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Intrinsic interpolation, near-circularity and maximal convergence

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Abstract

Let E be compact and connected with $\operatorname{cap} E > 0$ and connected complement $\Omega = \overline{\mathbb{C}} \setminus E$, let $g_\Omega(z, \infty)$ be the Green's function of Ω with pole at infinity and let

$$E_\sigma := \{z \in \Omega : g_\Omega(z, \infty) < \log \sigma\} \cup E, \ 1 < \sigma < \infty,$$

be the Green domains with boundaries Γ_σ. Let f be holomorphic on E and let $\rho(f)$ denote the maximal parameter of holomorphy of f and let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials converging maximally to f on E. If σ, $1 < \sigma < \rho(f) < \infty$, is fixed and if $m_n(\sigma)$ denotes the number of interpolation points of p_n to f in E_σ with normalized counting measure $\mu_{\sigma,n}$, then there exists a subset $\Lambda \subset \mathbb{N}$ such that

$$m_n(\sigma) = n + o(n) \text{ as } n \in \Lambda, n \to \infty,$$

$$\widehat{\mu_{\sigma,n}|_E} + \mu_{\sigma,n}|_\Omega \xrightarrow{*} \mu_E \text{ as } n \in \Lambda, n \to \infty,$$

where $\mu_{\sigma,n} = \mu_{\sigma,n}|_E + \mu_{\sigma,n}|_\Omega$, $\widehat{\mu_{\sigma,n}|_E}$ denotes the balayage measure of $\mu_{\sigma,n}|_E$ onto the boundary of E and μ_E is the equilibrium measure of E.

Moreover, there exists a sequence $\{\sigma_n\}_{n\in\Lambda}$ converging to σ such that the closed curves $\gamma_n = (f - p_n)(\Gamma_{\sigma_n})$ do not pass through the point 0 and the winding numbers $\operatorname{Ind}_{\gamma_n}(0)$ satisfy

$$\operatorname{Ind}_{\gamma_n}(0) = m_n(\sigma_n) = n + o(n) \text{ as } n \in \Lambda, n \to \infty.$$

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1. Introduction

For $B \subset \mathbb{C}$, we denote by $\overline{B}$ its closure and by ∂B the boundary of B and we use $\|\cdot\|_B$ for the supremum norm on B. Let $\mathcal{A}(B)$ be the class of functions that are holomorphic in a neighborhood of B.

Let K be a compact subset of the complex plane $\mathbb{C}$ and let $\mathcal{M}(K)$ be the collection of all probability measures supported on K, then the logarithmic potential of $\mu \in \mathcal{M}(K)$ is defined by

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, d\mu(t)$$

and the logarithmic energy $I(\mu)$ by

$$I(\mu) := \iint \log \frac{1}{|z-t|}\, d\mu(t)\, d\mu(z) = \int U^{\mu}(z)\, d\mu(z).$$

Let

$$V(K) := \inf\{I(\mu) : \mu \in \mathcal{M}(K)\},$$

then $V(K)$ is either finite or $V(K) = +\infty$. The quantity

$$\operatorname{cap} K = e^{-V(K)}$$

is called the *logarithmic capacity* or *capacity* of K.

Let E be compact in the complex plane $\mathbb{C}$ with connected complement $\Omega = \overline{\mathbb{C}} \setminus E$. Then $g_\Omega(z, \infty)$ is a Green's function of Ω with pole at ∞, if

(i) $g_\Omega(z, \infty)$ is positive and harmonic in $\Omega \setminus \{\infty\}$,
(ii) $\lim_{|z|\to\infty} \left(g_\Omega(z, \infty) - \log |z|\right) = -\log \operatorname{cap} \mathrm{E}$,
(iii) $\lim_{\zeta\in\Omega,\, \zeta\to z} g_\Omega(\zeta, \infty) = 0$ for quasi-every $z \in \partial\Omega$

If $\operatorname{cap} E > 0$, then there exists a unique Green's function (cf. Ransford [11]) and the complement Ω is called *regular* if property (iii) holds for all $z \in \partial\Omega$.

In the following, let E be compact and connected in the complex plane $\mathbb{C}$ with connected complement $\Omega = \overline{\mathbb{C}} \setminus E$ and $\operatorname{cap} E > 0$. Since E is connected, Ω is regular and there exists a unique measure $\mu_E \in \mathcal{M}(E)$ such that

$$I(\mu_E) = -\log \operatorname{cap} E = V(E),$$

and we have

$$U^{\mu_E}(z) = -g_\Omega(z, \infty) - \log \operatorname{cap} E, \quad z \in \Omega.$$

μ_E is called the *equilibrium measure* of E. For $\sigma > 1$, let us define the *Green domain* E_σ by

$$E_\sigma := \{z \in \Omega : g_\Omega(z, \infty) < \log \sigma\} \cup E$$

with boundary $\Gamma_\sigma := \partial E_\sigma$, and we denote by $\mu_\sigma := \mu_{\overline{E_\sigma}}$ the equilibrium measure of $\overline{E_\sigma}$.

Let $\mathcal{P}_n$ denote the collection of all algebraic polynomials of degree at most n. If $f \in \mathcal{A}(E)$, then there exists $\rho > 1$ and polynomials $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} \le \frac{1}{\rho},$$

due to a result of Walsh [15]. If $f \in \mathcal{A}(E)$ is not an entire function and if $\rho(f)$ denotes the maximal parameter $\rho > 1$, $1 < \rho < \infty$, such that f is holomorphic in E_ρ, then there exist polynomials $p_n \in \mathcal{P}_n$ such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} = \frac{1}{\rho(f)}.$$

Such a sequence $p_n \in \mathcal{P}_n, n \in \mathbb{N}$, is called *maximally convergent to f on E*. Moreover, Walsh [15] (Sect 4.7, Theorem 7, Theorem 8 and its Corollary, pp. 79–81) proved that for such maximally convergent polynomials

$$\limsup_{n\to\infty} \|f - p_n\|_{\Gamma_\sigma}^{1/n} = \frac{\sigma}{\rho(f)}, \quad 1 < \sigma < \rho(f).$$

For example, the polynomials p_n^* of best uniform approximation to f on E with respect to $\mathcal{P}_n$ are maximally convergent. Other examples of maximally convergent polynomials are best polynomial L^p-approximations or partial sums of power series and Faber series.

The best investigated prototype of maximally convergent polynomials are interpolating polynomials to f: Let

$$Z_n : z_{n,0}, z_{n,1}, \ldots, z_{n,n} \subset E_{\rho(f)}$$

be $n+1$ points (not necessarily distinct) and let us denote by $p_n \in \mathcal{P}_n$ the polynomial such that

$$p_n(z_{n,j}) = f(z_{n,j}), \quad 0 \le j \le n$$

(in the case of multiple points, Hermite interpolation is used). We introduce the *normalized counting measure ν_n of Z_n*, i.e.,

$$\nu_n(B) := \frac{\#\{z_{n,j} : z_{n,j} \in B\}}{n+1} \quad (B \subset \mathbb{C}),$$

and we decompose ν_n into $\nu_n = \nu_{n|_E} + \nu_{n|_\Omega}$. Moreover, we use the balayage measure $\widehat{\nu_{n|_E}}$ of $\nu_{n|_E}$ onto the boundary $\partial E = \partial \Omega$. Then a sufficient condition for maximal convergence of $\{p_n\}_{n\in\mathbb{N}}$ is well-known :

If $f \in \mathcal{A}(E)$ is not entire and if the point sets $Z_n, n \in \mathbb{N}$, have no limit point outside E, then these interpolating polynomials p_n converge maximally to f if

$$\widehat{\nu_{n|_E}} + \nu_{n|_\Omega} \xrightarrow[n\to\infty]{*} \mu_E$$

(cf. Walsh [15] (Chapter 7, Theorem 2)). Well-known examples for such interpolation sets Z_n are Fekete points and Leja points of E.

Conversely, if $f \in \mathcal{A}(E)$ is not entire and if $\{p_n\}_{n\in\mathbb{N}}$, $p_n \in \mathcal{P}_n$, is a sequence converging maximally to f on E and interpolating f on $Z_n \subset E$, then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\widehat{\nu_{n|_E}} \xrightarrow[n\in\Lambda, n\to\infty]{*} \mu_E.$$

This result was proven by Grothmann [9] for connected sets E and more generally in [4] for unconnected sets.

In [6] the distribution of interpolation points of maximally convergent polynomials p_n for the special case of rational functions $f = P/Q$ was investigated: If Z_n denotes in this case the set of all zeros of $f - p_n$ in $\mathbb{C}$, then Z_n consists of $n + o(n)$ points ($o(n)$ = Landau symbol) such that

$$\widehat{\nu_{n|E}} + \nu_{n|\Omega} \xrightarrow[n\to\infty]{*} \mu_E,$$

where ν_n denotes the normalized counting measure of Z_n (Theorem 1 in [6]).

In this paper we show that interpolation of p_n to f is an intrinsic property of maximally convergent polynomial sequences $\{p_n\}_{n\in\mathbb{N}}$ by investigating the distribution of the zeros of $f - p_n$ on E_σ, $1 < \sigma < \rho(f)$. Moreover, we obtain results about the winding numbers of the error curves $(f - p_n)(\Gamma_\sigma)$ with respect to 0, a phenomenon well-known for Carathéodory–Fejér approximations of power series (cf. Trefethen [14]).

