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On interpolation free disks of polynomials converging maximally to power series

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Dedicated to Wiesław Pleśniak
on the occasion of his 80th birthday

Abstract. We construct a power series f with radius R of convergence, $0 < R < \infty$, such that for any σ, $0 < \sigma < R$, there exists a subset $\Lambda \subset \mathbb{N}$, a parameter r_σ, $0 < r_\sigma < \sigma$, and a sequence $\{p_n\}_{n\in\mathbb{N}}$ of polynomials converging maximally to f on the disk

$$\overline{D}_{r_\sigma} = \{z \in \mathbb{C} : |z| \le r_\sigma\}$$

such that p_n has no points of interpolation to f on $\overline{D}_\sigma$ for $n \in \Lambda$.

1. Introduction. For $B \subset \mathbb{C}$, we denote by $\overline{B}$ its closure and by ∂B the boundary of B and we write $\|\cdot\|_B$ for the supremum norm on B. Let $\mathcal{A}(B)$ be the class of functions that are holomorphic in some neighborhood of B.

Let E be compact and connected in the complex plane $\mathbb{C}$ with connected complement $\Omega = \overline{\mathbb{C}} \setminus E$ and positive logarithmic capacity $\operatorname{cap} E$ and let $g_\Omega(z,\infty)$ denote the Green function of Ω with pole at ∞. For $\sigma > 1$, let

$$E_\sigma := \{z \in \Omega : g_\Omega(z,\infty) < \log\sigma\} \cup E$$

denote the *Green domains* with boundaries Γ_σ.

Let $\mathcal{P}_n$ denote the collection of algebraic polynomials of degree at most n. If $f \in \mathcal{A}(E)$, then there exists $\rho > 1$ and polynomials $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} \le \frac{1}{\rho},$$

due to a result of Walsh [6]. If f is holomorphic on E, but not an entire function, then there exists a maximal parameter $\rho(f) > 1$, $1 < \rho(f) < \infty$,

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such that f is holomorphic in E_ρ and there exist polynomials $p_n \in \mathcal{P}_n$ such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} = \frac{1}{\rho(f)}.$$

Such a sequence $p_n \in \mathcal{P}_n$ is said to *converge maximally* to f on E (see [6, Sect. 4.7, Theorem 7]).

Let $\{p_n\}_{n\in\mathbb{N}}$, $p_n \in \mathcal{P}_n$, be a sequence of polynomials converging maximally to f on E. We consider for $1 < \sigma < \rho(f)$ the point set

$$Z_n(\sigma) := \{z \in E_\sigma : p_n(z) = f(z)\},$$

i.e., $Z_n(\sigma)$ is the set of points of interpolation of p_n to f on E_σ, each point listed according to its multiplicity as a zero of $f - p_n$, and we denote by $m_n(\sigma)$ the number of points of $Z_n(\sigma)$. Since f is holomorphic in $E_{\rho(f)}$, the number $m_n(\sigma)$ is finite.

In the investigation of the asymptotic behavior of $Z_n(\sigma)$, *near-circularity in capacity* is the essential property, especially well-known for Carathéodory–Fejér approximations of power series (cf. Trefethen [5]).

Definition ([2, 3]). Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials converging maximally to f on E, let $\Lambda \subset \mathbb{N}$ be an infinite subset and let $1 < \sigma < \rho(f) < \infty$. Then the error functions $f - p_n$, $n \in \Lambda$, are called *near-circular at* σ_- if there exist two sequences

$$\{\sigma_n\}_{n\in\Lambda}, \quad 1 < \sigma_n \le \sigma < \rho(f), \lim_{n\in\Lambda,\, n\to\infty} \sigma_n = \sigma,$$

$$\{\varepsilon_n\}_{n\in\Lambda}, \quad \varepsilon_n > 0, \lim_{n\in\Lambda,\, n\to\infty} \varepsilon_n = 0,$$

such that

$$\frac{\sigma_n}{\rho(f)} e^{-\varepsilon_n} < \min_{z\in\Gamma_{\sigma_n}} |f(z) - p_n(z)|^{1/n} \le \|f - p_n\|_{\Gamma_{\sigma_n}}^{1/n} < \frac{\sigma_n}{\rho(f)} e^{\varepsilon_n}, \qquad n \in \Lambda.$$

Proposition (cf. [3, Lemma 4.1, Corollary 3.2]). *Let* $\{p_n\}_{n\in\mathbb{N}}$ *converge maximally to* f *on* E *and let* $1 < \sigma < \rho(f)$*. Then there exists* $\Lambda \subset \mathbb{N}$ *such that* $f - p_n$, $n \in \Lambda$, *are near-circular at* σ_- *with associated sequences* $\{\sigma_n\}_{n\in\Lambda}$ *and* $\{\varepsilon_n\}_{n\in\Lambda}$ *such that*

$$\limsup_{n\in\Lambda,\, n\to\infty} \frac{n}{m_n(\sigma)} \le \limsup_{n\in\Lambda,\, n\to\infty} \frac{n}{m_n(\sigma_n)} \le 1.$$

In other words, the Proposition implies that for any σ, $1 < \sigma < \rho(f)$, there exists $\Lambda \subset \mathbb{N}$ such that there are at least

$$n + o(n) \quad (n \in \Lambda,\, n \to \infty)$$

points of interpolation of p_n to f on E_σ.

