beta})^{\mu-1}(t^2-r^2)^{\nu-1} dt & \text{for } 0 \le r \le 1, \\ 0 & \text{for } r > 1. \end{cases} \tag{2.8}$$

Clearly, if we set Zastavnyi's parameters as

$$\nu = \lambda + 1, \quad \mu = \ell + 1, \quad \gamma = 2\alpha + 2, \quad \text{and} \quad \beta = 2\delta,$$

then $\mathcal{Z}_{d,\lambda+1,\ell+1,2\alpha+2,2\delta}(r) = 2\phi_{d,\lambda,\ell,\alpha,\delta}(r)$ and so coincide with Buhmann's functions. The conditions on the parameters that ensure existence, specific smoothness and positive definiteness (for a desired space dimension d where one wishes to employ the functions) can be found in [22].

2.1 The classical Buhmann functions

The classical Buhmann functions [3] were introduced in 1998 and form a specific instance of the wider Buhmann class, given above. Specifically, if we set $\delta = \frac{1}{2}$, choose k to be a positive integer such that $k \ge \frac{d-1}{2}$, then the following parameter choices

$$\alpha = k, \quad \lambda = 2k \text{ and } \ell \in \mathbb{Z}_+ \text{ (where } \ell \ge k),$$

give

$$\phi_{d,k,\ell,2k,\frac{1}{2}}(r) = k!2^k \mathcal{B}_{k,\ell}(r),$$

where $\mathcal{B}_{k,\ell}(r)$ denotes the classical Buhmann functions defined by

$$\mathcal{B}_{k,\ell}(r) = \begin{cases} \frac{1}{k!2^{k-1}} \int_r^1 t^{1-2k}(1-t)^\ell (t^2 - r^2)^k dt & \text{for } 0 \le r \le 1, \\ 0 & \text{for } r > 1. \end{cases} \tag{2.9}$$

When the condition $k \ge \frac{d-1}{2}$ is imposed, these functions are strictly positive definite on $\mathbb{R}^d$ as was shown in [3]. It was also shown in [3] that these functions belong to $C^{2k+1}(\mathbb{R}^d)$ and, within their support, they have the form that consists of a pure polynomial term augmented with a polynomial multiple of $r^2 \log(r)$, the thin-plate spline radial basis function. Explicit expressions for the polynomials appearing in this representation are not known; however, we fill this gap with the following theorem.

Theorem 2.3 *Let k and ℓ denote positive integers such that $\ell \ge k$. The classical Buhmann functions $\mathcal{B}_{k,\ell}(r)$ are strictly positive definite on $\mathbb{R}^d$, where $d \le 2k+1$ and they have the form*

$$\mathcal{B}_{k,\ell}(r) = P_{k,\ell}(r) + Q_{k,\ell}(r^2)\log(r), \quad 0 \le r \le 1,$$

where

$$\begin{aligned} P_{k,\ell}(r) = \frac{1}{k!2^k} \Bigg(& \sum_{j=0}^{k-1} \binom{\ell}{2j} \binom{k}{j+1} \Bigg[\sum_{i=0}^{j} \frac{(-1)^i r^{2i} (1-r^2)^{j+1-i}}{j+1-i} \\ & - (j+1)! \sum_{i=0}^{k-j-2} \frac{(k-j-2-i)!}{(k-i)!} (1-r^2)^{k-i} \Bigg] \\ & + \sum_{j=k}^{\lfloor \frac{\ell}{2} \rfloor} \binom{\ell}{2j} \sum_{i=0}^{j-k} \binom{j-k}{i} \frac{r^{2(j-k-i)}(1-r^2)^{k+1+i}}{(k+1+i)} \\ & + \sum_{i=0}^{k} \binom{k}{i} \left(\sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \binom{\ell}{2j+1} \frac{1}{j-i+\frac{3}{2}} \right) (-1)^i r^{2i} \\ & + (-1)^{k+1} k! \sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \binom{\ell}{2j+1} \frac{\Gamma\left(j+\frac{3}{2}-k\right)}{\Gamma\left(j+\frac{5}{2}\right)} r^{2j+3} \Bigg), \end{aligned} \tag{2.10}$$

$$Q_{k,\ell}(r^2) = \frac{1}{k!2^k} \sum_{j=0}^{k-1} (-1)^j \binom{\ell}{2j} \binom{k}{j+1} r^{2(j+1)}.$$

Proof See Appendix. □

We close this section with some examples.

◇ **Example 1** The function $\mathcal{B}_{1,4}(r)$ is strictly positive definite on $\mathbb{R}^d$ $(d \leq 3)$ and is given by

$$\mathcal{B}_{1,4}(r) = \begin{cases} \frac{1}{30} + \frac{19r^2}{12} - \frac{8r^3}{3} + \frac{3r^4}{2} - \frac{8r^5}{15} + \frac{r^6}{12} + r^2\log(r) & 0 \leq r \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

◇ **Example 2** The function $\mathcal{B}_{3,7}(r)$ is strictly positive definite on $\mathbb{R}^d$ $(d \leq 7)$ and is given by

$$\mathcal{B}_{3,7}(r) = \begin{cases} P_{3,7}(r) + Q_{3,7}(r^2)\log(r) & 0 \leq r \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$P_{3,7}(r) = \frac{1}{1728} + \frac{199r^2}{560} - \frac{14r^3}{9} - \frac{7r^4}{5} + \frac{14r^5}{3} - \frac{245r^6}{144} - \frac{2r^7}{5} + \frac{7r^8}{192} - \frac{2r^9}{945},$$

$$Q_{3,7}(r^2) = \frac{r^2}{8} - \frac{21r^4}{8} + \frac{35r^6}{24}.$$

◇ **Example 3** The function $\mathcal{B}_{5,8}(r)$ is strictly positive definite on $\mathbb{R}^d$ $(d \leq 11)$ and is given by

$$\mathcal{B}_{5,8}(r) = \begin{cases} P_{5,8}(r) + Q_{5,8}(r^2)\log(r) & 0 \leq r \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$P_{5,8}(r) = \frac{1}{172800} + \frac{1369r^2}{161280} - \frac{16r^3}{315} - \frac{469r^4}{2880} + \frac{112r^5}{225} - \frac{35r^6}{576} - \frac{16r^7}{45} + \frac{1253r^8}{11520} + \frac{16r^9}{945} - \frac{1621r^{10}}{806400},$$

$$Q_{5,8}(r^2) = \frac{r^2}{384} - \frac{7r^4}{48} + \frac{35r^6}{96} - \frac{7r^8}{96} + \frac{r^{10}}{1920}.$$

3 The generalised Wendland functions

A well-known sub-class of Buhmann's family is the generalised Wendland functions. They arise with the following choice of parameters

$$\lambda = \nu - 1, \quad \alpha = \nu - 1, \quad \text{and} \quad \delta = \frac{1}{2},$$

leading to

$$\phi_{d,\nu-1,\mu,\nu-1,\frac{1}{2}}(r) = \Gamma(\nu)2^{\nu}\psi^{gen}_{d,\nu,\mu}(r)$$

where $\psi^{gen}_{d,\nu,\mu}(r)$ denote the generalised Wendland functions defined by

$$\psi^{gen}_{d,\nu,\mu}(r) = \begin{cases} \frac{1}{\Gamma(\nu)2^{\nu-1}} \int_r^1 t(1-t)^{\mu}(t^2-r^2)^{\nu-1}dt & \text{for } 0 \le r \le 1, \\ 0 & \text{for } r > 1. \end{cases} \tag{3.11}$$

The d-dimensional Fourier transform of $\psi^{gen}_{d,\nu,\mu}(r)$ was computed in [9] and is given by

$$\widehat{\psi^{gen}_{d,\nu,\mu}}(z) = \frac{C_{\lambda,\mu}}{\sqrt{2\pi}} {}_1F_2\left[\begin{matrix} \lambda \\ \lambda+\frac{\mu}{2} \quad \lambda+\frac{\mu+1}{2} \end{matrix}; -\left(\frac{z}{2}\right)^2\right] \tag{3.12}$$

where

$$\lambda := \frac{d+1}{2} + \nu \quad \text{and} \quad C_{\lambda,\mu} := \frac{2^{\lambda}\Gamma(\lambda)\Gamma(\mu+1)}{\Gamma(2\lambda+\mu)} \tag{3.13}$$

and where ${}_1F_2(a; b, c; z)$ denotes the hypergeometric function (see [1, 15.1.1]). Hypergeometric functions will feature again in this work and so we briefly remind the reader that a general hypergeometric function is defined by

$${}_pF_q\left[\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{3.14}$$

where

$$(c)_n := c(c+1)\cdots(c+n-1) = \frac{\Gamma(c+n)}{\Gamma(c)}, \quad \text{for } n \ge 1, \tag{3.15}$$

denotes the Pochhammer symbol, with $(c)_0 := 1$.

It is known (see [9]) that $\widehat{\psi^{gen}_{d,\nu,\mu}}(z) > 0$ if and only if $\mu \ge \lambda$. Thus, with such a choice, $\widehat{\psi^{gen}_{d,\nu,\mu}}(z)$ induces a positive definite and compactly supported RBF on $\mathbb{R}^d$. Furthermore, as was shown in [11], there exist positive constants $\kappa_1 < \kappa_2$ such that

$$\frac{\kappa_1}{(1+z^2)^{\lambda}} \le \widehat{\psi^{gen}_{d,\nu,\mu}}(z) \le \frac{\kappa_2}{(1+z^2)^{\lambda}}, \quad z \ge 0, \tag{3.16}$$

and thus, following the discussion in the introductory section, the native space (1.3) of $\psi^{gen}_{d,\nu,\mu}$ is norm equivalent to the Sobolev space $H^{\lambda}(\mathbb{R}^d)$.