2. Distribution of interpolation points

Let E be compact and connected in $\mathbb{C}$ with $\operatorname{cap} E > 0$ and connected complement $\Omega = \overline{\mathbb{C}} \setminus E$, $f \in \mathcal{A}(E)$ with maximal parameter $\rho(f)$ of holomorphy and let $\{p_n\}_{n\in\mathbb{N}}$, $p_n \in \mathcal{P}_n$, be a sequence of polynomials. We consider the set $Z_n(\sigma)$ of interpolation points of p_n to f in E_σ, i.e.,

$$Z_n(\sigma) := \left\{ z \in E_\sigma : |f(z) - p_n(z)| = 0 \right\}, \ 1 < \sigma < \rho(f) < \infty,$$

each zero of $f - p_n$ listed according to its multiplicity. Denoting by $m_n(\sigma)$ the number of points of $Z_n(\sigma)$, the number $m_n(\sigma)$ is finite, since f is holomorphic in $E_{\rho(f)}$. Let $\mu_{\sigma,n}$

$$\mu_{\sigma,n}(B) := \frac{\# \left\{ z \in B \cap Z_n(\sigma) \right\}}{m_n(\sigma)} \quad (B \subset \mathbb{C})$$

denote the normalized counting measure of the point set $Z_n(\sigma)$ and let $\widehat{\mu_{\sigma,n}}$ denote the balayage measure of $\mu_{\sigma,n}$ onto the boundary of E_σ.

We investigate the error functions $f - p_n$ on level lines of the Green's function $g_\Omega(z, \infty)$ and we use the property of near-circularity.

Definition. Let $f \in \mathcal{A}(E)$, let $1 < \sigma < \rho(f) < \infty$ and let $\{p_n\}_{n\in\mathbb{N}}$, $p_n \in \mathcal{P}_n$, be a sequence of polynomials converging maximally to f on E. Then $f - p_n$, $n \in \Lambda$, are called *near-circular at* σ_- if there exists a sequence

$$\{\sigma_n\}_{n\in\Lambda}, \ 1 < \sigma_n \le \sigma < \rho(f), \ \lim_{n\in\Lambda, n\to\infty} \sigma_n = \sigma,$$

connected with a sequence

$$\{\varepsilon_n\}_{n\in\Lambda}, \ \varepsilon_n > 0, \ \lim_{n\in\Lambda, n\to\infty} \varepsilon_n = 0,$$

such that

$$\|f - p_n\|_E^{1/n} < \frac{1}{\rho(f)} e^{\varepsilon_n}, \ n \in \Lambda,$$

and

$$\frac{\sigma_n}{\rho(f)} e^{-\varepsilon_n} < \min_{z\in\Gamma_{\sigma_n}} |f(z) - p_n(z)|^{1/n} \le \|f - p_n\|_{\Gamma_{\sigma_n}}^{1/n} < \frac{\sigma_n}{\rho(f)} e^{\varepsilon_n}, \ n \in \Lambda.$$

If γ is a piecewise analytic closed curve and if γ does not pass through the point a, we use the notion $\operatorname{Ind}_\gamma(a)$ for the winding number (or index) of the curve γ with respect to the point a (cf. Ahlfors [1] or Rudin [12]).

Let us decompose $\mu_{\sigma,n}$ into

$$\mu_{\sigma,n} = \mu_{\sigma,n|_E} + \mu_{\sigma,n|_\Omega},$$

and let $\widehat{\mu_{\sigma,n|_E}}$ denote the balayage measure of $\mu_{\sigma,n|_E}$ onto ∂E.

Main Theorem. *Let $f \in \mathcal{A}(E)$ with $\rho(f) < \infty$, let $1 < \sigma < \rho(f)$ be fixed and let $\{p_n\}_{n\in\mathbb{N}}$, $p_n \in \mathcal{P}_n$, be a sequence of polynomials converging maximally to f on E. Then there exists $\Lambda \subset \mathbb{N}$ such that the functions $f - p_n$, $n \in \Lambda$, are near circular at σ_- with associated sequence $\{\sigma_n\}_{n\in\Lambda}$, connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$, and*

$$m_n(\sigma) = n + o(n) \text{ as } n \in \Lambda, n \to \infty, \tag{2.1}$$

$$m_n(\sigma_n) = n + o(n) \text{ as } n \in \Lambda, n \to \infty, \tag{2.2}$$

$$\widehat{\mu_{\sigma,n|_E}} + \mu_{\sigma,n|_\Omega} \xrightarrow{*} \mu_E \text{ as } n \in \Lambda, n \to \infty. \tag{2.3}$$

Moreover, the winding numbers $\mathrm{Ind}_{\gamma_n}(0)$ of the curves $\gamma_n = (f - p_n)(\Gamma_{\sigma_n})$ with respect to the point 0 satisfy

$$\mathrm{Ind}_{\gamma_n}(0) = m_n(\sigma_n) = n + o(n) \text{ as } n \in \Lambda, n \to \infty. \tag{2.4}$$

The auxiliary tools for the proofs will be the theory of condensers (Bagby [2]), outlined in Section 3.1, the property of near-circularity in capacity [5], outlined in Section 3.2, and asymptotic estimates of solutions of special Dirichlet problems, outlined in Section 3.3. These tools are the basis for investigating in Section 4 the asymptotic behavior of the numbers $m_n(\sigma)$, which count the number of points of interpolation of p_n to f on E_σ.

3. Auxiliaries

3.1. Condensers

We recall known facts about condensers, due to Bagby:

Let A, B be disjoint compact sets in $\overline{\mathbb{C}}$, then (A, B) is called a *condenser*. If

$$\mathcal{M}(A, B) := \left\{\sigma = \sigma_A - \sigma_B : \sigma_A \in \mathcal{M}(A), \sigma_B \in \mathcal{M}(B)\right\}$$

denotes the collection of signed measures with support in A, resp. B, then

$$u_\sigma(z) := \int \log\frac{1}{|z-t|}d\sigma(t) = \int \log\frac{1}{|z-t|}d\sigma_A(t) - \int \log\frac{1}{|z-t|}d\sigma_B(t)$$

is the logarithmic potential of σ and

$$J(\sigma) := \iint \log\frac{1}{|z-t|}d\sigma(t)d\sigma(z)$$

defines the logarithmic energy integral of σ. The *modulus of the condenser (A,B)* is defined by

$$\mathrm{mod}(A, B) := \inf\left\{J(\sigma) : \sigma \in \mathcal{M}(A, B)\right\}.$$

If $\mathrm{mod}(A, B) < \infty$, then there exists a unique signed measure $\tau \in \mathcal{M}(A, B)$ such that

$$J(\tau) = \mathrm{mod}(A, B),$$

and τ is called the *equilibrium measure of the condenser (A,B)*.

To decide whether $\sigma = \sigma_A - \sigma_B \in \mathcal{M}(A, B)$ is the equilibrium measure of the condenser (A, B), we use the following criterion of Bagby ([2], Theorem 1 and Theorem 2):

If there exist constants $V_A, V_B \in \mathbb{R}$ such that

(i) $V_B \le 0 \le V_A$,
(ii) $V_B \le u_\sigma(z) \le V_A$ for all $z \in \overline{\mathbb{C}}$,
(iii) $u_\sigma(z) = V_A$ for $z \in A$,
(iv) $u_\sigma(z) = V_B$ for $z \in B$,

then $\mathrm{mod}(A, B) = V_A - V_B$ and σ is the equilibrium measure of the condenser (A, B).

Finally, the definition of the modulus immediately yields the property of monotonicity: If

A', B' are compact sets such that $A' \subset A$ and $B' \subset B$,

then

$$\mathrm{mod}(A', B') \ge \mathrm{mod}(A, B).$$

3.2. *Near-circularity in capacity*

For $z \in E_{\rho(f)} \setminus E$ we define the functions

$$F_n(z) := \frac{1}{n} \log |f(z) - p_n(z)| - g_\Omega(z, \infty) + \log \rho(f),$$

which are subharmonic and therefore upper semicontinuous in $E_{\rho(f)} \setminus E$, and moreover harmonic outside the zeros of $f - p_n$.

If S is a compact set in $E_{\rho(f)} \setminus E$ and $\varepsilon > 0$, we define

$$K_n(S; \varepsilon) := \left\{ z \in S : F_n(z) \le -\varepsilon \right\}$$

and introduce for $1 < \kappa_1 \le \kappa_2 < \infty$ the annulus

$$D_{\kappa_1,\kappa_2} := \overline{E}_{\kappa_2} \setminus E_{\kappa_1}$$

between the level lines Γ_{κ_2} and Γ_{κ_1} of the Green's function $g_\Omega(z, \infty)$.

Then in [5] the following theorem was proved.

Theorem 3.1. *Let E be compact and connected with* cap $E > 0$ *and connected complement, $f \in \mathcal{A}(E)$ with maximal parameter $\rho(f)$ of holomorphy and let $\{p_n\}_{n\in\mathbb{N}}$ be maximally convergent to f on E.*

If $1 < \sigma_1 \le \sigma_2 < \rho(f) < \infty$, then the compact sets $K_n(D_{\sigma_1,\sigma_2}; \varepsilon)$ satisfy

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \operatorname{cap} K_n(D_{\sigma_1,\sigma_2}; \varepsilon) = 0,$$

or equivalently,

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \inf_{z\in D_{\sigma_1,\sigma_2} \setminus K_n(D_{\sigma_1,\sigma_2};\varepsilon)} F_n(z) = \limsup_{n\to\infty} \max_{z\in D_{\sigma_1,\sigma_2}} F_n(z) = 0.$$

Let $\{p_n\}_{n\in\mathbb{N}}$ be maximally convergent to f on E and let β be a fixed parameter with

$$1 < \beta < \rho(f).$$

Since E is compact and connected with cap $E > 0$ *and connected complement Ω, $g_\Omega(z, \infty)$ can be extended continuously to E by $g_\Omega(z, \infty) = 0$. Extending analogously $F_n(z)$ to E, the* *function $F_n(z)$ is upper semicontinuous in $E_{\rho(f)}$. Then $F_n(z)$ is bounded above on $\overline{E_\beta}$ and attains its bound on $\overline{E_\beta}$ (cf. Ransford [11], Theorem 2.1.2). Hence, for $1 < \beta < \rho(f)$*

$$\max_{z \in \overline{E_\beta}} F_n(z)$$

exists and the maximal convergence of p_n to f on E implies

$$\limsup_{n\to\infty} \max_{z \in \overline{E_\beta}} F_n(z) = 0. \tag{3.1}$$

Corollary 3.2. *Let*

$$1 < \sigma < \tau \le \beta < \rho(f) < \infty.$$

then there exists $\Lambda \subset \mathbb{N}$ such that the functions $f - p_n$, $n \in \Lambda$, are near-circular at σ_- and at τ_- with associated sequences $\{\sigma_n\}_{n\in\Lambda}$ and $\{\tau_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$.