The situation of the Proposition is well-understood for the special case of a rational function $f = P/Q$, holomorphic on E: Let P and Q have no common divisors and let k_n denote the number of zeros of $f - p_n$ in $\mathbb{C}$. Then

$$k_n = \max(\deg(P), \deg(p_n) + \deg(Q))$$

and

$$\lim_{n\to\infty} \frac{n}{k_n} = 1$$

(see [1, Lemma 2]). Moreover, if ν_n denotes the normalized zero-counting measure of the zeros of $f - p_n$ in $\mathbb{C}$, then

$$\widehat{\nu_n|_E} + \nu_n|_\Omega \xrightarrow[n\to\infty]{*} \mu_E,$$

where $\nu_n = \nu_n|_E + \nu_n|_\Omega$ and $\nu_n|_E$, resp. $\nu_n|_\Omega$, is the restriction of ν_n to E, resp. Ω, and $\widehat{\nu_n|_E}$ is the balayage of $\nu_n|_E$ onto the boundary ∂E of E (see [1, Theorem 1]).

Hence,

$$\lim_{n\to\infty} \nu_n(K) = 0$$

for any compact set K in $\overline{\mathbb{C}} \setminus E$, and

$$\lim_{n\to\infty} \frac{n}{m_n(\sigma)} = 1$$

for any σ, $1 < \sigma < \infty$.

The objective of the paper is to show that the Proposition is sharp, namely we show that

$$\limsup_{n\in\mathbb{N},\, n\to\infty} \frac{n}{m_n(\sigma)} \le 1$$

cannot be universally true, even for the approximation of power series by maximally converging polynomials.

Finally, let us mention that we will use in our main result a strengthened version of the Definition: $f - p_n$, $n \in \Lambda$, are called *near-circular at* σ if

$$\sigma_n = \sigma, \qquad n \in \Lambda,$$

in the Definition.

2. The crucial power series. Let

$$D_r := \{z : |z| < r\}, \qquad r > 0,$$

denote the open disc, and let $\overline{D}_r$ be its closure, with boundary Γ_r.

Let f be a power series with finite radius of convergence R. If s_n denotes the nth partial sum of f and if $m(s_n; r)$ denotes the number of zeros of $f - s_n$ in D_r, then $\{s_n\}_{n\in\mathbb{N}}$ converges maximally to f on every closed disk $\overline{D}_r$, $0 < r < R$, and

$$\limsup_{n\in\mathbb{N},\, n\to\infty} \frac{n}{m_n(s_n; r)} \le 1.$$

More specifically, let

$$\Lambda_0 := \{2^\nu\}_{\nu\in\mathbb{N}}, \tag{2.1}$$

and let R, $1 < R < \infty$, be fixed. Then we define

$$a_k = \begin{cases} (1/R)^k & \text{if } k \in \Lambda_0, \\ 0 & \text{if } k \notin \Lambda_0, \end{cases} \tag{2.2}$$

and so

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{2.3}$$

is a power series with radius of convergence R. As above, let s_n denote the nth partial sum of f,

$$s_n(z) = \sum_{k=0}^{n} a_k z^k.$$

Then for $\nu \in \mathbb{N}$,

$$s_{2^\nu} = s_{2^\nu+1} = \cdots = s_{2^{\nu+1}-1}.$$

Define

$$\Lambda_1 := \{2^{\nu+1} - 1\}_{\nu\in\mathbb{N}}, \tag{2.4}$$

and consider

$$2^\nu + \frac{2^{\nu+1} - 2^\nu}{2} = 2^\nu + 2^{\nu-1} = 3 \cdot 2^{\nu-1}.$$

Then the subset

$$\Lambda := \{2^\nu + 2^{\nu-1}\}_{\nu\in\mathbb{N}} = \{3 \cdot 2^{\nu-1}\}_{\nu\in\mathbb{N}} \tag{2.5}$$

satisfies

$$2^\nu < 2^\nu + 2^{\nu-1} = 3 \cdot 2^{\nu-1} < 2^{\nu+1}.$$

THEOREM. *Let f be the power series with radius of convergence R, defined by (2.1)–(2.3), with partial sums s_n. If Λ_1 is defined by (2.4), then for any ρ, $0 < \rho < R$, the error functions $f - s_n$, $n \in \Lambda_1$, are near-circular at ρ and*

$$\lim_{n\in\Lambda_1,\, n\to\infty} \frac{n}{m(s_n;\rho)} = 1. \tag{2.6}$$

Let $0 < \sigma < \rho < R$ be fixed and let

$$r_\sigma = \sigma\left(\frac{\rho}{R}\right)^{1/28}.$$

Then there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$, $p_n \in \mathcal{P}_n$, converging maximally to f on $\overline{D}_{r_\sigma}$ such that $f - p_n$, $n \in \Lambda$, have no zeros on $\overline{D}_\sigma$, where Λ is defined by (2.5).

The following Corollary defines a lower bound for parameters r, $0<r<\sigma$, such that $\{p_n\}_{n\in\mathbb{N}}$ converges maximally to f on $\overline{D}_r$.