3.1 The original Wendland functions (for *d* odd)

When the space dimension d is odd, the original Wendland functions arise with the following parameter choices

$$\nu = k \in \mathbb{Z}_+ \text{ and } \mu = \ell = k + \frac{d+1}{2} (= \lambda)$$

to give for $k > 0$

$$\psi_{d,k,\ell}(r) = \begin{cases} \frac{1}{(k-1)!2^{k-1}} \int_r^1 t(1-t)^\ell (t^2 - r^2)^{k-1} dt & \text{for } 0 \le r \le 1, \\ 0 & \text{for } r > 1, \end{cases} \tag{3.17}$$

and for $k = 0$, $\psi_{d,0,\ell}(r) = (1-r)_+^\ell$. A careful evaluation of (3.17) (see [9]) shows that this function is a polynomial within its support that is given by

$$\psi_{d,k,\ell}(r) = \frac{(-1)^k 2^k k! \ell!}{(2k+\ell)!} \sum_{j=0}^{2k+\ell} (-1)^j \binom{2k+\ell}{j} \binom{\frac{j-1}{2}}{k} r^j, \quad \text{for } 0 \le r \le 1. \tag{3.18}$$

These functions have a special place in the literature because of their simple form and also due to the fact that, for odd space dimensions, their corresponding native spaces are norm equivalent to the integer order Sobolev spaces $H^{\frac{d+1}{2}+k}(\mathbb{R}^d)$.

3.2 The missing Wendland functions (for *d* even)

When the space dimension is even, the so-called missing Wendland functions arise with the following parameter choices

$$\nu = k + \frac{1}{2} \text{ (where } k \in \mathbb{Z}_+) \text{ and } \mu = \ell = k + 1 + \frac{d}{2} (= \lambda),$$

to give

$$\phi_{d,k+\frac{1}{2},\ell,k+\frac{1}{2},\frac{1}{2}}(r) = \Gamma\left(k + \frac{1}{2}\right) 2^{k+\frac{1}{2}} \psi_{d,k+\frac{1}{2},\ell}(r),$$

where $\psi_{d,k+\frac{1}{2},\ell}(r)$ denote the missing Wendland functions defined by

$$\psi_{d,k+\frac{1}{2},\ell}(r) = \begin{cases} \frac{1}{\Gamma\left(k+\frac{1}{2}\right)2^{k-\frac{1}{2}}} \int_r^1 t(1-t)^\ell (t^2 - r^2)^{k-\frac{1}{2}} dt & \text{for } 0 \le r \le 1, \\ 0 & \text{for } r > 1. \end{cases} \tag{3.19}$$

We have seen that when the space dimension is odd, the native space of original Wendland functions is norm equivalent to a Sobolev space of integer order. Schaback [16] shone a light on the missing Wendland function as those with similar properties to the original class but whose native spaces are norm equivalent to a Sobolev space of integer order on even-dimensional spaces. Schaback demonstrated that these missing Wendland functions, within their support, have a form that consists of a polynomial multiple of the square root function $S(r) = \sqrt{1-r^2}$ augmented with a polynomial multiple of the logarithmic function $L(r) = \ln\left(\frac{r}{1+\sqrt{1-r^2}}\right)$. Closed-form representations for these polynomial multiples were given by the first author in [10]; however, the resulting expressions, although correct, are rather cumbersome. We improve these representations, providing cleaner, more accessible expressions with the following result.

Theorem 3.4 *Let d denote an even space dimension, k be a positive integer and define $\ell := k+1+\frac{d}{2}$. The missing Wendland functions $\psi_{d,k+\frac{1}{2},\ell}(r)$, given by (3.19) are strictly positive definite on $\mathbb{R}^d$, and they have the form*

$$\psi_{d,k+\frac{1}{2},\ell}(r) = p_{d,k,\ell}(r^2)\sqrt{1-r^2} + q_{d,k,\ell}(r^2)\ln\left(\frac{r}{1+\sqrt{1-r^2}}\right) \tag{3.20}$$

where

$$p_{d,k,\ell}(r^2) = \frac{1}{2^{k+\frac{1}{2}}}\left[\sum_{i=0}^{k-1} a_i(k) r^{2i} + \sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} b_j(k) r^{2(k+j)}\right],$$

with

$$\begin{aligned} a_i(k) &= \frac{(-1)^i \binom{k-1}{i}}{\Gamma\left(k+\frac{1}{2}\right)} \sum_{j=0}^{\ell} \frac{(-1)^j \binom{\ell}{j}}{\left(k+\frac{j+1}{2}\right)} {}_3F_2\left[\begin{matrix} 1 & -i & \frac{1}{2}-k \\ 1-k & -k-\frac{j}{2}+\frac{1}{2} & \end{matrix}; 1\right], \\ b_j(k) &= \frac{(-1)^k}{\sqrt{\pi}} \sum_{i=0}^{\ell-2j} (-1)^i \binom{\ell}{i+2j} \frac{\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(j+\frac{i+2}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)\Gamma\left(k+j+\frac{i+3}{2}\right)}, \end{aligned} \tag{3.21}$$

and

$$q_{d,k,\ell}(r^2) = \frac{1}{2^{k+\frac{1}{2}}}\left[\sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} c_j(k) r^{2(k+j+1)}\right],$$

$$c_j(k) = \frac{\ell!(-1)^k}{\sqrt{\pi}2^{2j} j!(\ell-2j-1)!(k+j+1)!}. \tag{3.22}$$

Proof See Appendix. □

We close this section with some examples:

◇ **Example 1** When $d=2$, $k=2$ and $\ell=4$ $(=\frac{d}{2}+k+1)$ the corresponding missing Wendland function is

$$\psi_{2,\frac{5}{2},4}(r) = \begin{cases} p_{2,2,4}(r)\sqrt{1-r^2} + q_{2,4,6}(r)\log\left(\frac{r}{1+\sqrt{1-r^2}}\right) & 0 \le r \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$p_{2,2,4}(r) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{945} - \frac{17r^2}{1890} + \frac{23r^4}{505} + \frac{377r^6}{2160} + \frac{32r^8}{945}\right),$$

$$q_{2,2,4}(r) = -\frac{1}{\sqrt{2\pi}}\left(\frac{r^6}{6} + \frac{r^8}{16}\right).$$

◇ **Example 2** When $d = 4$, $k = 3$ and $\ell = 6\ (= \frac{d}{2} + k + 1)$ the corresponding missing Wendland function is

$$\psi_{4,\frac{7}{2},6}(r) = \begin{cases} p_{4,3,6}(r)\sqrt{1-r^2} + q_{4,3,6}(r)\log\left(\frac{r}{1+\sqrt{1-r^2}}\right), & 0 \le r \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$p_{4,3,6}(r) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{90090} - \frac{5r^2}{36036} + \frac{37r^4}{40040} - \frac{63r^6}{11440} - \frac{238981r^8}{5765760} - \frac{6827r^{10}}{349440} - \frac{32r^{12}}{45045}\right),$$

$$q_{4,3,6}(r) = \frac{1}{\sqrt{2\pi}}\left(\frac{r^8}{32} + \frac{r^{10}}{32} + \frac{r^{12}}{256}\right).$$

◇ **Example 3** When $d = 6$, $k = 4$ and $\ell = 8\ (= \frac{d}{2} + k + 1)$ the corresponding missing Wendland function is

$$\psi_{6,\frac{9}{2},8}(r) = \begin{cases} p_{6,4,8}(r)\sqrt{1-r^2} + q_{6,4,8}(r)\log\left(\frac{r}{1+\sqrt{1-r^2}}\right) & 0 \le r \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$p_{6,4,8}(r) = \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{11486475} - \frac{r^2}{696150} + \frac{5r^4}{408408} - \frac{1307r^6}{16707600} + \frac{1607r^8}{2970240}$$

$$+ \frac{18549137r^{10}}{2940537600} + \frac{24491671r^{12}}{3920716800} + \frac{1841897r^{14}}{1568286720} + \frac{256r^{16}}{11486475}\Big),$$

$$q_{6,4,8}(r) = \frac{1}{\sqrt{2\pi}}\left(\frac{r^{10}}{240} + \frac{7r^{12}}{960} + \frac{r^{14}}{384} + \frac{r^{16}}{6144}\right).$$

4 The Wu functions

The final class of compactly supported RBFs that we study was introduced by Wu [21] in 1995. Recently, the native space of these functions has been characterised and a generalisation of the class, comparable to the missing Wendland functions, has been proposed, see [8]. The original definition of Wu's functions is

$$\varphi_{d,\mu,\nu}(r) := \mathcal{D}^{\nu}(f_{\mu} * f_{\mu})(2r), \tag{4.23}$$

where $f_\mu(r) = (1 - r^2)_+^\mu$ for μ, ν positive integers with $\nu \le \mu$ and

$$\mathcal{D}\phi(r) := \frac{-1}{r}\frac{\partial}{\partial r}\phi(r), \quad r \ge 0,\ \phi \in C^2(\mathbb{R}_+). \tag{4.24}$$

This was extended in [8] to the so-called generalised Wu functions, defined for $\nu \in \mathbb{N}/2$, $\mu \in \mathbb{R}_+$. The connection to the other compactly supported functions comes from the special case $\nu = \mu$, which is stated in Theorem 4.4. [8] can be expressed, following a rescaling, as:

$$\begin{aligned}\varphi_{d,\mu,\mu}(r) &= \Gamma(\mu+1)2^{\mu+1}\int_r^1 (1-x^2)^\mu dx,\\ &= \Gamma(\mu+1)2^{\mu+1}\mathcal{Z}_{d,1,\mu+1,1,2}(r)\\ &= \Gamma(\mu+1)2^{\mu}\phi_{d,0,\mu,-\frac{1}{2},1}(r).\end{aligned}$$