Proof. We choose

$$\alpha := 1 + \frac{\sigma - 1}{2},$$

then by Theorem 3.1

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \operatorname{cap} K_n(D_{\alpha,\beta}; \varepsilon) = 0.$$

Defining

$$\delta(\varepsilon) := \liminf_{n\to\infty} \operatorname{cap} K_n(D_{\alpha,\beta}; \varepsilon),$$

we get

$$\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0. \tag{3.2}$$

Set

$$D_n := D_{\sigma-1/n,\sigma} \cup D_{\tau-1/n,\tau},$$

then there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$D_n \subset D_{\alpha,\beta} \text{ and } D_{\sigma-1/n,\sigma} \cap D_{\tau-1/n,\tau} = \emptyset.$$

Let Φ be the conformal mapping $\Phi : \overline{\mathbb{C}} \setminus E \longrightarrow \{z : |z| > 1\}$, normalized by $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$, then

$$c(\alpha) := \max_{z\in\Gamma_\alpha} \left|\Phi'(z)\right| = \max_{z\in\Omega\setminus E_\sigma \Gamma_\alpha} \left|\Phi'(z)\right| > 0.$$

Because of (3.2) we can choose $\left\{\varepsilon_n^*\right\}_{n=n_0}^{\infty}$ such that

$$\delta(\varepsilon_n^*) \le \frac{1}{c(\alpha)} \frac{1}{16n}, \ 0 < \varepsilon_n^* \le \frac{1}{n}.$$

Together with (3.1), we define inductively $\{k_n\}_{n=n_0}^{\infty}$, $k_n < k_{n+1}$, such that

$$\max_{z\in\overline{E_\beta}} F_{k_n}(z) < \varepsilon_n^* \tag{3.3}$$

and

$$\operatorname{cap} K_{k_n}(D_{\alpha,\beta}; \varepsilon_n^*) \le \frac{1}{c(\alpha)} \frac{1}{8n}. \tag{3.4}$$

Let p_1 denote the projection $p_1 : \mathbb{C} \setminus \{0\} \to \mathbb{R}_+$,

$$p_1(z) = r = |z|, \quad z = re^{i\phi},$$

where we have used polar coordinates (r, ϕ) in $\mathbb{C} \setminus \{0\}$.

Then the contraction property of the capacity (cf. Pommerenke [10] or Ransford [11]), together with (3.4), yields

$$\begin{aligned} \text{cap } p_1(\Phi(K_{k_n}(D_{\alpha,\beta}; \varepsilon_n^*))) &\le \text{cap } \Phi(K_{k_n}(D_{\alpha,\beta}; \varepsilon_n^*)) \\ &\le c(\alpha) \text{ cap } K_{n_n}(D_{\alpha,\beta}; \varepsilon_n^*) \\ &\le \frac{1}{8n}. \end{aligned} \tag{3.5}$$

On the other hand,

$$\text{cap } p_1(\Phi(D_{\sigma-1/n,\sigma})) = \text{cap } p_1(\Phi(D_{\tau-1/n,\tau})) = \frac{1}{4n}. \tag{3.6}$$

Comparing (3.5) with (3.6), we conclude that for every $n \in \mathbb{N}$, $n \ge n_0$, there exists $\sigma_{k_n} \in [\sigma - 1/n, \sigma]$ such that

$$\Gamma_{\sigma_{k_n}} \cap K_{k_n}(D_{\alpha,\beta}; \varepsilon_n^*) = \emptyset,$$

and $\tau_{k_n} \in [\tau - 1/n, \tau]$ such that

$$\Gamma_{\tau_{k_n}} \cap K_{k_n}(D_{\alpha,\beta}; \varepsilon_n^*) = \emptyset.$$

Using the definition of $K_{k_n}(D_{\alpha,\beta}; \varepsilon_n^*)$, we obtain

$$-\varepsilon_n^* < \min_{z \in \Gamma_{\sigma_{k_n}}} F_{k_n}(z) \quad \text{and} \quad -\varepsilon_n^* < \min_{z \in \Gamma_{\tau_{k_n}}} F_{k_n}(z),$$

and together with (3.3)

$$-\varepsilon_n^* < \min_{z \in \Gamma_{\sigma_{k_n}}} F_{k_n}(z) \le \max_{z \in \Gamma_{\sigma_{k_n}}} F_{k_n}(z) < \varepsilon_n^*,$$

and

$$-\varepsilon_n^* < \min_{z \in \Gamma_{\tau_{k_n}}} F_{k_n}(z) \le \max_{z \in \Gamma_{\tau_{k_n}}} F_{k_n}(z) < \varepsilon_n^*.$$

Consequently,

$$\frac{\sigma_{k_n}}{\rho(f)} e^{-\varepsilon_n^*} < \min_{z \in \Gamma_{\sigma_{k_n}}} |f(z) - p_{k_n}(z)|^{1/k_n} \le \| f - p_{k_n} \|_{\Gamma_{\sigma_{k_n}}}^{1/k_n} < \frac{\sigma_{k_n}}{\rho(f)} e^{\varepsilon_n^*},$$

and

$$\frac{\tau_{k_n}}{\rho(f)} e^{-\varepsilon_n^*} < \min_{z \in \Gamma_{\tau_{k_n}}} |f(z) - p_{k_n}(z)|^{1/k_n} \le \| f - p_{k_n} \|_{\Gamma_{\tau_{k_n}}}^{1/k_n} < \frac{\tau_{k_n}}{\rho(f)} e^{\varepsilon_n^*}.$$

Define

$$\Lambda := \{k_n\}_{n=n_0}^{\infty} \subset \mathbb{N}$$

and

$$\varepsilon_{k_n} := \varepsilon_n^*.$$

Then (3.3) implies

$$\| f - p_{k_n} \|_E^{1/k_n} < \frac{1}{\rho(f)} e^{\varepsilon_{k_n}}$$

so that the error functions $f - p_n$, $n \in \Lambda$, are near-circular at σ_- and at τ_- with associated sequences $\{\sigma_n\}_{n\in\Lambda}$, $\{\tau_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$. $\square$

Finally, we need the following extension of the previous result.

Corollary 3.3. *Let*

$$1 < \rho < \sigma < \tau \le \beta < \rho(f) < \infty,$$

then there exists $\Lambda \subset \mathbb{N}$ such that the functions $f - p_n$, $n \in \Lambda$, are near-circular at ρ_-, at σ_- and at τ_- with associated sequences $\{\rho_n\}_{n\in\Lambda}$, $\{\sigma_n\}_{n\in\Lambda}$, $\{\tau_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$.

Proof. Let

$$\alpha := 1 + \frac{\rho - 1}{2}$$

and

$$D_n := D_{\rho-1/n,\rho} \cup D_{\sigma-1/n,\sigma} \cup D_{\tau-1/n,\tau},$$

then analogous arguments as in the proof of Corollary 3.2 lead to Corollary 3.3 by using Theorem 3.1 for $D_{\sigma_1,\sigma_2} = D_{\alpha,\beta}$. $\square$

Moreover, concerning $Z_n(\sigma)$ we show

Corollary 3.4. *Let*

$$1 < \sigma < \rho(f) < \infty,$$

and let $\Lambda \subset \mathbb{N}$ such that the functions $f - p_n, n \in \Lambda$, are near-circular at σ_- with associated sequence $\{\sigma_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$. Then $Z_n(\sigma_n) \neq \emptyset$ for all sufficiently large $n \in \Lambda$, or more precisely,

$$Z_n(\sigma_n) \neq \emptyset \text{ for } n \in \Lambda \text{ with } \log \sigma_n \ge 2\varepsilon_n.$$

Proof. Since the functions $f - p_n$, $n \in \Lambda$, are near-circular at σ_- with associated sequence $\{\sigma_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$.

$$\|f - p_n\|_E^{1/n} < \frac{1}{\rho(f)} e^{\varepsilon_n}, \; n \in \Lambda, \tag{3.7}$$

and

$$\frac{\sigma_n}{\rho(f)} e^{-\varepsilon_n} < \min_{z\in\Gamma_{\sigma_n}} |f(z) - p_n(z)|^{1/n} \le \|f - p_n\|_{\Gamma_{\sigma_n}}^{1/n} < \frac{\sigma_n}{\rho(f)} e^{\varepsilon_n}. \tag{3.8}$$

The left inequality in (3.8) implies $f(z) - p_n(z) \neq$ for $z \in \Gamma_{\sigma_n}$.