Corollary. *Let f, Λ, and $\{p_n\}_{n\in\mathbb{N}}$ be as in the Theorem. Then there exists a minimal parameter r_σ^*,*

$$0<\left(\frac{\sigma^4}{R}\right)^{1/3}\le r_\sigma^*\le\sigma,$$

such that $\{p_n\}_{n\in\mathbb{N}}$ converges maximally to f on $\overline{D}_{r_\sigma^*}$.

The Corollary is reminiscent of results of Saff and Totik [4] on the behavior of polynomials p_n of best uniform approximation of functions f on a compact set E, where the error $f-p_n$ does not decrease faster at interior points of E than on E itself.

3. Proofs

3.1. Proof of the Theorem. For $z\in D_R$ we obtain

$$\begin{aligned}(f-s_{2^\nu})(z)&=(f-s_{2^\nu+1})(z)=\cdots=(f-s_{2^{\nu+1}-1})(z)\\&=\left(\frac{z}{R}\right)^{2^{\nu+1}}\left(1+\left(\frac{z}{R}\right)^{2^{\nu+1}}+\left(\frac{z}{R}\right)^{2^{\nu+1}\cdot 3}+\left(\frac{z}{R}\right)^{2^{\nu+1}\cdot 7}+\cdots\right).\end{aligned}$$

Hence, for $z\in D_R$,

$$\begin{aligned}|(f-s_{2^\nu})(z)|&=|(f-s_{2^\nu+1})(z)|=\cdots=|(f-s_{2^{\nu+1}-1})(z)|\\&=\left|\frac{z}{R}\right|^{2^{\nu+1}}\left|1+\left(\frac{z}{R}\right)^{2^{\nu+1}}+\left(\frac{z}{R}\right)^{2^{\nu+1}\cdot 3}\left(\frac{z}{R}\right)^{2^{\nu+1}\cdot 7}+\cdots\right|\\&\le\left|\frac{z}{R}\right|^{2^{\nu+1}}\left(1+\left|\frac{z}{R}\right|^{2^{\nu+1}}\frac{R^{2^{\nu+1}}}{R^{2^{\nu+1}}-|z^{2^{\nu+1}}|}\right)\\&=\left|\frac{z}{R}\right|^{2^{\nu+1}}(1+B(z,\nu)),\end{aligned}$$

where

$$B(z,\nu):=\left|\frac{z}{R}\right|^{2^{\nu+1}}\frac{R^{2^{\nu+1}}}{R^{2^{\nu+1}}-|z^{2^{\nu+1}}|}\xrightarrow{\nu\to\infty}0.$$

We define, for $0<\rho<R$,

$$C(\rho,\nu):=\max_{z\in\Gamma_\rho}B(z,\nu)=\left(\frac{\rho}{R}\right)^{2^{\nu+1}}\frac{R^{2^{\nu+1}}}{R^{2^{\nu+1}}-\rho^{2^{\nu+1}}}.$$

Then

$$C(\rho,\nu)\xrightarrow{\nu\to\infty}0$$

and

$$\|f - s_{2^{\nu+1}-1}\|_{\Gamma_\rho} \le \left(\frac{\rho}{R}\right)^{2^{\nu+1}} (1 + C(\rho,\nu)). \tag{3.1}$$

On the other hand,

$$\begin{aligned}
|(f - s_{2^\nu})(z)| &= |(f - s_{2^\nu+1})(z)| = \cdots = |(f - s_{2^{\nu+1}-1})(z)| \\
&= \left|\frac{z}{R}\right|^{2^{\nu+1}} \left|1 + \left(\frac{z}{R}\right)^{2^{\nu+1}} + \left(\frac{z}{R}\right)^{2^{\nu+1}\cdot 2} + \cdots\right| \\
&\ge \left|\frac{z}{R}\right|^{2^{\nu+1}} \left(1 - \left|\frac{z}{R}\right|^{2^{\nu+1}} - \left|\frac{z}{R}\right|^{2^{\nu+1}\cdot 2} - \cdots\right) \\
&\ge \left|\frac{z}{R}\right|^{2^{\nu+1}} \left(1 - \left|\frac{z}{R}\right|^{2^{\nu+1}} \frac{R^{2^{\nu+1}}}{R^{2^{\nu+1}} - |z^{2^{\nu+1}}|}\right) \\
&= \left|\frac{z}{R}\right|^{2^{\nu+1}} (1 - B(z,\nu)).
\end{aligned}$$

Then

$$\min_{z \in \Gamma_\rho} |(f - s_{2^\nu})(z)| \ge \left(\frac{\rho}{R}\right)^{2^{\nu+1}} (1 - C(\rho,\nu)). \tag{3.2}$$

For $n = 2^{\nu+1} - 1 \in \Lambda_1$, let

$$\varepsilon_n(\rho) := \frac{2^{\nu+1}}{2^{\nu+1}-1}\left(\log(1 + C(\rho,\nu)) + \log\frac{1}{1 - C(\rho,\nu)}\right), \qquad \nu \in \mathbb{N}. \tag{3.3}$$

Then (3.1) and (3.2) yield, for $n \in \Lambda_1$,

$$\frac{\rho}{R} e^{-\varepsilon_n(\rho)} < \min_{z \in \Gamma_\rho} |f(z) - s_n(z)|^{1/n} \le \|f - s_n\|_{\Gamma_\rho}^{1/n} < \frac{\rho}{R} e^{\varepsilon_n(\rho)}. \tag{3.4}$$

Hence, the functions $f - s_n$, $n \in \Lambda_1$, are near-circular at ρ. Due to Rouché's Theorem, $m(s_n;\rho) = 2^{\nu+1}$ for $n \in \Lambda_1$ and (2.6) holds.