The general definition of the missing Wu function involves the use of the fractional differential operator derived from the fractional integral operator

$$I_\nu\psi(r) := \int_r^\infty \frac{(x-r)^{\nu-1}}{\Gamma(\nu)}\psi(x)dx, \quad r \in \mathbb{R}_+, \nu > 0 \tag{4.25}$$

combined with

$$\begin{aligned} I_{-n} &:= (-1)^n \frac{\partial^n}{\partial r^n}, \quad n \in \mathbb{N}\\ I_{-\nu} &:= I_{n-\nu}I_{-n}, \quad 0 < \nu \le n = \lceil \nu \rceil \end{aligned} \tag{4.26}$$

and the f-form and inverse f-form

$$f\psi(r) := \psi(\sqrt{2r}), \quad f^{-1}\psi = \psi(r^2/2). \tag{4.27}$$

Then, the definition of the original Wu functions extends to the generalised ones by setting

$$\mathcal{D}^\nu := f^{-1}I_{-\nu}f, \quad \nu \in \mathbb{N}/2.$$

In Theorem 5.1 of [8], a formula that connects the generalised Wu functions to the generalised Wendland functions is established. Namely for $\nu \in \mathbb{N}/2$ and $\mu \in \mathbb{N}$

$$\varphi_{d,\mu,\nu}(r) = 2^{2\mu-\nu+1}\Gamma(\mu+1)\sum_{n=0}^{\mu}\binom{\mu}{n}\frac{2^{\mu-n}(-1)^n}{\mu+n+1}\psi^{gen}_{\mu-\nu,\mu+n+1}(r). \tag{4.28}$$

Here, we omit the dimension parameter in the generalised Wendland functions because it is implicitly included in the other two parameters and does not correspond to the dimension the Wu function is applied in.

4.1 The original Wu functions (for *d* odd)

When the space dimension d is odd, the original Wu functions arise from the parameter selection:

$$\nu = \frac{d-1}{2} \quad \text{and} \quad \nu \leq \mu \in \mathbb{N}.$$

The original Wu functions give rise to functions that are in $C^{2(\mu-\nu)}(\mathbb{R}^d)$. In order to consider a framework that is consistent with the Wendland functions, we introduce a smoothness parameter $k \in \mathbb{N}$ and a corresponding dimension-dependent parameter $\ell = \frac{d-1}{2} + k \in \mathbb{N}$ and focus on Wu functions given by

$$\mathcal{W}_{d,k,\ell}(r) = \varphi_{d,\frac{d-1}{2}+k,\frac{d-1}{2}}(r),$$

where the parameter choices ensures that $\mathcal{W}_{d,k,\ell} \in C^{2k}(\mathbb{R}^d)$. Also, as was shown in [8] Theorem 3.14, the native space of $\mathcal{W}_{d,k,\ell}$ is a dense subset of $H^s(\mathbb{R}^d)$ for $\frac{d}{2} \leq s \leq \frac{d+1}{2} + k$.

The original Wu functions have a polynomial form on their support, and we will give an explicit representation using the connection to the Wendland functions (4.28) and our representation of the original Wendland function. We can reformulate (4.28) in terms of the original Wendland functions

$$\mathcal{W}_{d,k,\ell}(r) = 2^{\ell+k+1}\Gamma(\ell+1)\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}\psi_{k,\ell+n+1}(r), \tag{4.29}$$

where the parameters of the Wendland functions $\ell+n+1$ and k are both integers. Using the known expression (3.18) for the original Wendland function, we can formulate the following result for the original Wu functions.

Theorem 4.5 *Let d denote an odd space dimension, k be a positive integer and define $\ell = \frac{d-1}{2} + k$. The original Wu functions $\mathcal{W}_{d,k,\ell}(r)$ are strictly positive definite on $\mathbb{R}^d$ and have the form*

$$\mathcal{W}_{d,k,\ell}(r) = (-1)^k k! 2^{2k+\ell+1}\Gamma(\ell+1)\left(\sum_{j=0}^{2k+\ell+1}\tilde{a}_{k,\ell,j}r^j + \sum_{j=2k+\ell+2}^{2k+2\ell+1}\tilde{b}_{k,\ell,j}r^j\right), \tag{4.30}$$

where

$$\begin{aligned}\tilde{a}_{k,\ell,j} &= (-1)^j\binom{\frac{j-1}{2}}{k}\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(\ell+n)!(-1)^n}{j!(2k+\ell+n+1-j)!},\\ \tilde{b}_{k,\ell,j} &= (-1)^j\binom{\frac{j-1}{2}}{k}\sum_{n=j-2k-\ell-1}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(\ell+n)!(-1)^n}{(2k+\ell+n+1-j)!j!}.\end{aligned} \tag{4.31}$$

Proof Starting from the connection function of the Wu and Wendland functions (4.29) and the expression of the original Wendland functions (3.18), we find for $\ell, k \in \mathbb{N}$:

$$\begin{aligned}\mathcal{W}_{d,k,\ell}(r) =&(-1)^k k! 2^{2k+\ell+1}\Gamma(\ell+1)\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(\ell+n)!}{(2k+\ell+n+1)!}(-1)^n \\ &\sum_{j=0}^{2k+\ell+n+1}(-1)^j\binom{2k+\ell+n+1}{j}\binom{\frac{j-1}{2}}{k}r^j.\end{aligned} \tag{4.32}$$

Exchanging the finite sums provides the coefficients of the polynomial stated in the theorem. ◻

Using Theorem 4.5, we provide expressions for two non-standard examples:

◇ **Example 1** When $d = 3$, $k = 3$ and $\ell = 4 = \frac{d-1}{2} + k$, the function $\mathcal{W}_{3,3,4}$ is for $0 \le r \le 1$:

$$\mathcal{W}_{3,3,4}(r) = \frac{2048}{45045}(1-r)^8\left(7r^7 + 56r^6 + 192r^5 + 360r^4 + 384r^3 + 216r^2 + 64r + 8\right).$$

◇ **Example 2** When $d = 7$, $k = 3$ and $\ell = 6 (= \frac{d-1}{2} + k)$, the function $\mathcal{W}_{7,3,6}$ is for $0 \le r \le 1$:

$$\begin{aligned}\mathcal{W}_{7,3,6}(r) = \frac{16384}{323323}(1-r)^{10}(&320 + 3200r + 13952r^2 + 33920r^3 + 48832r^4 + \\ &44800r^5 + 26880r^6 + 10311r^7 + 2310r^8 + 231r^9).\end{aligned}$$

4.2 The missing Wu functions (for *d* even)

When the space dimension is even, we will stay consistent with the missing Wendland functions and refer to the generalised Wu functions with the parameter choice:

$$\nu := \frac{d-1}{2} \quad \text{and} \quad \nu \le \mu \in \mathbb{N}$$

as the missing Wu functions. The native spaces of the missing Wu functions are described in Theorem 4.3 of [8].

We mimic our approach to the classical Wu functions by introducing a smoothness parameter $k \in \mathbb{N}$ and a corresponding dimension-dependent parameter $\ell = \frac{d-1}{2} + k + \frac{1}{2} \in \mathbb{N}$ and describe the missing Wu functions by

$$\mathcal{W}_{d,k+\frac{1}{2},\ell}(r) = \varphi_{d,\frac{d-1}{2}+k+\frac{1}{2},\frac{d-1}{2}}(r).$$

The connection formula (4.28) now reads

$$\mathcal{W}_{d,k+\frac{1}{2},\ell}(r) = 2^{\ell+k+\frac{3}{2}}\Gamma(\ell+1)\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}\psi_{k+\frac{1}{2},\ell+n+1}(r), \tag{4.33}$$

where $\psi_{k+\frac{1}{2},\ell+n+1}$ are missing Wendland functions.

As a result of the above connection, the missing Wu function shares a common representation with the missing Wendland functions. Specifically, their form is also that of a polynomial multiple of the square root function $S(r)=\sqrt{1-r^2}$ augmented with a polynomial multiple of the logarithmic function $L(r)=\ln\left(\frac{r}{1+\sqrt{1-r^2}}\right)$. Closed-form representations for these polynomial multiples are not known, as far as the authors are aware; however, using Theorem 3.4, we can fill this gap with the following result.

Theorem 4.6 *Let d denote an even space dimension, k be a positive integer and define $\ell=\frac{d-1}{2}+k+\frac{1}{2}$. The missing Wu functions $\mathcal{W}_{d,k+\frac{1}{2},\ell}(r)$ given by (4.33) are strictly positive definite on $\mathbb{R}^d$, and they have the form*

$$
\begin{aligned}
\mathcal{W}_{d,k+\frac{1}{2},\ell}(r) &= 2^{\ell+1}\ell!\left[\sum_{i=0}^{k-1}\tilde{a}_i(k,\ell)r^{2i}+\sum_{j=0}^{\ell}\tilde{b}_j(k,\ell)r^{2(k+j)}\right]\sqrt{1-r^2}\\
&\quad+2^{\ell+1}\ell!\left[\sum_{j=0}^{\ell}\tilde{c}_j(k,\ell)r^{2(k+j+1)}\right]\log\left(\frac{r}{1+\sqrt{1-r^2}}\right)
\end{aligned}
\tag{4.34}
$$

where

$$
\begin{aligned}
\tilde{a}_i(k,\ell) = {} & \frac{(-1)^i\binom{k-1}{i}}{\Gamma\left(k+\frac{1}{2}\right)}\sum_{j=0}^{2\ell+1}\frac{(-1)^j}{\left(k+\frac{j+1}{2}\right)}\,{}_3F_2\left[\begin{matrix}1 & -i & \frac{1}{2}-k\\ 1-k & -k-\frac{j}{2}+\frac{1}{2} & \end{matrix};1\right]\\
&\times\sum_{n=\max\{0,j-\ell-1\}}^{\ell}\binom{\ell}{n}\frac{(-1)^n2^{\ell-n}}{\ell+n+1}\binom{\ell+n+1}{j},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{b}_j(k,\ell) = {} & \frac{(-1)^k}{\sqrt{\pi}}\sum_{i=0}^{2\ell+1-2j}(-1)^i\frac{\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(j+\frac{i+2}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)\Gamma\left(k+j+\frac{i+3}{2}\right)}\\
&\times\sum_{n=\max\{0,2j-\ell-1+i\}}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}\binom{\ell+n+1}{i+2j},
\end{aligned}
$$

and

$$
\tilde{c}_j(k,\ell)=\frac{(-1)^k2^{\ell-2j}}{\sqrt{\pi}\,j!(k+j+1)!}\sum_{n=\max\{0,2j-\ell\}}^{\ell}\left(-\frac{1}{2}\right)^n\frac{\ell!(\ell+n)!}{(\ell-n)!n!(\ell+n-2j)!}.
$$