Let

$$h_n(z) := \frac{1}{n} \log |f(z) - p_n(z)|,$$

and let us assume that $n \in \Lambda$ with $Z_n(\sigma_n) = \emptyset$:

Then $h_n(z)$ is harmonic in E_{σ_n} and continuous on the boundary Γ_{σ_n} and we obtain

$$\min_{z\in\overline{E}_{\sigma_n}} h_n(z) = \min_{z\in\Gamma_{\sigma_n}} h_n(z) > \log \sigma_n - \log \rho(f) - \varepsilon_n.$$

Hence, by (3.7)

$$\begin{aligned}\log\sigma_n - \log\rho(f) - \varepsilon_n &< \min_{z\in\overline{E}_{\sigma_n}} h_n(z)\\ &\le \min_{z\in E} h_n(z) \le \max_{z\in E} h_n(z)\\ &< -\log\rho(f) + \varepsilon_n,\end{aligned}$$

or

$$\log\sigma_n < 2\varepsilon_n.$$

But this inequality is only possible for a finite number of elements of Λ and Corollary 3.4 is proven. □

3.3. Characteristic Dirichlet problems

Let E be compact and connected with $\operatorname{cap} E > 0$ and connected complement and let us fix parameters r and R such that

$$1 < \sigma < r < R < \infty.$$

We consider the equilibrium measure μ_σ of $\overline{E_\sigma}$, resp. Γ_σ, and let $\nu \in \mathcal{M}(\Gamma_\sigma)$ with $\nu \neq \mu_\sigma$. Then the difference

$$(U^\nu - U^{\mu_\sigma})(z)$$

is harmonic in $\overline{\mathbb{C}} \setminus \overline{E_\sigma}$ and the maximum of $U^\nu - U^{\mu_\sigma}$ on the level curves Γ_s, $\sigma < s < \infty$, is increasing with decreasing s. Hence

$$\max_{t\in\Gamma_r} (U^\nu - U^{\mu_\sigma})(t) > \max_{t\in\Gamma_R} (U^\nu - U^{\mu_\sigma})(t). \tag{3.9}$$

Let $\Lambda \subset \mathbb{N}$ and let $\{\nu_n\}_{n\in\Lambda}$ be a sequence with $\nu_n \in \mathcal{M}(\Gamma_\sigma)$ and $\nu_n \neq \mu_\sigma$ for $n \in \Lambda$. Then we consider for $n \in \Lambda$ the solution $\phi(\nu_n;\cdot)$ of the Dirichlet problem in $E_R \setminus \overline{E}_r$ with boundary conditions

$$\phi(\nu_n; z) = 0,\ z \in \Gamma_R, \tag{3.10}$$

and

$$\phi(\nu_n; z) = \min\left(0,\ c(\nu_n;\Gamma_R) - (U^{\nu_n}(z) - U^{\mu_\sigma}(z))\right),\ z \in \Gamma_r, \tag{3.11}$$

where

$$c(\nu_n;\Gamma_R) := \max_{t\in\Gamma_R} (U^{\nu_n} - U^{\mu_\sigma})(t). \tag{3.12}$$

The boundary functions are continuous and ≤ 0. Because of $\nu_n \neq \mu_\sigma$, resp. (3.9), the boundary function in (3.11) is not identically 0. Hence, the maximum principle for harmonic functions implies that $\phi(\nu_n; z) < 0$ for all $z \in E_R \setminus \overline{E}_r$.

Let $K \subset E_R \setminus \overline{E}_r$ be compact, then

$$\delta_{\nu_n}(K) := \max_{z\in K} \phi(\nu_n; z) < 0,$$

and we obtain in the following lemma an upper bound for the sequence $\left\{\delta_{\nu_n}(K)\right\}_{n\in\Lambda}$, using a modified reasoning of Grothmann ([9], Proof of Theorem 2.5).

Lemma 3.5. *Let*

$$1 < \sigma < r < R < \infty,$$

let $\Lambda \subset \mathbb{N}$ and let $\{\nu_n\}_{n\in\Lambda}$ be a sequence of measures $\nu_n \in \mathcal{M}(\Gamma_\sigma)$ such that μ_σ is not a weak limit point of $\{\nu_n\}_{n\in\Lambda}$. If K is compact in $E_R \setminus \overline{E}_r$, then*

$$\limsup_{n\in\Lambda, n\to\infty} \delta_{\nu_n}(K) = \limsup_{n\in\Lambda, n\to\infty} \max_{z\in K} \phi(\nu_n; z) < 0.$$

Proof. Let us assume that the Lemma is false: Then by Helly's Theorem, there exists a subsequence $\Lambda_1 \subset \Lambda$ and $\nu \in \mathcal{M}(\Gamma_\sigma)$ such that

$$\nu_n \underset{n\in\Lambda_1, n\to\infty}{\overset{*}{\longrightarrow}} \nu \neq \mu_\sigma$$

and

$$\lim_{n\in\Lambda_1, n\to\infty} \delta_{\nu_n}(K) = 0.$$

Let $\phi(\nu;\cdot)$ be the solution of the Dirichlet problem in the annulus $E_R \setminus \overline{E}_r$ with boundary conditions (3.10)–(3.12), where ν_n is replaced by ν. Then the same arguments as above show that $\phi(\nu; z) < 0$ for all z in $E_R \setminus \overline{E}_r$.

For $n \in \Lambda_1$ and $z \in E_R \setminus \overline{E}_r$, we define the function

$$\widetilde{h}_n(z) := \min\left(0,\ c(\nu_n; \Gamma_R) - (U^{\nu_n}(z) - U^{\mu_\sigma}(z))\right).$$

Since the functions U^{ν_n}, $n \in \Lambda_1$, converge uniformly on compact sets of $\mathbb{C} \setminus \overline{E}_\sigma$, the functions $\widetilde{h}_n(z)$ converge uniformly on Γ_r to the continuous function

$$\widetilde{h}(z) = \min\left(0,\ c(\nu; \Gamma_R) - (U^{\nu}(z) - U^{\mu_\sigma}(z))\right),\ z \in \Gamma_r.$$

Then due to a well-known theorem (cf. Behnke–Sommer [3], chapter II, Theorem 57), the harmonic functions $\phi(\nu_n;\cdot)$ converge uniformly on compact sets of $E_R \setminus \overline{E}_r$ to $\phi(\nu;\cdot)$. Consequently,

$$\limsup_{n\in\Lambda_1, n\to\infty} \max_{z\in K} \phi(\nu_n; z) = \limsup_{n\in\Lambda_1, n\to\infty} \max_{z\in K} \phi(\nu; z) < 0,$$

contradicting our assumption that the Lemma is false. □

4. Asymptotics of interpolation points

In the following, the sequence $\{p_n\}_{\in\mathbb{N}}$ converges maximally to f on E and

$$1 < \sigma < \tau \le \beta < \rho(f) < \infty.$$

The proof of the Main Theorem will be based on four lemmas, where $f - p_n$, $n \in \Lambda$, are near-circular at σ_- (resp. τ_-) with associated sequences $\{\sigma_n\}_{n\in\Lambda}$ (resp. $\{\tau_n\}_{n\in\Lambda}$) connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$. Since $\lim_{n\in\Lambda, n\to\infty} \sigma_n = \sigma$, we may assume that

$$\log \sigma_n > 2\varepsilon_n,\ n \in \Lambda.$$

Notations.

(1) $1 < \sigma < \rho(f)$:

$\Lambda(\sigma) \subset \mathbb{N}$ denotes the collection of subsets $\Lambda \subset \mathbb{N}$ such that $f - p_n$, $n \in \Lambda$, are near-circular at σ_- with associated sequence $\{\sigma_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$.

(2) $1 < \sigma < \tau < \rho(f)$:

$\Lambda(\sigma, \tau) \subset \mathbb{N}$ denotes the collection of subsets $\Lambda \subset \mathbb{N}$ such that $f - p_n$, $n \in \Lambda$, are near-circular at σ_- and at τ_- with associated sequences $\{\sigma_n\}_{n\in\Lambda}$ and $\{\tau_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$.

Lemma 4.1. *Let $\Lambda \in \Lambda(\sigma)$, then*

$$\limsup_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma)} \le \limsup_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma_n)} \le 1.$$

For $\kappa > 1$ we denote by $\lfloor \kappa n \rfloor$ the greatest entire number $\le \kappa n$ and define for $\Lambda \in \Lambda(\tau)$ the subset

$$\Lambda_{\sigma,\kappa} := \left\{ n \in \Lambda : m_n(\sigma) > \lfloor \kappa n \rfloor \right\}, \quad \kappa > 1.$$

Lemma 4.2. *Let $\Lambda \in \Lambda(\tau)$ and $\kappa > 1$, then $\Lambda_{\sigma,\kappa}$ is a finite subset of Λ and*

$$\liminf_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma_n)} \ge \liminf_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma)} \ge 1.$$

Lemma 4.3. *Let $\Lambda \in \Lambda(\sigma, \tau)$, then*

$$m_n(\sigma) = n + o(n) \text{ as } n \in \Lambda, n \to \infty, \tag{4.1}$$

$$m_n(\sigma_n) = n + o(n) \text{ as } n \in \Lambda, n \to \infty, \tag{4.2}$$

$$\widehat{\mu_{\sigma,n}} \xrightarrow{*} \mu_\sigma \text{ as } n \in \Lambda, n \to \infty. \tag{4.3}$$

Lemma 4.4. *Let $\Lambda \in \Lambda(\sigma, \tau)$, then the winding numbers $Ind_{\gamma_n}(0)$ of the curves $\gamma_n = (f - p_n)(\Gamma_{\sigma_n})$ with respect to the point 0 satisfy*

$$Ind_{\gamma_n}(0) = m_n(\sigma_n) = n + o(n) \text{ as } n \in \Lambda, n \to \infty.$$

By Corollary 3.2, $\Lambda(\sigma)$, $\Lambda(\tau)$ and $\Lambda(\sigma, \tau)$ are nonempty sets.