Concerning the second part of the Theorem, let us fix the parameters σ and ρ such that

$$0 < \sigma < \rho < R.$$

For $n \in \mathbb{N}$, let

$$z_0^{(n)}, z_1^{(n)}, \ldots, z_n^{(n)}$$

denote the $(n+1)$th roots of unity, and define

$$\xi_i^{(n)} := \sigma z_i^{(n)}, \qquad 0 \le i \le n.$$

We construct for $n = 3 \cdot 2^{\nu-1} \in \Lambda$ the polynomial

$$p_{0,n}(z) = z^{2^\nu} q_{2^\nu - 1}(z) \in \mathcal{P}_n,$$

by interpolating the function

$$f - s_{2^\nu} = f - s_{2^\nu+1} = \cdots = f - s_{2^{\nu+1}-1}$$

at the points

$$\xi_i^{(2^{\nu-1})}, \qquad 0 \le i \le 2^{\nu-1},$$

and at 0 with multiplicity 2^ν.

Then the error formula for Lagrange–Hermite interpolation of $f - s_n$ by $p_{0,n}$, $n = 3 \cdot 2^{\nu-1} \in \Lambda$, at $z \in D_\rho$ yields

$$f(z) - s_n(z) - p_{0,n}(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{z^{2^\nu}}{t^{2^\nu}} \frac{w_{2^{\nu-1}}(z)}{w_{2^{\nu-1}}(t)} \frac{f(t) - s_n(t)}{t - z}\, dt \tag{3.5}$$

with

$$w_{2^{\nu-1}}(z) = \prod_{i=0}^{2^{\nu-1}} (z - \xi_i^{(2^{\nu-1})}), \qquad z \in \mathbb{C}.$$

Let us denote by μ_n the normalized counting measure of the point set

$$\{\xi_i^{(2^{\nu-1})} : 0 \le i \le 2^{\nu-1}\} \subset \Gamma_\sigma, \qquad n = 3 \cdot 2^{\nu-1} \in \Lambda.$$

Then

$$\mu_n \xrightarrow{*} \mu_\sigma \qquad \text{as} n = 3 \cdot 2^{\nu-1},\ \nu \to \infty,$$

where μ_σ is the equilibrium measure of Γ_σ (resp. $\overline{D}_\sigma$).

Next, we fix r such that

$$0 < \sigma < r < \rho < R.$$

Then the logarithmic potentials U^{μ_n} converge uniformly to U^{μ_σ} on compact subsets of $\overline{\mathbb{C}} \setminus \Gamma_\sigma$ as $n \to \infty$. Hence, for $\varepsilon > 0$, there exists $\nu_0 \in \mathbb{N}$ such that

$$|U^{\mu_n}(z) - U^{\mu_\sigma}(z)| < \varepsilon, \qquad z \in \overline{D}_\rho \setminus D_r, \tag{3.6}$$

for $n = 3 \cdot 2^{\nu-1} \in \Lambda$ and $\nu \ge \nu_0$. By (3.5) we deduce for $n = 3 \cdot 2^{\nu-1} \in \Lambda$ and $z \in \Gamma_r$ that

$$\begin{aligned} \frac{1}{n+1} \log |(f - s_n - p_{0,n})(z)| &\le \frac{1}{n+1} (2^{\nu-1} + 1) \Big(\max_{t \in \Gamma_\rho} U^{\mu_n}(t) - U^{\mu_n}(z) \Big) \\ &\quad + \frac{1}{n+1} \Big(\log \|f - s_{2^\nu}\|_{\Gamma_\rho} + 2^\nu \log \frac{r}{\rho} + c_1 \Big), \end{aligned} \tag{3.7}$$

where

$$c_1 := \log \max_{t \in \Gamma_\rho} \max_{t \in \Gamma_r} \frac{1}{|t - z|} + \log \frac{\text{length}(\Gamma_\rho)}{2\pi} = \log \frac{1}{\rho - r} + \log \rho, \tag{3.8}$$

and by (3.1),

$$\log \|f - s_{2^\nu}\|_{\Gamma_\rho} \le 2^{\nu+1} \log \frac{\rho}{R} + \log(1 + C(\rho,\nu)) \le 2^{\nu+1} \log \frac{\rho}{R} + C(\rho,\nu). \tag{3.9}$$