Proof See Appendix. □

We close this section with two non-standard examples:

◇ **Example 1** When $d=2$, $k=3$ and $\ell=4(=\frac{d-1}{2}+k+\frac{1}{2})$, the function $\mathcal{W}_{2,3+\frac{1}{2},4}$ is:

$$\mathcal{W}_{3,3+\frac{1}{2},4}(r) = \begin{cases} \tilde{p}_{3,4}(r)\sqrt{1-r^2} + \tilde{q}_{3,4}(r)\log\left(\frac{r}{\sqrt{1-r^2}+1}\right)\Big), & 0 \le r \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

with

$$\tilde{p}_{3,4}(r) = \frac{1}{29400\sqrt{\pi}}(3675r^{14} - 31150r^{12} + 120680r^{10} - 298480r^8 - 646784r^6 + 224512r^4 - 54272r^2 + 6144),$$

$$\tilde{q}_{3,4}(r) = \frac{-105}{29400\sqrt{\pi}}r^8\left(35r^8 - 320r^6 + 1344r^4 - 3584r^2 + 8960\right).$$

◇ **Example 2** When $d = 8$, $k = 1$ and $\ell = 5(=\frac{d-1}{2}+1+\frac{1}{2})$, the function $\mathcal{W}_{8,1+\frac{1}{2},5}$ is:

$$\mathcal{W}_{8,1+\frac{1}{2},5}(r) = \begin{cases} \tilde{p}_{1,5}(r)\sqrt{1-r^2} + \tilde{q}_{1,5}(r)\log\left(\frac{r}{\sqrt{1-r^2}+1}\right)\Big), & 0 \le r \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

with

$$\tilde{p}_{1,5}(r) = \frac{1}{21\sqrt{\pi}}\Big(-945r^{12} + 6720r^{10} - 20804r^8 + 37088r^6 - 44064r^4 - 25600r^2 + 2560\Big)$$

$$\tilde{q}_{1,5}(r) = \frac{105}{21\sqrt{\pi}}\left(9r^{10} - 70r^8 + 240r^6 - 480r^4 + 640r^2 - 768\right)r^4.$$

5 Discussion

In this paper, we have examined three classes of compactly supported RBFs and, using their integral representations, we have provided closed-form expressions for certain parameter classes. We summarise the parameters and properties in (Table 1), giving additional explanations for the missing entries in the corresponding sections of the following discussion.

◇ **The Buhmann class**

The Buhmann class of compactly supported functions is arguably the most rich; we have observed that this class contains the generalised Wendland functions, and it is also stated in [5] that it contains the original Wu functions. The general class has been studied in detail in [12], where the Fourier transform and its rate of decay are

Table 1 Functions investigated with parameters and references to the closed forms

Basis function	Symbol	Parameters	$C^m(\mathbb{R}^d)$	Native space	Closed form eq
cl. Buhmann	$\mathcal{B}_{k,\ell}(r)$	$\ell \geq k, d \leq 2k+1$	$m = 2k+1$	$*$	(2.10)
orig. Wend	$\psi_{d,k,\ell}(r)$	$\ell = k + \frac{d+1}{2}$, d odd	$m = 2k$	$\sim H^{\frac{d+1}{2}+k}$	(3.4)
miss. Wend	$\psi_{d,k+\frac{1}{2},\ell}(r)$	$\ell = k + \frac{d+1}{2}$, d even	$m = 2k$	$\sim H^{\frac{d}{2}+k}$	(3.20)
orig. Wu	$\mathcal{W}_{d,k,\ell}(r)$	$\ell = \frac{d-1}{2} + k$, d odd	$m = 2k$	$\subset H^{\frac{d+1}{2}+k}$	(4.30)
miss. Wu	$\mathcal{W}_{d,k+\frac{1}{2},\ell}(r)$	$\ell = \frac{d}{2} + k$, d even	$*$	$*$	(4.34)

established across a wide range of parameters. The focus here is on the form of a particular subclass, termed the classical Buhmann functions, which are reminiscent of the famous thin-plate splines. These were highlighted in [3] where theoretical error bounds are established, including statements on the native space. Other closed-form expressions are certainly possible; indeed, it is stated in [3] that considering α and ℓ to be non-negative integers is merely a simplifying assumption which places no limitation on the applicability of the results. To illustrate this, we can choose $\alpha = \frac{3}{4}$, $\ell = 1$ and $\lambda = 2$ and then, via evaluating integral (2.6), we have the following function

$$\phi_{2,2,1,\frac{3}{4},\frac{1}{2}}(r) = \frac{8}{63} - \frac{16r^2}{15} - 8r^4 + \frac{128r^{\frac{7}{2}}}{21} + \frac{128r^{\frac{9}{2}}}{45}, \quad 0 \leq r \leq 1,$$

which is positive definite on $\mathbb{R}^2$. This example suggests there are further sub-classes, the above involving a polynomial augmented with fractional powers. The authors suspect that a more encompassing representation for the general Buhmann functions could be possible by using hypergeometric function identities. This is left for future work and could open the door to more amenable expressions for applications; the following recent work by Emery et al. [7] could prove to be a helpful starting point.

◇ The Wendland class

The Wendland class is perhaps the most popular among those we have considered. In this case, a master formula via hypergeometric functions is known. Specifically, using formula (3.5) of [6], we have

$$\psi_{d,\nu,\mu}(r) := \frac{B(\mu, \nu+1)\left(1-r^2\right)^{\nu+\mu}}{2^{\nu+1}B(2\mu, \nu+1)} {}_2F_1\left[\begin{matrix} \frac{\nu}{2} & \frac{\nu+1}{2} \\ \nu+\mu+1 \end{matrix}; 1-r^2\right], \tag{5.35}$$

for $0 \leq r \leq 1$, where $\nu > 0$, $\mu > 0$ and $B(x, y)$ denotes the beta function. We have noted in Sect. 3 that $\psi_{d,\nu,\mu}(r)$ is positive definite on $\mathbb{R}^d$ provided that $\nu \geq \frac{d+1}{2} + \mu$ and that the corresponding native space is norm equivalent to the Sobolev space $H^{\frac{d+1}{2}+\mu}(\mathbb{R}^d)$. In our work, we have focused on functions that are considered natural for odd and even dimensional spaces separately. When d is odd, we consider integer values (k) for μ; this leads to the original Wendland functions having a polynomial form (3.18), whose native spaces correspond to Sobolev spaces of integer order $\frac{d+1}{2}+k$. When d is even, we consider half-integer values ($k+\frac{1}{2}$) for μ, this leads to the missing Wendland functions which have a non-polynomial form (3.20), and whose native spaces also correspond to Sobolev spaces of integer order $\frac{d}{2}+k+1$.

The authors cannot rule out the existence of further sub-classes of the generalised Wendland functions that have an amenable closed-form representation. A confirmation of this would require a deeper investigation of the properties of the hypergeometric function appearing in (5.35) and is left as an open question.

◇ The Wu class

Interest in the Wu class of compactly supported functions has been revived with the recent paper [8] where the following highlighted results were established:

1. Following Schaback's proposal of the missing Wendland functions [16], an analogous framework for the missing Wu functions, for the even-dimensional spaces, is established.
2. A connection formula that links expressions for the original and missing Wu functions to the original and missing Wendland functions is established.
3. The native spaces of the original and missing Wu functions are characterised, and they are shown to be dense in subsets of a range of smoothness orders of Sobolev spaces.
4. Unlike the case for the Wendland class, a direct norm equivalence to a particular Sobolev space cannot be established due to the fact that the Fourier transform of the Wu functions is non-negative but possesses zeroes; these occur at the locations of the zeroes of the Bessel function that appears in the Fourier transform representation.

In view of the second point above, we are able to use earlier results for the original/missing Wendland classes to deliver closed-form expressions for the original/missing Wu functions. There are several open questions regarding the Wu functions, and again, the authors suspect that further progress can be made by seeking general characterisations in terms of hypergeometric functions.