4.1. *Proof of Lemma 4.1*

4.1.1. *The crucial condenser*

For $n \in \Lambda$ let

$$G_n^0 := \left\{ z \in E_{\sigma_n} : h_n(z) = \frac{1}{n} \log |f(z) - p_n(z)| < \log \frac{\sigma_n}{\rho(f)} - \varepsilon_n \right\}. \tag{4.4}$$

Since $h_n(z)$ is subharmonic in E_{σ_n} and therefore upper semicontinuous, the set G_n^0 is open in E_{σ_n} (cf. Ransford [11] (Definition 2.1.1, p.25)). By Corollary 3.4, $Z_n(\sigma_n) \neq \emptyset$ and $Z_n(\sigma_n) \subset G_n^0$ implies that $G_n^0 \neq \emptyset$, $n \in \Lambda$. We set

$$Z_n(\sigma_n) = \left\{ \xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,m_n(\sigma_n)} \right\}.$$

For any $1 \leq i \leq m_n(\sigma_n)$, there exists a connected component $G^0_{n,i} \subset G^0_n$ with $\xi_{n,i} \in G^0_{n,i}$. Then we claim that

$$G^0_n = \bigcup_{i=1}^{m_n(\sigma_n)} G^0_{n,i}. \tag{4.5}$$

Otherwise, there exists a connected component $G^*_n \neq \emptyset$, $G^*_n \subset G^0_n$, such that

$$G^*_n \cap \bigcup_{i=1}^{m_n(\sigma_n)} G^0_{n,i} = \emptyset.$$

Then

$$h_n(z) = \frac{1}{n} \log |f(z) - p_n(z)|, \ z \in G^*_n,$$

is harmonic in G^*_n, continuous on $\overline{G^*_n}$ and constant on the boundary ∂G^*_n, namely

$$h_n(z) = \log \frac{\sigma_n}{\rho(f)} - \varepsilon_n, \ z \in \partial G^*_n.$$

Then the maximum principle for harmonic (resp. holomorphic) functions yields that $h_n(z)$ (resp. the function $f - p_n$) is a constant on G^*_n, contradicting the fact that $\rho(f) < \infty$. Hence, (4.5) holds.

$\overline{\mathbb{C}} \setminus G^0_n$ will take the role of one plate of our crucial condenser.

As counterpart to (4.4), let us define for $n \in \Lambda$

$$G^1_n := \left\{ z \in G^0_n : h_n(z) = \frac{1}{n} \log |f(z) - p_n(z)| < \log \frac{1}{\rho(f)} + \varepsilon_n \right\}.$$

Then G^1_n is open and

$$\|f - p_n\|_E^{1/n} < \frac{1}{\rho(f)} e^{\varepsilon_n}, \ n \in \Lambda,$$

so that $E \subset G^1_n, n \in \Lambda$.

Let $G^1_{n,i}$ denote the connected component of G^1_n with $\xi_{n,i} \in G^1_{n,i}$. Then the same arguments as above for (4.5) show that

$$G^1_n = \bigcup_{i=1}^{m_n(\sigma_n)} G^1_{n,i}.$$

Now, the definitions imply that

$$E \subset G^1_n \subset G^0_n \subset E_{\sigma_n}.$$

For $z \in \overline{G^1_n}$ we have

$$h_n(z) = \frac{1}{n} \log |f(z) - p_n(z)| \leq \log \frac{1}{\rho(f)} + \varepsilon_n$$

and for $z \in \partial G^0_n$

$$h_n(z) = \frac{1}{n} \log |f(z) - p_n(z)| = \log \frac{\sigma_n}{\rho(f)} - \varepsilon_n.$$

Since $\log \sigma_n > 2\varepsilon_n$, $n \in \Lambda$, we obtain

$$\overline{G^1_n} \cap (\overline{\mathbb{C}} \setminus G^0_n) = \emptyset.$$

Therefore, $(\overline{G_n^1}, \overline{\mathbb{C}} \setminus G_n^0)$ is a condenser for all $n \in \Lambda$ with

$\overline{G_n^1}$ as **first plate** and $\overline{\mathbb{C}} \setminus G_n^0$ as **second plate**.

Finally, we define

$$\Gamma_n^0 = \partial G_n^0 \text{ and } \Gamma_n^1 = \partial G_n^1,$$

and we will calculate the modulus of the condenser

$$(\overline{G_n^1}, \overline{\mathbb{C}} \setminus G_n^0), \text{ resp. } (\Gamma_n^1, \Gamma_n^0),$$

via appropriate probability measures on Γ_n^1 and Γ_n^0.

The probability measure ν_n^0 on Γ_n^0

Let δ_i be the Dirac measure at the point $\xi_{n,i} \in Z_n(\sigma_n)$, then we define

$$\nu_n^0 := \frac{1}{m_n(\sigma_n)} \sum_{i=1}^{m_n(\sigma_n)} \widehat{\delta_i},$$

where $\widehat{\delta_i}$ denotes the balayage measure of δ_i onto the boundary of $G_{n,i}^0$.

It is well known that

$$U^{\widehat{\delta_i}}(z) = \begin{cases} \log \frac{1}{|z-\xi_{n,i}|}, & z \in \mathbb{C} \setminus G_{n,i}^0, \\ \\ \log \frac{1}{|z-\xi_{n,k}|} - g_{n,i}^0(z, \xi_{n,i}), & z \in G_{n,i}^0, \end{cases}$$

where $g_{n,i}^0(z, \xi_{n,i})$ denotes the Green's function of $G_{n,i}^0$ with pole at $\xi_{n,i} \in G_{n,i}^0$ (cf. [13], Chapter II, Theorem 4.1).

Extending the definition of $g_{n,i}^0(z, \xi_{n,i})$ to $\mathbb{C} \setminus G_{n,i}^0$ by

$$g_{n.i}^0(z, \xi_{n,i}) = 0\ ,\ z \in \mathbb{C} \setminus G_{n,i}^0, \tag{4.6}$$

we obtain

$$U^{\nu_n^0}(z) = \frac{1}{m_n(\sigma_n)} \sum_{i=1}^{m_n(\sigma_n)} \left\{ \log \frac{1}{|z-\xi_{n,i}|} - g_{n,i}^0(z, \xi_{n,i}) \right\}, \quad z \in G_n^0, \tag{4.7}$$

and

$$U^{\nu_n^0}(z) = \frac{1}{m_n(\sigma_n)} \sum_{i=1}^{m_n(\sigma_n)} \log \frac{1}{|z-\xi_{n,i}|}, \quad z \in \mathbb{C} \setminus G_n^0. \tag{4.8}$$

The probability measure ν_n^1 on Γ_n^1

We define

$$\nu_n^1 := \frac{1}{m_n(\sigma_n)} \sum_{i=1}^{m_n(\sigma_n)} \widehat{\widehat{\delta_i}},$$

where $\widehat{\widehat{\delta_i}}$ denotes the balayage measure of δ_i onto the boundary of the connected component $G^1_{n,i}$ of G^1_n. Then

$$U^{\widehat{\widehat{\delta_i}}}(z) = \begin{cases} \log \frac{1}{|z-\xi_{n,i}|}, & z \in \mathbb{C} \setminus G^1_{n,i}, \\ \log \frac{1}{|z-\xi_{n,i}|} - g^1_{n,i}(z, \xi_{n,i}), & z \in G^1_{n,i}, \end{cases}$$

where $g^1_{n,i}(z, \xi_{n,i})$ is the Green's function of $G^1_{n,i}$ with pole at $\xi_{n,i} \in G^1_{n,i}$.

Extending the definition of $g^1_{n,i}(z, \xi_{n,i})$ to $\mathbb{C} \setminus G^1_{n,i}$ by

$$g^1_{n.i}(z, \xi_{n,i}) = 0 \ , \ z \in \mathbb{C} \setminus G^1_{n,i}, \tag{4.9}$$

we can write

$$U^{\nu^1_n}(z) = \frac{1}{m_n(\sigma_n)} \sum_{i=1}^{m_n(\sigma_n)} \left\{ \log \frac{1}{|z-\xi_{n,i}|} - g^1_{n,i}(z, \xi_{n,i}) \right\}, \quad z \in G^1_n, \tag{4.10}$$

and

$$U^{\nu^1_n}(z) = \frac{1}{m_n(\sigma_n)} \sum_{i=1}^{m_n(\sigma_n)} \log \frac{1}{|z-\xi_{n,i}|}, \quad z \in \mathbb{C} \setminus G^1_n. \tag{4.11}$$

The logarithmic potential of the signed measure $\nu^1_n - \nu^0_n$

Using (4.8) and (4.11), we obtain

$$U^{\nu^1_n}(z) - U^{\nu^0_n}(z) = 0, \quad z \in \overline{\mathbb{C}} \setminus G^0_n, \tag{4.12}$$

and by (4.7) and (4.10) for $z \in \overline{G^1_n}$

$$U^{\nu^1_n}(z) - U^{\nu^0_n}(z) = \frac{1}{m_n(\sigma_n)} \sum_{i=1}^{m_n(\sigma_n)} \left(g^0_{n,i}(z, \xi_{n,i}) - g^1_{n,i}(z, \xi_{n,i}) \right). \tag{4.13}$$

Let

$$R^0_n(z) := h_n(z) + \frac{1}{n} \sum_{i=1}^{m_n(\sigma_n)} g^0_{n,i}(z, \xi_{n,i}), \ z \in \overline{G^0_n},$$

where $g^0_{n,i}(z, \xi_{n,i})$ is the extended Green's function, defined in (4.6). Then

$$R^0_n(z) = h_n(z) = \frac{1}{n} \log |f(z) - p_n(z)| = \log \frac{\sigma_n}{\rho(f)} - \varepsilon_n, \quad z \in \Gamma^0_n.$$

Hence, the function $R^0_n(z)$ is harmonic on G^0_n, continuous on $\overline{G^0_n}$ and constant on the boundary Γ^0_n. Since G^0_n consists of a finite number of disjoint regions of type $G^0_{n,i}$, by the maximum principle

$$R^0_n(z) = \log \frac{\sigma_n}{\rho(f)} - \varepsilon_n, \quad z \in \overline{G^0_n}. \tag{4.14}$$

Analogously, let

$$R^1_n(z) := h_n(z) + \frac{1}{n} \sum_{i=1}^{m_n(\sigma_n)} g^1_{n,i}(z, \xi_{n,i}), \ z \in \overline{G^1_n},$$

where $g^1_{n,i}(z,\xi_{n,i})$ is the extended Green's function, defined in (4.9). Then

$$R^1_n(z) = \log\frac{1}{\rho(f)} + \varepsilon_n, \quad z \in \overline{G^1_n}, \tag{4.15}$$

and by (4.14) and (4.15) we obtain for $z \in \overline{G^1_n}$

$$R^0_n(z) - R^1_n(z) = \frac{1}{n}\sum_{i=1}^{m_n(\sigma_n)}\left(g^0_{n,i}(z,\xi_{n,i}) - g^1_{n,i}(z,\xi_{n,i})\right) = \log\sigma_n - 2\varepsilon_n > 0.$$