Since

$$U^{\mu_\sigma}(z) = \log \frac{\sigma}{r}, \quad z \in \Gamma_r, \quad \text{and} \quad U^{\mu_\sigma}(z) = \log \frac{\sigma}{\rho}, \quad z \in \Gamma_\rho,$$

by the uniform convergence of U^{μ_n} to U^{μ_σ} we obtain, for $z \in \Gamma_r$,

$$\begin{aligned} &\max_{t\in\Gamma_\rho} U^{\mu_n}(t) - U^{\mu_n}(z) \\ &= \max_{t\in\Gamma_\rho} (U^{\mu_n}(t) - U^{\mu_\sigma}(t)) - (U^{\mu_n}(z) - U^{\mu_\sigma}(z)) + \log \frac{r}{\rho} \le \log \frac{r}{\rho} + 2\varepsilon. \end{aligned} \tag{3.10}$$

Using (3.7)–(3.10), for $n = 3 \cdot 2^{\nu-1} \in \Lambda$ and $\nu \ge \nu_0$ we get

$$\begin{aligned} &\frac{1}{n+1} \log \|f - s_n - p_{0,n}\|_{\Gamma_r} \\ &\le \frac{1}{n+1}\left((2^{\nu-1}+1)\left(\log\frac{r}{\rho} + 2\varepsilon\right) + 2^{\nu+1}\log\frac{\rho}{R}\right) \\ &\quad + \frac{1}{n+1}\left(C(\rho,\nu) + 2^\nu \log\frac{r}{\rho} + c_1\right) \\ &= \frac{1}{n+1}\left((2^{\nu-1}+1)\log\frac{r}{\rho} + 2^{\nu+1}\log\frac{\rho}{R}\right) \\ &\quad + \frac{1}{n+1}\left(C(\rho,\nu) + (2^\nu+2)\varepsilon + 2^\nu\log\frac{r}{\rho} + c_1\right) \\ &= \frac{1}{n+1}\left((2^{\nu-1}+2^\nu+1)\log\frac{r}{\rho} + 2^{\nu+1}\log\frac{\rho}{R}\right) \\ &\quad + \frac{1}{n+1}((2^\nu+2)\varepsilon + C(\rho,\nu) + c_1) \\ &= \log\frac{r}{\rho} + \frac{2^{\nu+1}}{2^\nu + 2^{\nu-1}+1}\log\frac{\rho}{R} + \frac{1}{n+1}((2^\nu+2)\varepsilon + C(\rho,\nu)+c_1) \\ &\le \log\frac{r}{\rho} + \log\frac{\rho}{R} + \frac{1}{n+1}((2^\nu+2)\varepsilon + C(\rho,\nu)+c_1). \end{aligned}$$

Since ε can be chosen to be arbitrarily small and r close to σ, we deduce for $\sigma \le r < \rho < R$ and $n = 3 \cdot 2^{\nu-1} \in \Lambda$ that

$$\limsup_{n\in\Lambda,\, n\to\infty} \frac{1}{n+1} \log \|f - s_n - p_{0,n}\|_{\Gamma_r} \le \log\frac{r}{R} + \log\frac{\rho}{R}.$$

Next, we define the sequence $\{p_{1,n}\}_{n\in\mathbb{N}}$ by

$$p_{1,n} := \begin{cases} s_n + p_{0,n} & \text{if } n \in \Lambda, \\ s_n & \text{if } n \notin \Lambda. \end{cases} \tag{3.11}$$

This sequence converges maximally to the power series f on $\overline{D}_\sigma$.

Next, we consider a parameter r^* such that

$$0 < r^* < \sigma < R$$

and let $\varepsilon > 0$ be such that $U^{\mu_n} - U^{\mu_\sigma}$ satisfies (3.6). Then there exists $\nu_1 \geq \nu_0$, $\nu_1 \in \mathbb{N}$, such that

$$|U^{\mu_n}(z) - U^{\mu_\sigma}(z)| < \varepsilon, \qquad z \in \Gamma_{r^*} \cup \Gamma_\rho, \tag{3.12}$$

for $n = 3 \cdot 2^{\nu-1} \in \Lambda$ and $\nu \geq \nu_1$ and we deduce by (3.5) for $n = 3 \cdot 2^{\nu-1} \in \Lambda$ and $z \in \Gamma_{r^*}$ that

$$\begin{aligned} \frac{1}{n+1} \log \bigl|f(z) - p_{1,n}(z)\bigr| &= \frac{1}{n+1} \log |f(z) - s_n(z) - p_{0,n}(z)| \\ &\leq \frac{1}{n+1}(2^{\nu-1}+1)\Bigl(\max_{t\in\Gamma_\rho} U^{\mu_n}(t) - U^{\mu_n}(z)\Bigr) \\ &\quad + \frac{1}{n+1}\biggl(\log \|f - s_{2^\nu}\|_{\Gamma_\rho} + 2^\nu \log \frac{r^*}{\rho} + c_2\biggr), \end{aligned} \tag{3.13}$$

where

$$c_2 := \log \max_{t\in\Gamma_\rho} \max_{t\in\Gamma_{r^*}} \frac{1}{|t-z|} + \log \frac{\operatorname{length}(\Gamma_\rho)}{2\pi} = \log \frac{\rho}{r^*} + \log \rho \tag{3.14}$$

and, by (3.1),

$$\begin{aligned} \log \|f - s_{2^\nu}\|_{\Gamma_\rho} &\leq 2^{\nu+1} \log \frac{\rho}{R} + \log(1 + C(\rho,\nu)) \\ &\leq 2^{\nu+1} \log \frac{\rho}{R} + C(\rho,\nu), \end{aligned} \tag{3.15}$$