Appendix. Proofs

The classical Buhmann functions

Proof of Theorem 2.3

We are reminded (2.9) that the classical Buhmann functions are given by

$$\mathcal{B}_{k,\ell}(r) = \frac{1}{k!2^{k-1}} \int_r^1 t^{1-2k}(1-t)^{\ell}(t^2-r^2)^k dt \quad \text{for } 0 \le r \le 1,$$

where ℓ and k are positive integers such that $\ell \geq k \geq \frac{d-1}{2}$. Applying the binomial theorem to the $(1-t)^\ell$ factor and then applying the change of variable $z^2 = t^2 - r^2$, we find that

$$\mathcal{B}_{k,\ell}(r) = \frac{1}{k!2^{k-1}} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \int_0^{\sqrt{1-r^2}} (z^2 + r^2)^{\frac{j-2k}{2}} z^{2k+1} dz.$$

Write

$$I_{k,j}(r^2) = \int_0^{\sqrt{1-r^2}} (z^2 + r^2)^{\frac{j-2k}{2}} z^{2k+1} dz,$$

then

$$\mathcal{B}_{k,\ell}(r) = \frac{1}{k!2^{k-1}} \left[\sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} \binom{\ell}{2j} I_{k,2j}(r^2) - \sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \binom{\ell}{2j+1} I_{k,2j+1}(r^2) \right].$$

Let us consider the even integrals under the assumption that $j \leq k-1$, then

$$I_{k,2j}(r^2) = \int_0^{\sqrt{1-r^2}} (z^2 + r^2)^{j-k} z^{2k+1} dz = \int_0^{\sqrt{1-r^2}} \frac{z^{2k+1}}{(z^2 + r^2)^{k-j}} dz.$$

Integrating by parts leads to

$$I_{k,2j}(r^2) = -\frac{(1-r^2)^k}{2((k-j)-1)} + \frac{k}{((k-j)-1)} \int_0^{\sqrt{1-r^2}} \frac{z^{2k-1}}{(z^2 + r^2)^{(k-j)-1}} dz.$$

Continuing to integrate by parts for $k-j-1$ steps, then we arrive at

$$\begin{aligned} I_{k,2j}(r^2) = &-\frac{1}{2} \sum_{i=0}^{k-j-2} \frac{k!(k-j-2-i)!}{(k-i)!(k-1-j)!} (1-r^2)^{k-i} \\ &+ \frac{k!}{(j+1)!(k-j-1)!} \int_0^{\sqrt{1-r^2}} \frac{z^{2j+3}}{(z^2+r^2)} dz. \end{aligned}$$

The following formula is taken from [14] 1.2.10.10

$$\int \frac{x^{2m+1}}{(x^2 + a^2)} dx = \frac{1}{2} \sum_{k=0}^{m-1} (-1)^k \frac{a^{2k} x^{2m-2k}}{m-k} + \frac{1}{2} (-1)^m a^{2m} \log |x^2 - a^2|.$$

Using this with $a = r$ and $m = j+1$ to give

$$\int_0^{\sqrt{1-r^2}} \frac{z^{2j+3}}{(z^2 + r^2)} dz = \frac{1}{2} \sum_{i=0}^{j} (-1)^i \frac{r^{2i}(1-r^2)^{j+1-i}}{j+1-i} + \frac{(-1)^j}{2} r^{2(j+1)} \log(r^2).$$

This then enables us to conclude that, when $j \leq k-1$, we have

$$I_{k,2j}(r^2) = \frac{1}{2}\binom{k}{j+1}\Bigg(-(j+1)!\sum_{i=0}^{k-j-2}\frac{k!(k-j-2-i)!}{(k-i)!(k-1-j)!}(1-r^2)^{k-i} + \sum_{i=0}^{j}(-1)^i\frac{r^{2i}(1-r^2)^{j+1-i}}{j+1-i} + (-1)^j r^{2(j+1)}\log(r^2)\Bigg). \tag{6.36}$$

For the case where $j \geq k$, we can apply the following identity that is taken from [14] 1.2.2.7

$$\int x^p(ax^m+b)^n dx = \sum_{i=0}^{n}\binom{n}{i}\frac{a^k b^{n-i} x^{p+im+1}}{p+im+1}.$$

Using this with $n = j-k$, $a = 1$, $p = 2k+1$ and $m = 2$, we have

$$I_{k,2j}(r^2) = \int_0^{\sqrt{1-r^2}}(z^2+r^2)^{j-k}z^{2k+1}dz = \frac{1}{2}\sum_{i=0}^{j-k}\binom{j-k}{i}\frac{r^{2(j-k-i)}(1-r^2)^{k+1+i}}{(k+1+i)}. \tag{6.37}$$

Using (6.36) and (6.37), we can define

$$\begin{aligned}\mathcal{B}_{k,\ell}^{(even)}(r) &= \frac{1}{k!2^{k-1}}\sum_{j=0}^{\lfloor\frac{\ell}{2}\rfloor}\binom{\ell}{2j}I_{k,2j}(r^2)\\ &= \frac{1}{k!2^k}\Bigg(\sum_{j=0}^{k-1}\binom{\ell}{2j}\binom{k}{j+1}\Bigg[\sum_{i=0}^{j}\frac{(-1)^i r^{2i}(1-r^2)^{j+1-i}}{j+1-i}\\ &- (j+1)!\sum_{i=0}^{k-j-2}\frac{(k-j-2-i)!}{(k-i)!}(1-r^2)^{k-i}\Bigg]\\ &+ \sum_{j=0}^{k-1}\binom{\ell}{2j}\binom{k}{j+1}(-1)^j r^{2(j+1)}\log(r^2)\\ &+ \sum_{j=k}^{\lfloor\frac{\ell}{2}\rfloor}\binom{\ell}{2j}\sum_{i=0}^{j-k}\binom{j-k}{i}\frac{r^{2(j-k-i)}(1-r^2)^{k+1+i}}{(k+1+i)}\Bigg).\end{aligned} \tag{6.38}$$

We note that if $k > \lfloor\frac{\ell}{2}\rfloor$ then the final sum in the above expression is null and does not contribute.

For the odd integrals $I_{k,2j+1}(r^2)$, we can apply the following identity that is taken from [14] 1.2.41.3

$$\int x^{2m+1}(x^2+a^2)^{n+\frac{1}{2}}dx = \sum_{i=0}^{m}\binom{m}{i}(-1)^i\frac{a^{2i}(x^2+a^2)^{m+n-i+\frac{3}{2}}}{2n+2m-2i+3}.$$

Setting $m = k$, $a = r$ and $n = j - k$, we can deduce that

$$I_{k,2j+1}(r^2) = \int_0^{\sqrt{1-r^2}} (z^2 + r^2)^{j-k+\frac{1}{2}} z^{2k+1} dz = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{r^{2i} - r^{2(j+1)+1}}{2(j-i+1)+1}$$
$$= \frac{1}{2} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i r^{2i}}{j - i + \frac{3}{2}} + \frac{r^{2j+3}}{2} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{i - j - \frac{3}{2}}.$$

The following identity is taken from [14] 4.2.2.45

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+a} \binom{n}{i} = \frac{n!}{a(a+1)\cdots(a+n)}.$$

Applying this with $n = k$ and $a = -j - \frac{3}{2}$ allows us to deduce that

$$I_{k,2j+1}(r^2) = \frac{1}{2} \left(\sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i r^{2i}}{j - i + \frac{3}{2}} + \frac{(-1)^{k+1} k! \Gamma\left(j + \frac{3}{2} - k\right) r^{2j+3}}{\Gamma\left(j + \frac{5}{2}\right)} \right). \quad (6.39)$$

Using this, we can define

$$\mathcal{B}_{k,\ell}^{(odd)}(r) = \frac{1}{k! 2^{k-1}} \sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} \binom{\ell}{2j+1} I_{k,2j+1}(r^2)$$
$$= \frac{1}{k! 2^k} \left(\sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \binom{\ell}{2j+1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i r^{2i}}{j - i + \frac{3}{2}} \right. \quad (6.40)$$
$$\left. + (-1)^{k+1} k! \sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \binom{\ell}{2j+1} \frac{\Gamma\left(j + \frac{3}{2} - k\right)}{\Gamma\left(j + \frac{5}{2}\right)} r^{2j+3} \right).$$

Recombining the parts, we find

$$\mathcal{B}_{k,\ell}(r) = P_{k,\ell}(r) + Q_{k,\ell}(r^2) \log(r),$$
$$P_{k,\ell}(r) = \frac{1}{k! 2^k} \left(\sum_{j=0}^{k-1} \binom{\ell}{2j} \binom{k}{j+1} \left[\sum_{i=0}^{j} \frac{(-1)^i r^{2i} (1 - r^2)^{j+1-i}}{j + 1 - i} \right. \right.$$
$$\left. - (j+1)! \sum_{i=0}^{k-j-2} \frac{(k - j - 2 - i)!}{(k - i)!} (1 - r^2)^{k-i} \right]$$

$$+\sum_{j=k}^{\lfloor \frac{\ell}{2} \rfloor}\binom{\ell}{2j}\sum_{i=0}^{j-k}\binom{j-k}{i}\frac{r^{2(j-k-i)}(1-r^2)^{k+1+i}}{(k+1+i)}$$

$$+\sum_{i=0}^{k}\binom{k}{i}\left(\sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor}\binom{\ell}{2j+1}\frac{1}{j-i+\frac{3}{2}}\right)(-1)^i r^{2i}$$

$$+(-1)^{k+1}k!\sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor}\binom{\ell}{2j+1}\frac{\Gamma\left(j+\frac{3}{2}-k\right)}{\Gamma\left(j+\frac{5}{2}\right)}r^{2j+3}\Bigg),$$

$$Q_{k,\ell}(r^2)=\frac{1}{k!2^k}\sum_{j=0}^{k-1}(-1)^j\binom{\ell}{2j}\binom{k}{j+1}r^{2(j+1)}, \tag{6.41}$$

as required.

The missing Wendland functions

Proof of Theorem 3.4

A closed-form expression for the missing Wendland function was not given in Schaback's original paper; however, the starting point, which was used to show the general form (3.20), was to apply the binomial theorem to the $(1-t)^\ell$ factor of the integrand (3.19) and then to make the change of variable $z^2=t^2-r^2$ to give

$$\psi_{d,k+\frac{1}{2},\ell}(r)=\frac{1}{\Gamma\left(k+\frac{1}{2}\right)2^{k-\frac{1}{2}}}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\int_0^{\sqrt{1-r^2}}(z^2+r^2)^{\frac{j}{2}}z^{2k}dz. \tag{6.42}$$

Integration by parts gives

$$\int_0^{\sqrt{1-r^2}}(z^2+r^2)^{\frac{j}{2}}z^{2k}dz$$
$$=\sqrt{1-r^2}\frac{(1-r^2)^{k-1}}{2+j}+\frac{\left(k-\frac{1}{2}\right)}{\left(\frac{j+2}{2}\right)}\int_0^{\sqrt{1-r^2}}(z^2+r^2)^{\frac{j}{2}+1}z^{2k-2}dz.$$

Continuing integrating by parts, in the same fashion, for k steps yields

$$\int_0^{\sqrt{1-r^2}}(z^2+r^2)^{\frac{j}{2}}z^{2k}dz$$
$$=\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{j+2}{2}\right)(-1)^{k+1}\Bigg[\frac{\sqrt{1-r^2}}{2}\sum_{i=0}^{k-1}\frac{(-1)^i(1-r^2)^i}{\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(\frac{j+2}{2}+k-i\right)}$$

$$-\frac{1}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{j+2}{2}+k\right)}\int_0^{\sqrt{1-r^2}}(z^2+r^2)^{\frac{j}{2}+k}dz\Bigg] \tag{6.43}$$

The following result demonstrates that the sum appearing in (6.43) can be represented as a linear combination of two hypergeometric functions.