Because of (4.13), we have got for $z \in \overline{G^1_n}$

$$U^{\nu^1_n}(z) - U^{\nu^0_n}(z) = \frac{n}{m_n(\sigma_n)}(\log\sigma_n - 2\varepsilon_n) > 0. \tag{4.16}$$

Summarized, for $n \in \Lambda$ we realize by (4.12), (4.13) and (4.16) that $\nu^1_n - \nu^0_n$ is the equilibrium measure of the condenser

$$(\overline{G^1_n}, \overline{\mathbb{C}} \setminus G^0_n), \text{ resp. } (\Gamma^1_n, \Gamma^0_n),$$

with modulus

$$\operatorname{mod}(\overline{G^1_n}, \overline{\mathbb{C}} \setminus G^0_n) = \operatorname{mod}(\Gamma^1_n, \Gamma^0_n) = \frac{n}{m_n(\sigma_n)}(\log\sigma_n - 2\varepsilon_n). \tag{4.17}$$

4.1.2. Comparison of $\operatorname{mod}(\overline{G^1_n}, \overline{\mathbb{C}} \setminus G^0_n)$ *and* $\operatorname{mod}(E, \overline{\mathbb{C}} \setminus E_{\sigma_n})$

Let us compare the condenser $(\overline{G^1_n}, \overline{\mathbb{C}} \setminus G^0_n)$ with the condenser $(E, \overline{\mathbb{C}} \setminus E_{\sigma_n})$. Since

$$E \subset \overline{G^1_n} \text{ and } \overline{\mathbb{C}} \setminus E_{\sigma_n} \subset \overline{\mathbb{C}} \setminus G^0_n,$$

the property of monotonicity of the modulus of condensers implies

$$\log\sigma_n = \operatorname{mod}(E, \overline{\mathbb{C}} \setminus E_{\sigma_n}) \ge \operatorname{mod}(\overline{G^1_n}, \overline{\mathbb{C}} \setminus G^0_n)$$

and therefore (4.17) yields

$$\log\sigma_n \ge \frac{n}{m_n(\sigma_n)}(\log\sigma_n - 2\varepsilon_n).$$

Since $\log\sigma_n > 2\varepsilon_n$, $n \in \Lambda$, we obtain

$$\frac{n}{m_n(\sigma_n)} \le 1 + \frac{2\varepsilon_n}{\log\sigma_n - 2\varepsilon_n}, \quad n \in \Lambda,$$

and

$$\limsup_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma_n)} \le 1.$$

And finally,

$$m_n(\sigma) \ge m_n(\sigma_n)$$

leads to

$$\limsup_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma)} \le \limsup_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma_n)} \le 1.$$

Hence, Lemma 4.1 is proven.

4.2. Proof of Lemma 4.2

Let

$$\kappa = 1 + \delta,$$

then $\delta > 0$ and

$$\frac{1}{\kappa}(\delta \log \tau + \log \rho(f))$$

is a convex combination of $\log \tau$ and $\log \rho(f)$. Since the logarithm function is concave on $(0, \infty)$, we can fix R, $\tau < R < \rho(f)$, such that

$$\log R > \frac{1}{\kappa}(\delta \log \tau + \log \rho(f)). \tag{4.18}$$

If we define

$$\varepsilon^* := \frac{\kappa \log R - \delta \log \tau - \log \rho(f)}{4\kappa + 1}, \tag{4.19}$$

then (4.18) implies $\varepsilon^* > 0$.

Since $\Lambda \in \Lambda(\tau)$, the functions $f - p_n, n \in \Lambda$, are near-circular at τ_- with associated sequence $\{\tau_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$,

$$\frac{\tau_n}{\rho(f)} e^{-\varepsilon_n} < \min_{z\in\Gamma_{\tau_n}} |f(z) - p_n(z)|^{1/n} \leq \|f - p_n\|_{\Gamma_{\tau_n}}^{1/n} < \frac{\tau_n}{\rho(f)} e^{\varepsilon_n}, \quad n \in \Lambda. \tag{4.20}$$

Moreover, there exists n_0 such that

$$\tau_n \geq \tau^* := \sigma + \frac{\tau - \sigma}{2}, \ n \in \Lambda, \ n \geq n_0. \tag{4.21}$$

Next, we choose r such that

$$1 < \sigma < r < \tau^* \leq \tau < R < \rho(f) < \infty,$$

where R satisfies (4.18).

Since $1 < \sigma < \rho(f)$ is fixed and $\kappa > 1$, the definition of $\Lambda_{\sigma,\kappa}$ implies that for $n \in \Lambda_{\sigma,\kappa}$ there exists a point set $Z^*_{\kappa,n} \subset Z_n(\sigma)$ of $\lfloor \kappa n \rfloor + 1$ points and let

$$Z^*_{\kappa,n} = \left\{\zeta^*_{\kappa,0}, \zeta^*_{\kappa,1}, \ldots, \zeta^*_{\kappa,\lfloor \kappa n \rfloor}\right\}.$$

We denote by $\mu_{\kappa,n}$ the normalized counting measure of $Z^*_{\kappa,n}$ and let $\widehat{\mu_{\kappa,n}}$ be the balayage of $\mu_{\kappa,n}$ onto Γ_σ.

We may interpret $f - p_n$ as the error of interpolating f by p_n on the point set $Z^*_{\kappa,n}$. Then the Lagrange–Hermite formula for $z \in E_R$ yields as

$$f(z) - p_n(z) \ = \ \frac{1}{2\pi i} \int_{\Gamma_R} \frac{w_{\lfloor \kappa\, n \rfloor}(z)}{w_{\lfloor \kappa\, n \rfloor}(t)} \, \frac{f(t)}{t - z} \, dt \tag{4.22}$$

with

$$w_{\lfloor \kappa\, n \rfloor}(t) \ = \ \prod_{i=0}^{\lfloor \kappa\, n \rfloor} (t - \zeta^*_{\kappa,i}), \quad t \in \mathbb{C}$$

(cf. Walsh [15], Chapter 3, §3.1). (4.22) can be rewritten as

$$f(z) - p_n(z) \ = \ \frac{1}{2\pi i} \int_{\Gamma_R} \frac{w_{\lfloor \kappa\, n \rfloor}(z)}{w_{\lfloor \kappa\, n \rfloor}(t)} \, \frac{f(t) - p_n(t)}{t - z} \, dt. \tag{4.23}$$

Let us assume that $\Lambda_{\kappa,n}$ is not a finite subset of $\Lambda \in \Lambda(\tau)$, contradicting the statement of Lemma 4.2. Then we distinguish two cases:

$$(\mathbf{a})\ \mu_\sigma \text{ is a weak* limit point of } \{\widehat{\mu_{\kappa,n}}\}_{n\in\Lambda_{\sigma,\kappa}}$$

or

$$(\mathbf{b})\ \mu_\sigma \text{ is not a weak* limit point of } \{\widehat{\mu_{\kappa,n}}\}_{n\in\Lambda_{\sigma,\kappa}}.$$

Case (a) μ_σ is a weak* limit point of $\{\widehat{\mu_{\kappa,n}}\}_{n\in\Lambda_{\sigma,\kappa}}$

In this case, we choose $\Lambda_1 \subset \Lambda_{\sigma,\kappa}$ such that

$$\widehat{\mu_{\kappa,n}} \xrightarrow[n\in\Lambda_1, n\to\infty]{*} \mu_\sigma.$$

By (4.22) we obtain for $z \in E_R$

$$|f(z) - p_n(z)| \le \frac{1}{2\pi}\left(\max_{t\in\Gamma_R}\frac{|w_{\lfloor\kappa n\rfloor}(z)|}{|w_{\lfloor\kappa n\rfloor}(t)|}\ \|f\|_{\Gamma_R}\ \max_{t\in\Gamma_R}\frac{1}{|t-z|}\right)\text{length}(\Gamma_R),$$

or for $z \in \overline{E_r}$

$$\frac{1}{\lfloor\kappa n\rfloor+1}\ \log|f(z)-p_n(z)| \le \max_{t\in\Gamma_R} U^{\widehat{\mu_{\kappa,n}}}(t) - U^{\widehat{\mu_{\kappa,n}}}(z) + \frac{c_1}{\lfloor\kappa n\rfloor+1}, \tag{4.24}$$

where

$$c_1 = \log\left[\max_{z\in\overline{E_\tau}}\max_{t\in\Gamma_R}\frac{1}{|t-z|}\right] + \log\|f\|_{\Gamma_R} + \log\frac{\text{length}(\Gamma_R)}{2\pi}.$$

Because of the uniform convergence of $U^{\widehat{\mu_{\kappa,n}}}$ to U^{μ_σ} on compact sets of $\mathbb{C}\setminus\overline{E_\sigma}$, there exists $n_1(\varepsilon^*) \in \mathbb{N}$, $n_1(\varepsilon^*) \ge n_0$, such that

$$|U^{\widehat{\mu_{\kappa,n}}}(z) - U^{\mu_\sigma}(z)| \le \varepsilon^* \quad \text{for } z \in D_{\tau^*,R},\ n\in\Lambda_1, n \ge n_1(\varepsilon^*).$$

Hence, for $z \in \Gamma_R$ and $n \in \Lambda_1, n \ge n_1(\varepsilon^*)$,

$$\max_{t\in\Gamma_R} U^{\widehat{\mu_{\kappa,n}}}(t) \le \max_{t\in\Gamma_R} U^{\mu_\sigma}(t) + \varepsilon^* = -\log\text{cap}\,E - \log R + \varepsilon^*$$

and for $z \in \Gamma_{\tau_n}$

$$U^{\widehat{\mu_{\kappa,n}}}(z) \ge U^{\mu_\sigma}(z) - \varepsilon^* = -\log\text{cap}\,E - \log\tau_n - \varepsilon^*.$$