We see that

$$\begin{aligned} U^{\mu_\rho}(z) &= -\log \operatorname{cap} D_\rho = -\log \rho, \qquad z \in \Gamma_\rho, \\ U^{\mu_\sigma}(z) &= -\log \operatorname{cap} D_\sigma = -\log \sigma, \qquad z \in \overline{D}_\sigma. \end{aligned}$$

Therefore, by the uniform convergence of U^{μ_n} to U^{μ_σ} in (3.12) we obtain, for $z \in \Gamma_{r^*}$,

$$\begin{aligned} &\max_{t\in\Gamma_\rho} U^{\mu_n}(t) - U^{\mu_n}(z) \\ &= \max_{t\in\Gamma_\rho}(U^{\mu_n}(t) - U^{\mu_\sigma}(t)) - (U^{\mu_n}(z) - U^{\mu_\sigma}(z)) + \log \frac{\sigma}{\rho} \leq \log \frac{\sigma}{\rho} + 2\varepsilon. \end{aligned} \tag{3.16}$$

Using (3.13)–(3.16), for $n = 3 \cdot 2^{\nu-1} \in \Lambda$ and $\nu \geq \nu_1$ we get

$$\begin{aligned}
(3.17)\qquad \frac{1}{n+1}\log\|f-p_{1,n}\|_{\Gamma_{r^*}} &\\
\leq \frac{1}{n+1}(2^{\nu-1}+1)\left(\log\frac{\sigma}{\rho}+2\varepsilon\right)&\\
+\frac{1}{n+1}\left(2^{\nu+1}\log\frac{\rho}{R}+C(\rho,\nu)\right)+\frac{1}{n+1}\left(2^{\nu}\log\frac{r^*}{\rho}+c_2\right)&\\
=\frac{1}{n+1}\left((2^{\nu-1}+1)\log\frac{\sigma}{\rho}+2^{\nu+1}\log\frac{\rho}{R}+2^{\nu}\log\frac{r^*}{\rho}\right)&\\
+\frac{1}{n+1}((2^{\nu}+2)\varepsilon+C(\rho,\nu)+c_2)&\\
=C_0(r^*,\nu)+C_1(\varepsilon,\nu),&
\end{aligned}$$

where

$$(3.18)\qquad C_0(r^*,\nu):=\frac{1}{n+1}\left((2^{\nu-1}+1)\log\frac{\sigma}{\rho}+2^{\nu+1}\log\frac{\rho}{R}+2^{\nu}\log\frac{r^*}{\rho}\right)$$

and

$$C_1(\varepsilon,\nu):=\frac{1}{n+1}((2^{\nu}+2)\varepsilon+C(\rho,\nu)+c_2)$$

with

$$(3.19)\qquad \lim_{\varepsilon\to 0}\lim_{\nu\to\infty}C_1(\varepsilon,\nu)=0.$$

Consequently,

$$\begin{aligned}
C_0(r^*,\nu)&=\frac{1}{n+1}\left((2^{\nu-1}+1)\log\frac{\sigma}{\rho}+2^{\nu+1}\log\frac{\rho}{R}+2^{\nu}\log\frac{r^*}{\rho}\right)\\
&=\frac{1}{n+1}\left((n+1)\log\frac{r^*}{\rho}-2^{\nu-1}\log\frac{r^*}{\rho}-\log\frac{r^*}{\rho}\right)\\
&\quad+\frac{1}{n+1}\left(2^{\nu+1}\log\frac{\rho}{R}+(2^{\nu-1}+1)\log\frac{\sigma}{\rho}\right)\\
&=\log\frac{r^*}{\rho}+\frac{1}{n+1}(2^{\nu-1}+1)\log\frac{\rho}{r^*}\\
&\quad+\frac{1}{n+1}\left(2^{\nu+1}\log\frac{\rho}{R}+(2^{\nu-1}+1)\log\frac{\sigma}{\rho}\right)
\end{aligned}$$

or

$$(3.20)\qquad C_0(r^*,\nu)=\log\frac{r^*}{\rho}+\frac{1}{n+1}\left((2^{\nu-1}+1)\log\frac{\sigma}{r^*}+2^{\nu+1}\log\frac{\rho}{R}\right).$$

Since

$$\frac{2^{\nu-1}+1}{n+1}\leq\frac{1}{2}\quad\text{and}\quad\frac{2^{\nu+1}}{n+1}\geq\frac{8}{7}\qquad\text{for } n=3\cdot 2^{\nu-1},\ \nu\in\mathbb{N},\ \nu\geq 2,$$

for $\nu \in \mathbb{N}$, $\nu \geq 2$, we obtain

$$C_0(r^*, \nu) \leq C_0(r^*) := \log \frac{r^*}{\rho} + \frac{1}{2} \log \frac{\sigma}{r^*} + \frac{8}{7} \log \frac{\rho}{R}. \tag{3.21}$$