Lemma 6.7 *For* $r \in [0, 1]$, *we have that*

$$\sum_{i=0}^{k-1}\frac{(-1)^i(1-r^2)^i}{\Gamma\left(\frac{3}{2}+i\right)\Gamma\left(\frac{j+2}{2}+k-i\right)} = \frac{(-1)^{k+1}(1-r^2)^k\,{}_2F_1\left(1,-\frac{j}{2};\frac{3}{2}+k;1-r^2\right)}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(\frac{j+2}{2}\right)} + \frac{{}_2F_1\left(1,-\frac{j}{2}-k;\frac{3}{2};1-r^2\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{j+2}{2}+k\right)}. \tag{6.44}$$

Proof Follows by the employing the definition of the hypergeometric function and some elementary manipulation. □

The following is a distant recurrence relation for the b parameter in the hypergeometric function, it is taken from [15] formula 7.3.1.120 (part 1)

$${}_2F_1\left(1,b-k;c;z\right) = (1-z)^k\frac{(1-b)_k}{(c-b)_k}\left[{}_2F_1\left(1,b;c;z\right)+\frac{c-1}{1-b}\sum_{i=0}^{k-1}\frac{(c-b)_i}{(2-b)_i}(1-z)^{-(i+1)}\right].$$

Setting $z=1-r^2$, $b=-\frac{j}{2}$ and $c=\frac{3}{2}$ into the above one can show, with some elementary manipulation, that

$${}_2F_1\left(1,-\frac{j}{2}-k;\frac{3}{2};1-r^2\right) = \frac{r^{2k}\Gamma\left(\frac{j+2}{2}+k\right)}{\Gamma\left(\frac{j+3}{2}+k\right)}\left[\frac{\Gamma\left(\frac{j+3}{2}\right)}{\Gamma\left(\frac{j+2}{2}\right)}\,{}_2F_1\left(1,-\frac{j}{2};\frac{3}{2};1-r^2\right)+\frac{1}{2}\sum_{i=0}^{k-1}\frac{\Gamma\left(\frac{j+3}{2}+i\right)}{\Gamma\left(\frac{j+4}{2}+i\right)}\frac{1}{r^{2(i+1)}}\right]. \tag{6.45}$$

The following is a distant recurrence relation for the c parameter in the hypergeometric function, it is taken from [15] formula 7.3.1.120 (part 2)

$${}_2F_1\left(1,b;c+k;z\right) = \left(\frac{1-z}{z}\right)^k\frac{(1-c-k)_k}{(b-c-k+1)_k}\,{}_2F_1\left(1,b;c;z\right)+\frac{1}{z}\sum_{i=1}^{k}\frac{(1-c-k)_i}{(b-c-k+1)_i}(-1)^i\left(\frac{1-z}{z}\right)^i.$$

Setting $z = 1 - r^2$, $b = -\frac{j}{2}$ and $c = \frac{3}{2}$ into the above one can show, with some elementary manipulation, that

$$
{}_2F_1\left(1, -\frac{j}{2}; \frac{3}{2} + k; 1 - r^2\right) = \frac{\Gamma\left(\frac{3}{2} + k\right)}{\Gamma\left(\frac{j+3}{2} + k\right)} \left[\frac{\Gamma\left(\frac{j+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \frac{(-1)^k r^{2k}}{(1 - r^2)^k} {}_2F_1\left(1, -\frac{j}{2}; \frac{3}{2}; 1 - r^2\right) \right.
$$
$$
\left. + \frac{1}{1 - r^2} \sum_{i=1}^{k} \frac{\Gamma\left(k + \frac{j+3}{2} - i\right)}{\Gamma\left(k + \frac{3}{2} - i\right)} (-1)^{i-1} \left(\frac{r^2}{1 - r^2}\right)^{i-1}\right]. \tag{6.46}
$$

Inserting (6.45) and (6.46) into the expression (6.44), one notices that the contributions involving the hypergeometric function terms cancel out and so leaving

$$
\sum_{i=0}^{k-1} \frac{(-1)^i (1 - r^2)^i}{\Gamma\left(\frac{3}{2} + i\right) \Gamma\left(\frac{j+2}{2} + k - i\right)}
$$
$$
= \frac{1}{\Gamma\left(k + \frac{j+3}{2}\right) \Gamma\left(\frac{j+2}{2}\right)} \sum_{i=0}^{k-1} \frac{\Gamma\left(\frac{j+3}{2} + i\right)}{\Gamma\left(\frac{3}{2} + i\right)} (-1)^i (1 - r^2)^i r^{2(k-1-i)}
$$
$$
+ \frac{1}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(k + \frac{j+3}{2}\right)} \sum_{i=0}^{k-1} \frac{\Gamma\left(\frac{j+3}{2} + i\right)}{\Gamma\left(\frac{j+4}{2} + i\right)} r^{2(k-1-i)}.
$$

Returning to (6.43) and using the above result, we have

$$
\int_0^{\sqrt{1-r^2}} (z^2 + r^2)^{\frac{j}{2}} z^{2k} dz
$$
$$
= \frac{\Gamma\left(k + \frac{1}{2}\right) \Gamma\left(\frac{j+2}{2}\right) (-1)^{k+1} \sqrt{1 - r^2}}{2\Gamma\left(k + \frac{j+3}{2}\right)} \left[\frac{1}{\Gamma\left(\frac{j+2}{2}\right)} \sum_{i=0}^{k-1} \frac{\Gamma\left(\frac{j+3}{2} + i\right)}{\Gamma\left(\frac{3}{2} + i\right)} (-1)^i (1 - r^2)^i r^{2(k-1-i)} \right.
$$
$$
\left. + \frac{1}{\Gamma\left(\frac{1}{2}\right)} \sum_{i=0}^{k-1} \frac{\Gamma\left(\frac{j+3}{2} + i\right)}{\Gamma\left(\frac{j+4}{2} + i\right)} r^{2(k-1-i)}\right] - \frac{\Gamma\left(k + \frac{1}{2}\right) \Gamma\left(\frac{j+2}{2}\right) (-1)^{k+1}}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(k + \frac{j+2}{2}\right)} \int_0^{\sqrt{1-r^2}} (z^2 + r^2)^{\frac{j}{2}+k} dz. \tag{6.47}
$$

Focusing upon the integral in the above expression and integrating by parts k-times, in the same fashion as before, we have that

$$
\int_0^{\sqrt{1-r^2}} (z^2 + r^2)^{\frac{j}{2}+k} dz = \frac{\sqrt{1 - r^2}}{2} \frac{\Gamma\left(\frac{j+2}{2} + k\right)}{\Gamma\left(\frac{j+3}{2} + k\right)} \sum_{i=0}^{k-1} \frac{\Gamma\left(\frac{j+1}{2} + k - i\right)}{\Gamma\left(\frac{j+2}{2} + k - i\right)} r^{2i}
$$
$$
+ \frac{\Gamma\left(\frac{j+3}{2}\right) \Gamma\left(\frac{j+2}{2} + k\right)}{\Gamma\left(\frac{j+2}{2}\right) \Gamma\left(\frac{j+3}{2} + k\right)} r^{2k} \int_0^{\sqrt{1-r^2}} (z^2 + r^2)^{\frac{j}{2}} dz.
$$

Re-substituting this into (6.47) the integral leads to

$$
\begin{aligned}
&\int_0^{\sqrt{1-r^2}} (z^2+r^2)^{\frac{j}{2}} z^{2k} dz \\
&= \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{j+2}{2}\right)(-1)^{k+1}\sqrt{1-r^2}}{2\Gamma\left(k+\frac{j+3}{2}\right)}\Bigg[\frac{1}{\Gamma\left(\frac{j+2}{2}\right)}\sum_{i=0}^{k-1}\frac{\Gamma\left(\frac{j+3}{2}+i\right)}{\Gamma\left(\frac{3}{2}+i\right)}(-1)^i(1-r^2)^i r^{2(k-1-i)} \\
&+ \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{i=0}^{k-1}\frac{\Gamma\left(\frac{j+3}{2}+i\right)}{\Gamma\left(\frac{j+4}{2}+i\right)} r^{2(k-1-i)} - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{i=0}^{k-1}\frac{\Gamma\left(\frac{j+1}{2}+k-i\right)}{\Gamma\left(\frac{j+2}{2}+k-i\right)} r^{2i}\Bigg] \\
&+ \frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{j+3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(k+\frac{j+3}{2}\right)}\int_0^{\sqrt{1-r^2}} (z^2+r^2)^{\frac{j}{2}} dz \\
&= \frac{\Gamma\left(k+\frac{1}{2}\right)(-1)^{k+1}\sqrt{1-r^2}}{2\Gamma\left(k+\frac{j+3}{2}\right)}\sum_{i=0}^{k-1}\frac{\Gamma\left(\frac{j+3}{2}+i\right)}{\Gamma\left(\frac{3}{2}+i\right)}(-1)^i(1-r^2)^i r^{2(k-1-i)} \\
&+ \frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{j+3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(k+\frac{j+3}{2}\right)} r^{2k}\int_0^{\sqrt{1-r^2}} (z^2+r^2)^{\frac{j}{2}} dz.
\end{aligned}
\tag{6.48}
$$