By (4.24) we obtain for $z \in \Gamma_{\tau_n}$ and $n \ge n_1(\varepsilon^*)$

$$\frac{1}{\lfloor\kappa n\rfloor+1}\ \log|f(z)-p_n(z)| \le \log\frac{\tau_n}{R} + 2\varepsilon^* + \frac{c_1}{\lfloor\kappa n\rfloor+1}.$$

Then there exists $n_2(\varepsilon^*) \ge n_1(\varepsilon^*)$ such that for $n \in \Lambda_1, n \ge n_2(\varepsilon^*)$,

$$\frac{1}{\lfloor\kappa n\rfloor+1}\log|f(z)-p_n(z)| \le \log\frac{\tau_n}{R} + 3\varepsilon^*,\ z\in\Gamma_{\tau_n}$$

or for $n \in \Lambda_1, n \ge n_2(\varepsilon^*)$,

$$\frac{1}{n}\log|f(z)-p_n(z)| \le \frac{\lfloor\kappa n\rfloor+1}{n}\left[\log\frac{\tau_n}{R} + 3\varepsilon^*\right],\ z\in\Gamma_{\tau_n}.$$

Using

$$\kappa n - 1 < \lfloor\kappa n\rfloor \le \kappa n \text{ and } \log\frac{\tau_n}{R} < 0,$$

we get

$$\frac{1}{n}\log|f(z)-p_n(z)| < \kappa \log\frac{\tau_n}{R} + 3\varepsilon^*\frac{\kappa n+1}{n},\ z\in\Gamma_{\tau_n}.$$

Then there exists $n_3(\varepsilon^*) \geq n_2(\varepsilon^*)$ such that for $n \in \Lambda_1, n \geq n_3(\varepsilon^*)$,

$$\frac{1}{n}\log|f(z)-p_n(z)| < \kappa \log\frac{\tau_n}{R} + 4\kappa\varepsilon^*,\ z\in\Gamma_{\tau_n},\ z\in\Gamma_{\tau_n}$$

or

$$\frac{1}{n}\log|f(z)-p_n(z)| < \log\frac{\tau_n}{\rho(f)} + A_n(\kappa,\varepsilon^*),\ z\in\Gamma_{\tau_n}, \tag{4.25}$$

where

$$A_n(\kappa,\varepsilon^*) := \delta\,\log\frac{\tau_n}{\rho(f)} + \kappa\log\frac{\rho(f)}{R} + 4\kappa\varepsilon^*.$$

Then some calculations, together with (4.19), show that

$$A_n(\kappa,\varepsilon^*) \leq A(\kappa,\varepsilon^*) := \delta\,\log\frac{\tau}{\rho(f)} + \kappa\log\frac{\rho(f)}{R} + 4\kappa\varepsilon^* = -\varepsilon^*.$$

Hence, by (4.25) for $n \in \Lambda_1, n \geq n_3(\varepsilon^*)$,

$$\max_{z\in\Gamma_{\tau_n}} |f(z)-p_n(z)|^{1/n} < \frac{\tau_n}{\rho(f)}e^{-\varepsilon^*}.$$

On the other hand, $f - p_n$, $n \in \Lambda$, are near-circular at τ_- and by (4.20)

$$\min_{z\in\Gamma_{\tau_n}} |f(z)-p_n(z)|^{1/n} > \frac{\tau_n}{\rho(f)}e^{-\varepsilon_n},\ z\in\Gamma_{\tau_n}\ n\in\Lambda.$$

Summarizing, the following inequalities must hold for all $n \in \Lambda_1, n \geq n_3(\varepsilon^*)$:

$$\frac{\tau_n}{\rho(f)}e^{-\varepsilon^*} > \max_{z\in\Gamma_{\tau_n}} |f(z)-p_n(z)|^{1/n} \geq \min_{z\in\Gamma_{\tau_n}} |f(z)-p_n(z)|^{1/n} > \frac{\tau_n}{\rho(f)}e^{-\varepsilon_n},$$

which is a contradiction to $\varepsilon^* > 0$ for sufficiently large $n \in \Lambda_1$.

Hence, Case (a) cannot occur.

Case (b) μ_σ is not a weak* limit point of $\{\widehat{\mu_{\kappa,n}}\}_{n\in\Lambda_{\sigma,\kappa}}$

Consider the sequence $\{\widehat{\mu_{\kappa,n}}\}_{n\in\Lambda_{\sigma,\kappa}}$ and let $\phi(\widehat{\mu_{\kappa,n}};\cdot)$ be the harmonic function in $E_R \setminus \overline{E}_r$ with boundary conditions (3.10)–(3.12), where ν_n is replaced by $\widehat{\mu_{\kappa,n}}$ and Λ by $\Lambda_{\sigma,\kappa}$. Since μ_σ is not a weak* limit point of $\{\widehat{\mu_{\kappa,n}}\}_{n\in\Lambda_{\sigma,\kappa}}$, we obtain for $K = D_{\tau^*,\tau}$ by Lemma 3.5

$$\limsup_{n\in\Lambda_{\sigma,\kappa},n\to\infty}\ \max_{z\in D_{\tau^*,\tau}} \phi(\widehat{\mu_{\kappa,n}};z) < 0,$$

where τ^* is defined by (4.21). Let n_0 satisfy (4.21), then there exists $\varepsilon > 0$ and $n_1(\varepsilon) \in \mathbb{N}$, $n_1(\varepsilon) \geq n_0$, such that for $n \geq n_1(\varepsilon)$, $n \in \Lambda_{\sigma,\kappa}$,

$$\max_{z\in D_{\tau^*,\tau}} \phi(\widehat{\mu_{\kappa,n}};z) \leq -3\varepsilon. \tag{4.26}$$

Since $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, converge maximally to f on E, there exists $n_2(\varepsilon) \geq n_1(\varepsilon)$ such that

$$\frac{1}{n}\log\|f-p_n\|_{\Gamma_R} \leq \log\frac{R}{\rho(f)} + \varepsilon,\quad n \geq n_2(\varepsilon).$$

Because of

$$0 < \frac{n}{\lfloor \kappa n \rfloor + 1} < 1, \tag{4.27}$$

we get

$$\frac{1}{\lfloor \kappa n \rfloor + 1} \log \|f - p_n\|_{\Gamma_R} \leq \frac{n}{\lfloor \kappa n \rfloor + 1} \log \frac{R}{\rho(f)} + \varepsilon, \quad n \geq n_2(\varepsilon). \tag{4.28}$$

Using (4.23), we obtain for $z \in \Gamma_r$

$$\begin{aligned} \frac{1}{\lfloor \kappa n \rfloor + 1} \log |f(z) - p_n(z)| \leq & \max_{t \in \Gamma_R} U^{\widehat{\mu_{\kappa,n}}}(t) - U^{\widehat{\mu_{\kappa,n}}}(z) \\ & + \frac{1}{\lfloor \kappa n \rfloor + 1} \left(\log \|f - p_n\|_{\Gamma_R} + c_2 \right), \end{aligned} \tag{4.29}$$

where

$$c_2 = \log \left[\max_{z \in \Gamma_r} \max_{t \in \Gamma_R} \frac{1}{|t - z|} \right] + \log \frac{\text{length}(\Gamma_R)}{2\pi}.$$

By (4.28) and (4.29), we can choose $n_3(\varepsilon) \geq n_2(\varepsilon)$ such that for $z \in \Gamma_r$ and $n \geq n_3(\varepsilon), n \in \Lambda_{\sigma,\kappa}$,

$$\begin{aligned} & \frac{1}{\lfloor \kappa n \rfloor + 1} \log |f(z) - p_n(z)| \\ & \quad \leq \max_{t \in \Gamma_R} U^{\widehat{\mu_{\kappa,n}}}(t) - U^{\widehat{\mu_{\kappa,n}}}(z) + \frac{n}{\lfloor \kappa n \rfloor + 1} \log \frac{R}{\rho(f)} + 2\varepsilon. \end{aligned} \tag{4.30}$$

Now,

$$U^{\mu_\sigma}(z) = -\log \operatorname{cap} E - \log r, \quad z \in \Gamma_r$$

and

$$U^{\mu_\sigma}(z) = -\log \operatorname{cap} E - \log R, \quad z \in \Gamma_R.$$

Therefore,

$$\begin{aligned} & \max_{t \in \Gamma_R} U^{\widehat{\mu_{\kappa,n}}}(t) - U^{\widehat{\mu_{\kappa,n}}}(z) \\ & \quad = \max_{t \in \Gamma_R} (U^{\widehat{\mu_{\kappa,n}}} - U^{\mu_\sigma})(t) - (U^{\widehat{\mu_{\kappa,n}}}(z) - U^{\mu_\sigma}(z)) + \log \frac{r}{R}, \end{aligned}$$

and by (4.30) we get for $z \in \Gamma_r$ and $n \geq n_3(\varepsilon)$, $n \in \Lambda_{\sigma,\kappa}$,

$$\begin{aligned} & \frac{1}{\lfloor \kappa n \rfloor + 1} \log |f(z) - p_n(z)| \\ & \quad \leq \max_{t \in \Gamma_R} (U^{\widehat{\mu_{\kappa,n}}} - U^{\mu_\sigma})(t) - (U^{\widehat{\mu_{\kappa,n}}}(z) - U^{\mu_\sigma}(z)) \\ & \qquad + \frac{n}{\lfloor \kappa n \rfloor + 1} \log \frac{R}{\rho(f)} + 2\varepsilon + \log \frac{r}{R} \\ & \quad \leq \max_{t \in \Gamma_R} (U^{\widehat{\mu_{\kappa,n}}} - U^{\mu_\sigma})(t) - (U^{\widehat{\mu_{\kappa,n}}}(z) - U^{\mu_\sigma}(z)) \\ & \qquad + \frac{n}{\lfloor \kappa n \rfloor + 1} \log \frac{R}{\rho(f)} + 2\varepsilon \\ & \qquad + \frac{n}{\lfloor \kappa n \rfloor + 1} \log \frac{r}{R} + \log \frac{r}{R} - \frac{n}{\lfloor \kappa n \rfloor + 1} \log \frac{r}{R} \end{aligned}$$

or

$$\begin{aligned}\frac{1}{\lfloor\kappa n\rfloor+1}&\log|f(z)-p_n(z)|\\&\le \max_{t\in\Gamma_R}(U^{\widehat{\mu_{\kappa,n}}}-U^{\mu_\sigma})(t)-(U^{\widehat{\mu_{\kappa,n}}}(z)-U^{\mu_\sigma}(z))\\&\quad+\frac{n}{\lfloor\kappa n\rfloor+1}\log\frac{r}{\rho(f)}+2\varepsilon+\log\frac{r}{R}\left[1-\frac{n}{\lfloor\kappa n\rfloor+1}\right].\end{aligned}\tag{4.31}$$