Since ε can be chosen to be arbitrarily small, by (3.17)–(3.21),

$$\limsup_{n \in \Lambda,\, n \to \infty} \frac{1}{n+1} \log \|f - p_{1,n}\|_{\Gamma_{r^*}} \leq \log \frac{r^*}{\rho} + \frac{1}{2} \log \frac{\sigma}{r^*} + \frac{8}{7} \log \frac{\rho}{R}. \tag{3.22}$$

REMARK 1. The sequence $\{p_{1,n}\}_{n \in \mathbb{N}}$ converges maximally to f on $\overline{D}_{r^*}$ if

$$C_0(r^*) = \log \frac{r^*}{\rho} + \frac{1}{2} \log \frac{\sigma}{r^*} + \frac{8}{7} \log \frac{\rho}{R} \leq \log \frac{r^*}{R}$$
$$\iff \frac{1}{2} \log \frac{\sigma}{r^*} \leq \frac{1}{7} \log \frac{R}{\rho} \iff r^* \geq \sigma \left(\frac{\rho}{R} \right)^{2/7}.$$

REMARK 2.

$$C_0(r^*) = \log \frac{r^*}{\rho} + \frac{1}{2} \log \frac{\sigma}{r^*} + \frac{8}{7} \log \frac{\rho}{R} \leq \log \frac{r^*}{R} + \frac{1}{14} \log \frac{\rho}{R}$$
$$\iff \frac{1}{2} \log \frac{\sigma}{r^*} \leq \frac{1}{14} \log \frac{R}{\rho} \iff r^* \geq \sigma \left(\frac{\rho}{R} \right)^{1/7}.$$

Let us fix

$$r^* := \sigma \left(\frac{\rho}{R} \right)^{1/7}.$$

Then

$$0 < r^* < \sigma < \rho < R,$$

and by Remark 1 the sequence $\{p_{1,n}\}_{n \in \mathbb{N}}$ converges maximally to f on $\overline{D}_{r^*}$.

By (3.22) and Remark 2 we obtain, for all r with $r^* \leq r \leq \sigma$,

$$\limsup_{n \in \Lambda,\, n \to \infty} \frac{1}{n+1} \log \|f - p_{1,n}\|_{\Gamma_r} \leq \log \frac{r}{R} + \frac{1}{14} \log \frac{\rho}{R}. \tag{3.23}$$

Now, we modify the sequence $\{p_{1,n}\}_{n \in \mathbb{N}}$ by

$$p_n := \begin{cases} p_{1,n} + 2\|f - p_{1,n}\|_{\Gamma_\sigma} & \text{if } n \in \Lambda, \\ p_{1,n} = s_n & \text{if } n \notin \Lambda. \end{cases}$$

Let us fix an auxiliary parameter $\varepsilon > 0$, which will be specified later. Then for $z \in \Gamma_r$, $r^* \leq r \leq \sigma$, by (3.23) we obtain

$$\begin{aligned} |f(z) - p_n(z)| &\leq |f(z) - p_{1,n}(z)| + 2\|f - p_{1,n}\|_{\Gamma_\sigma} \\ &\leq \left(\frac{r}{R} \left(\frac{\rho}{R} \right)^{1/14} e^{\varepsilon} \right)^{n+1} + 2 \left(\frac{\sigma}{R} \left(\frac{\rho}{R} \right)^{1/14} e^{\varepsilon} \right)^{n+1} \\ &\leq 3 \left(\frac{\sigma}{R} \left(\frac{\rho}{R} \right)^{1/14} e^{\varepsilon} \right)^{n+1} \end{aligned}$$

for all sufficiently large $n = 2^\nu + 2^{\nu-1} \in \Lambda$, or

$$\frac{1}{n+1}\log|f(z)-p_n(z)| \le \log\frac{\sigma}{R} + \frac{1}{14}\log\frac{\rho}{R} + \varepsilon + \frac{\log 3}{n+1} = \log\frac{r}{R} + \log\frac{\sigma}{r} + \frac{1}{14}\log\frac{\rho}{R} + \varepsilon + \frac{\log 3}{n+1}. \tag{3.24}$$

Now, we specify ε by

$$\varepsilon := \frac{1}{28}\log\frac{R}{\rho}$$

and fix the parameter r_σ by

$$\log\frac{\sigma}{r_\sigma} + \frac{1}{14}\log\frac{\rho}{R} + \varepsilon = \log\frac{\sigma}{r_\sigma} + \frac{1}{28}\log\frac{\rho}{R} = 0.$$

Then

$$r_\sigma = \sigma\left(\frac{\rho}{R}\right)^{1/28}$$

and $r^* < r_\sigma < \sigma$. Moreover, by (3.24),

$$\limsup_{n\in\Lambda, n\to\infty}\frac{1}{n+1}\log\|f-p_n\|_{\Gamma_r} \le \log\frac{r}{R}, \qquad r_\sigma \le r \le \sigma.$$

Consequently, the sequence $\{p_n\}_{n\in\mathbb{N}}$ converges maximally to f on $\overline{D}_{r_\sigma}$.

On the other hand,

$$f(z) - p_n(z) = f(z) - p_{1,n}(z) - 2\|f-p_{1,n}\|_{\Gamma_\sigma},$$

and therefore, for $n \in \Lambda$ and $z \in \overline{D}_\sigma$,

$$|f(z)-p_n(z)| \ge 2\|f-p_{1,n}\|_{\Gamma_\sigma} - |f(z)-p_{1,n}(z)| \ge \|f-p_{1,n}\|_{\Gamma_\sigma} > 0, \tag{3.25}$$

since f is not a polynomial.