The final line follows from its predecessor, where we note that the sums involving even powers of r are identical and hence cancel out. The following identity is a particular instance of formula 1.2.2.1 of [14]

$$I(\beta) = \int_0^{\sqrt{1-r^2}} (z^2+r^2)^{\beta} dz = \frac{\sqrt{1-r^2}}{2\left(\beta+\frac{1}{2}\right)} + \frac{\beta r^2}{\left(\beta+\frac{1}{2}\right)} I(\beta-1).$$

Applying this recursively to the integrals above, together with the fact that

$$I(0) = \sqrt{1-r^2} \quad \text{and} \quad I\left(\frac{1}{2}\right) = \frac{1}{2}\left(\sqrt{1-r^2} - r^2\ln\left(\frac{r}{1+\sqrt{1-r^2}}\right)\right)$$

we can deduce that

$$
\begin{aligned}
\int_0^{\sqrt{1-r^2}} (z^2+r^2)^{\frac{j}{2}} dz = \frac{\sqrt{1-r^2}}{2}\frac{\Gamma\left(\frac{j+2}{2}\right)}{\Gamma\left(\frac{j+3}{2}\right)}\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor}\frac{\Gamma\left(\frac{j+1}{2}-i\right)}{\Gamma\left(\frac{j+2}{2}-i\right)} r^{2i} \\
- \Delta_{j,2}\frac{\Gamma\left(\frac{j+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{j+3}{2}\right)} r^{j+1}\ln\left(\frac{r}{1+\sqrt{1-r^2}}\right),
\end{aligned}
$$

where

$$\Delta_{j,2} = \begin{cases} 0 & \text{if } j \text{ is even;} \\ 1 & \text{if } j \text{ is odd.} \end{cases}$$

This allows us to conclude that

$$
\begin{aligned}
&\int_0^{\sqrt{1-r^2}} (z^2+r^2)^{\frac{j}{2}} z^{2k} dz \\
&\quad = \frac{\Gamma\left(k+\frac{1}{2}\right)(-1)^{k+1}}{2\Gamma\left(k+\frac{j+3}{2}\right)}\Bigg[\sum_{i=0}^{k-1}\frac{\Gamma\left(\frac{j+3}{2}+i\right)}{\Gamma\left(\frac{3}{2}+i\right)}(-1)^i(1-r^2)^i r^{2(k-1-i)} \\
&\quad - \frac{\Gamma\left(\frac{j+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\sum_{i=0}^{\lfloor\frac{j}{2}\rfloor}\frac{\Gamma\left(\frac{j+1}{2}-i\right)}{\Gamma\left(\frac{j+2}{2}-i\right)}r^{2(k+i)}\Bigg]\sqrt{1-r^2} \\
&\quad + \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{j+2}{2}\right)(-1)^{k+1}}{\Gamma\left(\frac{1}{2}\right)^2\Gamma\left(k+\frac{j+3}{2}\right)}\Delta_{j,2}r^{2k+j+1}\ln\left(\frac{r}{1+\sqrt{1-r^2}}\right).
\end{aligned}
$$

Applying the binomial theorem to the first sum and re-indexing in the above expression yields

$$
\begin{aligned}
&\sum_{i=0}^{k-1}\frac{\Gamma\left(\frac{j+3}{2}+i\right)}{\Gamma\left(\frac{3}{2}+i\right)}(-1)^i(1-r^2)^i r^{2(k-1-i)} \\
&= r^{2(k-1)}\sum_{i=0}^{k-1}\frac{\Gamma\left(\frac{j+3}{2}+i\right)}{\Gamma\left(\frac{3}{2}+i\right)}\left(1-\frac{1}{r^2}\right)^i \\
&= r^{2(k-1)}\sum_{i=0}^{k-1}\left(\sum_{p=i}^{k-1}\frac{\Gamma\left(p+\frac{j+3}{2}\right)}{\Gamma\left(p+\frac{3}{2}\right)}\binom{p}{i}\right)\frac{(-1)^i}{r^{2i}} \\
&= (-1)^{k-1}r^{2(k-1)}\sum_{i=0}^{k-1}\left(\sum_{p=k-1-i}^{k-1}\frac{\Gamma\left(p+\frac{j+3}{2}\right)}{\Gamma\left(p+\frac{3}{2}\right)}\binom{p}{k-1-i}\right)\frac{(-1)^i}{r^{2i}} \\
&= (-1)^{k-1}\sum_{i=0}^{k-1}\left(\sum_{p=0}^{i}\frac{\Gamma\left(k-1-p+\frac{j+3}{2}\right)}{\Gamma\left(k-1-p+\frac{3}{2}\right)}\binom{k-1-p}{i-p}\right)(-1)^i r^{2i}.
\end{aligned}
$$

In an effort to improve notation, we can observe that the inner sum of the above expression can be written as a multiple of a certain unit argument ${}_3F_2$ hypergeometric function. The following development verifies this:

$$
{}_3F_2\left[\begin{matrix}1,\ -i,\ \frac{1}{2}-k \\ 1-k,\ -k-\frac{j}{2}+\frac{1}{2}\end{matrix};1\right] = \sum_{p=0}^{\infty}\frac{(1)_p(-i)_p\left(\frac{1}{2}-k\right)_p}{(1-k)_p\left(-k-\frac{j}{2}+\frac{1}{2}\right)p!}
$$

$$
\begin{aligned}
&= \sum_{p=0}^{\infty} \frac{(k)_{-p}\left(k-1+\frac{j+3}{2}\right)_{-p}}{(i+1)_{-p}\left(k+\frac{1}{2}\right)_{-p}} \\
&= \sum_{p=0}^{i} \frac{(k-1-p)!}{(k-1)!} \frac{\Gamma\left(k-1-p+\frac{j+3}{2}\right)}{\Gamma\left(k+\frac{j+1}{2}\right)} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k-p+\frac{1}{2}\right)} \frac{i!}{(i-p)!} \\
&= \frac{\Gamma\left(k+\frac{1}{2}\right)}{\binom{k-1}{i}\Gamma\left(k+\frac{j+1}{2}\right)} \sum_{p=0}^{i} \frac{\Gamma\left(k-1-p+\frac{j+3}{2}\right)}{\Gamma\left(k-1-p+\frac{3}{2}\right)} \binom{k-1-p}{i-p},
\end{aligned}
$$

where the second line uses $(1)_p = p!$ and also relies on the application of the Pochhammer identity $(-a)_p = \frac{(-1)^p}{(1+a)_{-p}}$, the third line terminates the series under the assumption that $i \leq k-1$. This observation then allows us to write

$$
\begin{aligned}
&\int_0^{\sqrt{1-r^2}} (z^2+r^2)^{\frac{j}{2}} z^{2k} dz \\
&\quad = \frac{1}{2}\Bigg[\frac{1}{\left(k+\frac{j+1}{2}\right)} \sum_{i=0}^{k-1} \binom{k-1}{i} {}_3F_2\left[\begin{matrix} 1,\ -i,\ \frac{1}{2}-k \\ 1-k,\ -k-\frac{j}{2}+\frac{1}{2} \end{matrix};1\right] (-1)^i r^{2i} \\
&\quad + (-1)^k \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{j+2}{2}\right)}{\Gamma\left(k+\frac{j+3}{2}\right)\Gamma\left(\frac{1}{2}\right)} \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \frac{\Gamma\left(\frac{j+1}{2}-i\right)}{\Gamma\left(\frac{j+2}{2}-i\right)} r^{2(k+i)} \Bigg] \sqrt{1-r^2} \\
&\quad + \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{j+2}{2}\right)(-1)^{k+1}}{\Gamma\left(\frac{1}{2}\right)^2\Gamma\left(k+\frac{j+3}{2}\right)} \Delta_{j,2} r^{2k+j+1} \ln\left(\frac{r}{1+\sqrt{1-r^2}}\right).
\end{aligned}
$$

Returning to (6.42), we can use the above to conclude that

$$
\begin{aligned}
\psi_{d,k+\frac{1}{2},\ell}(r) &= \frac{1}{\Gamma\left(k+\frac{1}{2}\right)2^{k-\frac{1}{2}}} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \int_0^{\sqrt{1-r^2}} (z^2+r^2)^{\frac{j}{2}} z^{2k} dz \\
&= \frac{1}{2^{k+\frac{1}{2}}} \Bigg[\sum_{i=0}^{k-1} \frac{(-1)^i\binom{k-1}{i}}{\Gamma\left(k+\frac{1}{2}\right)} \sum_{j=0}^{\ell} \frac{(-1)^j\binom{\ell}{j}}{\left(k+\frac{j+1}{2}\right)} {}_3F_2\left[\begin{matrix} 1,\ -i,\ \frac{1}{2}-k \\ 1-k,\ -k-\frac{j}{2}+\frac{1}{2} \end{matrix};1\right] r^{2i} \\
&+ \frac{(-1)^k r^{2k}}{\sqrt{\pi}} \sum_{j=0}^{\ell} \frac{(-1)^j\binom{\ell}{j}\Gamma\left(\frac{j+2}{2}\right)}{\Gamma\left(k+\frac{j+3}{2}\right)} \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \frac{\Gamma\left(\frac{j+1}{2}-i\right)}{\Gamma\left(\frac{j+2}{2}-i\right)} r^{2i} \Bigg] \sqrt{1-r^2} \\
&+ \frac{(-1)^k \ell!}{2^{k+\frac{1}{2}}\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \frac{r^{2(k+j+1)}}{2^{2j} j!(\ell-2j-1)!(k+j+1)!} \log\left(\frac{r}{1+\sqrt{1-r^2}}\right).
\end{aligned}
$$

By considering the odd and even contributions to the mixed sum above and rearranging its terms, one can show

$$\psi_{d,k+\frac{1}{2},\ell}(r) = \frac{1}{\Gamma\left(k+\frac{1}{2}\right)2^{k-\frac{1}{2}}}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\int_0^{\sqrt{1-r^2}}(z^2+r^2)^{\frac{j}{2}}z^{2k}dz$$
$$= \frac{1}{2^{k+\frac{1}{2}}}\left[\sum_{i=0}^{k-1}a_i(k)r^{2i} + \sum_{j=0}^{\lfloor\frac{\ell}{2}\rfloor}b_j(k)r^{2(k+j)}\right]\sqrt{1-r^2}$$
$$+ \frac{1}{2^{k+\frac{1}{2}}}\left[\sum_{j=0}^{\lfloor\frac{\ell-1}{2}\rfloor}c_j(k)r^{2(k+j+1)}\right]\log\left(\frac{r}{1+\sqrt{1-r^2}}\right)$$

where

$$a_i(k) = \frac{(-1)^i\binom{k-1}{i}}{\Gamma\left(k+\frac{1}{2}\right)}\sum_{j=0}^{\ell}\frac{(-1)^j\binom{\ell}{j}}{\left(k+\frac{j+1}{2}\right)}{}_3F_2\left[\begin{matrix}1,\ -i,\ \frac{1}{2}-k\\ 1-k,\ -k-\frac{j}{2}+\frac{1}{2}\end{matrix};1\right],$$
$$b_j(k) = \frac{(-1)^k}{\sqrt{\pi}}\sum_{i=0}^{\ell-2j}(-1)^i\binom{\ell}{i+2j}\frac{\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(j+\frac{i+2}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)\Gamma\left(k+j+\frac{i+3}{2}\right)},$$
$$c_j(k) = \frac{\ell!(-1)^k}{\sqrt{\pi}2^{2j}j!(\ell-2j-1)!(k+j+1)!},$$

as required.

The missing Wu functions

Proof of Theorem 4.6

Starting from the connection function of the Wu and Wendland functions (4.33), we find for $\ell, k \in \mathbb{N}$:

$$\mathcal{W}_{d,k+\frac{1}{2},\ell}(r) = 2^{\ell+k+\frac{3}{2}}\Gamma(\ell+1)\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}\psi_{k+\frac{1}{2},\ell+n+1}(r). \qquad (6.49)$$

The representation of $\mathcal{W}_{d,k+\frac{1}{2},\ell}(r)$ as a linear combination of missing Wendland functions implies that

$$\mathcal{W}_{d,k+\frac{1}{2},\ell}(r) = \tilde{p}_{k,\ell}(r)\sqrt{1-r^2} + \tilde{q}_{k,\ell}(r)\log\left(\frac{r}{1+\sqrt{1-r^2}}\right),$$

with polynomials $\tilde{p}_{k,\ell}$ and $\tilde{q}_{k,\ell}$. Our aim will be to deduce a closed form for these polynomials. We first review the representation of the appearing missing Wendland functions:

$$\psi_{k+\frac{1}{2},\ell+n+1}(r) = \frac{1}{2^{k+\frac{1}{2}}}\left[\sum_{i=0}^{k-1} a_i(k,\ell,n) r^{2i} + \sum_{j=0}^{\lfloor\frac{\ell+n+1}{2}\rfloor} b_j(k,\ell,n) r^{2(k+j)}\right]\sqrt{1-r^2} + \frac{1}{2^{k+\frac{1}{2}}}\left[\sum_{j=0}^{\lfloor\frac{\ell+n}{2}\rfloor} c_j(k,\ell,n) r^{2(k+j+1)}\right]\log\left(\frac{r}{1+\sqrt{1-r^2}}\right) \tag{6.50}$$

where

$$\begin{aligned} a_i(k,\ell,n) &= \frac{(-1)^i\binom{k-1}{i}}{\Gamma\left(k+\frac{1}{2}\right)}\sum_{j=0}^{\ell+n+1}\frac{(-1)^j\binom{\ell+n+1}{j}}{\left(k+\frac{j+1}{2}\right)}\,{}_3F_2\left[\begin{matrix}1,\ -i,\ \frac{1}{2}-k\\ 1-k,\ -k-\frac{j}{2}+\frac{1}{2}\end{matrix};1\right],\\ b_j(k,\ell,n) &= \frac{(-1)^k}{\sqrt{\pi}}\sum_{i=0}^{\ell+n+1-2j}(-1)^i\binom{\ell+n+1}{i+2j}\frac{\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(j+\frac{i+2}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)\Gamma\left(k+j+\frac{i+3}{2}\right)},\\ c_j(k,\ell,n) &= \frac{(\ell+n+1)!(-1)^k}{\sqrt{\pi}2^{2j}j!(\ell+n-2j)!(k+j+1)!}. \end{aligned} \tag{6.51}$$

We now keep the same structure for the missing Wu functions:

$$\begin{aligned} \mathcal{W}_{d,k+\frac{1}{2},\ell}(r) =& 2^{\ell+1}\ell!\Bigg\{\Bigg[\sum_{i=0}^{k-1}\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}a_i(k,\ell,n)r^{2i}\\ &+\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}\sum_{j=0}^{\lfloor\frac{\ell+n+1}{2}\rfloor} b_j(k,\ell,n)r^{2(k+j)}\Bigg]\sqrt{1-r^2}\\ &+\Bigg[\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}\sum_{j=0}^{\lfloor\frac{\ell+n}{2}\rfloor} c_j(k,\ell,n)r^{2(k+j+1)}\Bigg]L(r)\Bigg\}\\ =& 2^{\ell+1}\ell!\Bigg\{\Bigg[\sum_{i=0}^{k-1}\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}a_i(k,\ell,n)r^{2i}\\ &+\sum_{j=0}^{\ell}\sum_{n=\max\{0,2j-\ell-1\}}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}b_j(k,\ell,n)r^{2(k+j)}\Bigg]\sqrt{1-r^2}\\ &+\Bigg[\sum_{j=0}^{\ell}\sum_{n=\max\{0,2j-\ell\}}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}c_j(k,\ell,n)r^{2(k+j+1)}\Bigg]L(r)\Bigg\}, \end{aligned} \tag{6.52}$$

where $L(r) := \log\left(\frac{r}{1+\sqrt{1-r^2}}\right)$.

Which we can simplify to give

$$\begin{aligned}\mathcal{W}_{d,k+\frac{1}{2},\ell}(r) &= 2^{\ell+1}\ell!\left[\sum_{i=0}^{k-1}\tilde{a}_i(k,\ell)r^{2i} + \sum_{j=0}^{\ell}\tilde{b}_j(k,\ell)r^{2(k+j)}\right]\sqrt{1-r^2} \\ &\quad + 2^{\ell+1}\ell!\left[\sum_{j=0}^{\ell}\tilde{c}_j(k,\ell)r^{2(k+j+1)}\right]\log\left(\frac{r}{1+\sqrt{1-r^2}}\right)\end{aligned} \tag{6.53}$$

where

$$\begin{aligned}\tilde{a}_i(k,\ell) = &\sum_{n=0}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}}{\ell+n+1}(-1)^n\frac{(-1)^i\binom{k-1}{i}}{\Gamma\left(k+\frac{1}{2}\right)}\sum_{j=0}^{\ell+n+1}\frac{(-1)^j\binom{\ell+n+1}{j}}{\left(k+\frac{j+1}{2}\right)} \\ &\times {}_3F_2\left[\begin{matrix}1,\ -i,\ \frac{1}{2}-k \\ 1-k,\ -k-\frac{j}{2}+\frac{1}{2}\end{matrix};1\right] \\ = &\frac{(-1)^i\binom{k-1}{i}}{\Gamma\left(k+\frac{1}{2}\right)}\sum_{j=0}^{2\ell+1}\frac{(-1)^j}{\left(k+\frac{j+1}{2}\right)}{}_3F_2\left[\begin{matrix}1,\ -i,\ \frac{1}{2}-k \\ 1-k,\ -k-\frac{j}{2}+\frac{1}{2}\end{matrix};1\right] \\ &\times\sum_{n=\max\{0,j-\ell-1\}}^{\ell}\binom{\ell}{n}\frac{(-1)^n2^{\ell-n}}{\ell+n+1}\binom{\ell+n+1}{j},\end{aligned} \tag{6.54}$$

$$\begin{aligned}\tilde{b}_j(k,\ell) = &\sum_{n=\max\{0,2j-\ell-1\}}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}}{\ell+n+1}(-1)^n\frac{(-1)^k}{\sqrt{\pi}} \\ &\times\sum_{i=0}^{\ell+n+1-2j}(-1)^i\binom{\ell+n+1}{i+2j}\frac{\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(j+\frac{i+2}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)\Gamma\left(k+j+\frac{i+3}{2}\right)} \\ = &\frac{(-1)^k}{\sqrt{\pi}}\sum_{i=0}^{2\ell+1-2j}(-1)^i\frac{\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(j+\frac{i+2}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)\Gamma\left(k+j+\frac{i+3}{2}\right)} \\ &\times\sum_{n=\max\{0,2j-\ell-1+i\}}^{\ell}\binom{\ell}{n}\frac{2^{\ell-n}(-1)^n}{\ell+n+1}\binom{\ell+n+1}{i+2j},\end{aligned} \tag{6.55}$$

$$\tilde{c}_j(k,\ell) = \frac{(-1)^k2^{\ell-2j}}{\sqrt{\pi}\,j!(k+j+1)!}\sum_{n=\max\{0,2j-\ell\}}^{\ell}\left(-\frac{1}{2}\right)^n\frac{\ell!(\ell+n)!}{(\ell-n)!n!(\ell+n-2j)!}. \tag{6.56}$$

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Declarations

Conflict of interest The authors declare no competing interests.

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