Summarizing, the error function $f - p_n$, $n \in \Lambda$, has no zeros in $\overline{D}_\sigma$ and $\{p_n\}_{n\in\mathbb{N}}$ converges maximally to f on $\overline{D}_{r_\sigma}$, and the Theorem is proven.

3.2. Proof of the Corollary. Because of (3.25),

$$\min_{z\in\overline{D}_\sigma}|f(z)-p_n(z)| \ge \|f-p_{1,n}\|_{\Gamma_\sigma}, \qquad n\in\Lambda.$$

Since $p_{1,n} \in \mathcal{P}_{3\cdot 2^{\nu-1}} \subset \mathcal{P}_m$ with $m := 2^{\nu+1}-1 \in \Lambda_1$, we obtain

$$\min_{z\in\overline{D}_\sigma}|f(z)-p_n(z)| \ge \|f-p_{1,n}\|_{\Gamma_\sigma} \ge \min_{q_m\in\mathcal{P}_m}\|f-q_m\|_{\Gamma_\sigma}. \tag{3.26}$$

By (3.3) and (3.4),

$$\frac{\sigma}{R}e^{-\varepsilon_m(\sigma)} < \min_{z\in\Gamma_\sigma}|f(z)-s_m(z)|^{1/m} \le \|f-s_m\|_{\Gamma_\sigma}^{1/m} < \frac{\sigma}{R}e^{\varepsilon_m(\sigma)}, \qquad m\in\Lambda_1, \tag{3.27}$$

where $\lim_{m\in\Lambda_1, m\to\infty}\varepsilon_m(\sigma) = 0$. Now, we claim that, for $m = 2^{\nu+1}-1 \in \Lambda_1$,

$$\|f - \widetilde{q}_m\|_{\Gamma_\sigma} = \min_{q_m \in \mathcal{P}_m} \|f - q_m\|_{\Gamma_\sigma} > \left(\frac{\sigma}{R}\right)^m e^{-m\varepsilon_m(\sigma)}, \tag{3.28}$$

where $\widetilde{q}_m$ is the best uniform approximation of f on Γ_σ with respect to $\mathcal{P}_m$.

Let us assume that the claim is false; then

$$\|f - \widetilde{q}_m\|_{\Gamma_\sigma} \le \left(\frac{\sigma}{R}\right)^m e^{-m\varepsilon_m(\sigma)}.$$

Using (3.27), Rouché's Theorem implies that the functions $f - s_m$ and

$$f - s_m - (f - \widetilde{q}_m) = \widetilde{q}_m - s_m \in \mathcal{P}_m$$

have $2^{\nu+1} = m + 1$ zeros in D_σ, contradicting $s_m \neq \widetilde{q}_m$. Hence (3.28) holds.

By (3.26) and (3.28), for $m \in \Lambda_1$,

$$\min_{z \in \overline{D}_\sigma} |f(z) - p_n(z)| \ge \min_{q_m \in \mathcal{P}_m} \|f - q_m\|_{\Gamma_\sigma} > \left(\frac{\sigma}{R}\right)^m e^{-m\varepsilon_m(\sigma)}$$

or

$$\left(\frac{\sigma}{R}\right)^{m/n} e^{-\varepsilon_m(\sigma)m/n} < \min_{z \in \overline{D}_\sigma} |f(z) - p_n(z)|^{1/n} \le \min_{z \in \overline{D}_r} |f(z) - p_n(z)|^{1/n}.$$

Then

$$\lim_{m \in \Lambda_1,\, m \to \infty} \varepsilon_m(\sigma) = 0 \text{ and } 1 \le \frac{m}{n} = \frac{2^{\nu+1} - 1}{3 \cdot 2^{\nu-1}} \le \frac{4}{3}, \quad \nu \in \mathbb{N}.$$

If the sequence $\{p_n\}_{n \in \mathbb{N}}$ converges maximally to f on $\overline{D}_r$ for $r \le \sigma$, we obtain

$$\begin{aligned}\left(\frac{\sigma}{R}\right)^{4/3} &\le \liminf_{n \in \Lambda,\, n \to \infty} \min_{z \in \overline{D}_\sigma} |f(z) - p_n(z)|^{1/n} \\ &\le \liminf_{n \in \Lambda,\, n \to \infty} \min_{z \in \overline{D}_r} |f(z) - p_n(z)|^{1/n} \le \limsup_{n \in \mathbb{N},\, n \to \infty} \|f - p_n\|_{\Gamma_r}^{1/n} = \frac{r}{R}.\end{aligned}$$

Hence,

$$r \ge \left(\frac{\sigma^4}{R}\right)^{1/3},$$

and there exists a minimal r_σ^*, with

$$0 < \left(\frac{\sigma^4}{R}\right)^{1/3} \le r_\sigma^* \le \sigma,$$

such that $\{p_n\}_{n \in \mathbb{N}}$ converges maximally to f on $\overline{D}_{r_\sigma^*}$, and the Corollary is proven.

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