Since (4.27) holds and $\log(r/R)<0$, we obtain finally for $z\in\Gamma_r$ and $n\ge n_3(\varepsilon)$, $n\in\Lambda_{\sigma,\kappa}$,

$$\begin{aligned}\frac{1}{\lfloor\kappa n\rfloor+1}&\log|f(z)-p_n(z)|\\&< \max_{t\in\Gamma_R}(U^{\widehat{\mu_{\kappa,n}}}-U^{\mu_\sigma})(t)-(U^{\widehat{\mu_{\kappa,n}}}(z)-U^{\mu_\sigma}(z))\\&\quad+\frac{n}{\lfloor\kappa n\rfloor+1}\log\frac{r}{\rho(f)}+2\varepsilon.\end{aligned}$$

Let

$$c(\widehat{\mu_{\kappa,n}};\Gamma_R):=\max_{t\in\Gamma_R}(U^{\widehat{\mu_{\kappa,n}}}-U^{\mu_\sigma})(t),$$

then for $z\in\Gamma_r$ and $n\ge n_3(\varepsilon)$, $n\in\Lambda_{\sigma,\kappa}$,

$$\begin{aligned}\frac{1}{\lfloor\kappa n\rfloor+1}&\log|f(z)-p_n(z)|\\&\le c(\widehat{\mu_{\kappa,n}};\Gamma_R)-(U^{\widehat{\mu_{\kappa,n}}}(z)-U^{\mu_\sigma}(z))+\frac{n}{\lfloor\kappa n\rfloor+1}\log\frac{r}{\rho(f)}+2\varepsilon.\end{aligned}\tag{4.32}$$

For $z\in\Gamma_R$ and $n\ge n_3(\varepsilon)$, $n\in\Lambda_{\sigma,\kappa}$, the inequality (4.28) yields

$$\frac{1}{\lfloor\kappa n\rfloor+1}\log|f(z)-p_n(z)|\le\frac{n}{\lfloor\kappa n\rfloor+1}\log\frac{R}{\rho(f)}+2\varepsilon.\tag{4.33}$$

Hence,

$$\frac{1}{\lfloor\kappa n\rfloor+1}\log|f(z)-p_n(z)|,\ z\in E_R\setminus\overline{E}_r,$$

is subharmonic and satisfies the boundary conditions (4.32) and (4.33) for $n\ge n_3(\varepsilon)$, $n\in\Lambda_{\sigma,\kappa}$. Therefore, the definition of $\phi(\widehat{\mu_{\kappa,n}};z)$ implies that

$$\frac{n}{\lfloor\kappa n\rfloor+1}(g_\Omega(z,\infty)-\log\rho(f))+\phi(\widehat{\mu_{\kappa,n}};z)+2\varepsilon$$

is a harmonic majorant for the subharmonic function

$$\frac{1}{\lfloor\kappa n\rfloor+1}\log|f(z)-p_n(z)|,\ z\in E_R\setminus\overline{E}_r.$$

Hence, for $z\in E_R\setminus\overline{E}_r$ and $n\in\Lambda_{\sigma,\kappa}$

$$\frac{1}{\lfloor\kappa n\rfloor+1}\log|f(z)-p_n(z)|\le\frac{n}{\lfloor\kappa n\rfloor+1}(g_\Omega(z,\infty)-\log\rho(f))+\phi(\widehat{\mu_{\kappa,n}};z)+2\varepsilon.$$

and by (4.26) we obtain for $z\in D_{\tau^*,\tau}$ and $n\ge n_3(\varepsilon)$, $n\in\Lambda_{\sigma,\kappa}$,

$$\frac{1}{\lfloor\kappa n\rfloor+1}\log|f(z)-p_n(z)|\le\frac{n}{\lfloor\kappa n\rfloor+1}(g_\Omega(z,\infty)-\log\rho(f))-\varepsilon.$$

Since $\Gamma_{\tau_n} \subset D_{\tau^*,\tau}$ for all $n \in \Lambda_{\sigma,\kappa}$ with $n \geq n_0$, we get finally for $n \geq n_3(\varepsilon)$

$$\|f - p_n\|_{\Gamma_{\tau_n}}^{1/n} \leq \frac{\tau_n}{\rho(f)} \left(e^{-\varepsilon}\right)^{\frac{\lfloor \kappa n \rfloor + 1}{n}} < \frac{\tau_n}{\rho(f)} e^{-\varepsilon}.$$

Since $f - p_n$, $n \in \Lambda$, are near-circular at τ_- with associated sequence $\{\tau_n\}_{n\in\Lambda}$ connected with $\{\varepsilon_n\}_{n\in\Lambda}$, we obtain for $n \in \Lambda_{\sigma,\kappa}$, $n \geq n_3(\varepsilon)$, with (4.20)

$$\frac{\tau_n}{\rho(f)} e^{-\varepsilon} > \|f - p_n\|_{\Gamma_{\tau_n}}^{1/n} \geq \min_{z\in\Gamma_{\tau_n}} |f(z) - p_n(z)|^{1/n} > \frac{\tau_n}{\rho(f)} e^{-\varepsilon_n},$$

which leads to a contradiction for all sufficiently large $n \in \Lambda_{\sigma,\kappa}$.

Hence, Case (b) cannot occur.

Summarizing, $\Lambda_{\sigma,\kappa}$ has to be a finite subset of $\Lambda \in \Lambda(\tau)$ and the first part of Lemma 4.2 is proven.

Concerning the second part of Lemma 4.2, let us assume that

$$\liminf_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma)} \leq \gamma < 1, \tag{4.34}$$

and let

$$\frac{1}{\kappa} := \gamma + \frac{1-\gamma}{2}.$$

Then $\kappa > 1$ and there are infinitely many indices

$$n_1 < n_2 < n_3 < \cdots$$

such that

$$\frac{n_j}{m_{n_j}(\sigma)} < \frac{1}{\kappa}, \quad j = 1, 2, 3, \ldots,$$

or

$$m_{n_j}(\sigma) > \kappa n_j \geq \lfloor \kappa n_j \rfloor, \quad j = 1, 2, 3, \ldots,$$

in contrast to the first part of Lemma 4.2.

Hence, the assumption (4.34) above is false, and Lemma 4.2 is proven.

4.3. *Proof of Lemma 4.3*

Since $\Lambda \in \Lambda(\sigma, \tau)$, by Lemmas 4.1 and 4.2 we obtain

$$\lim_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma)} = \lim_{n\in\Lambda, n\to\infty} \frac{n}{m_n(\sigma_n)} = 1,$$

which is equivalent to (4.1) and (4.2). It remains to prove (4.3).

As in the proof of Lemma 4.2, we fix $n_0 \in \mathbb{N}$ such that

$$\tau_n \geq \tau^* := \sigma + \frac{\tau - \sigma}{2}, \ n \geq n_0.$$

Next, we choose r and R such that

$$1 < \sigma < r < \tau^* \leq \tau_n \leq \tau < R < \rho(f), \ n \geq n_0.$$

In contrast to (4.3), let μ_σ be not the weak* limit of $\left\{\widehat{\mu_{\sigma,n}}\right\}_{n\in\Lambda}$:

Then there exists an infinite subset $\Lambda_1 \subset \Lambda$ such that μ_σ is not a weak* limit point of $\{\widehat{\mu_{\sigma,n}}\}_{n\in\Lambda_1}$ and we consider for $n \in \Lambda_1$ the solution $\phi(\widehat{\mu_{\sigma,n}};\cdot)$ of the Dirichlet problem in $E_R \setminus \overline{E_r}$ with boundary conditions

$$\phi(\widehat{\mu_{\sigma,n}}; z) = 0,\ z \in \Gamma_R,$$

and

$$\phi(\widehat{\mu_{\sigma,n}}; z) = \min\left(0,\ c(\widehat{\mu_{\sigma,n}}; \Gamma_R) - (U^{\widehat{\mu_{\sigma,n}}}(z) - U^{\mu_\sigma}(z))\right),\ z \in \Gamma_r,$$

where

$$c(\widehat{\mu_{\sigma,n}}; \Gamma_R) := \max_{t\in\Gamma_R} (U^{\widehat{\mu_{\sigma,n}}} - U^{\mu_\sigma})(t). \tag{4.35}$$

According to Lemma 3.5, there exists $\varepsilon > 0$ and $n_1(\varepsilon) \in \mathbb{N}$, $n_1(\varepsilon) \geq n_0$, such that for $n \geq n_1(\varepsilon)$, $n \in \Lambda_1$,

$$\max_{z\in D_{\tau^*,\tau}} \phi(\widehat{\mu_{\sigma,n}}; z) \leq -4\varepsilon. \tag{4.36}$$

Since $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, converge maximally to f on E, there exists $n_2(\varepsilon) \geq n_1(\varepsilon)$ such that

$$\frac{1}{n}\log\|f - p_n\|_{\Gamma_R} \leq \log\frac{R}{\rho(f)} + \varepsilon,\quad n \geq n_2(\varepsilon), n \in \Lambda_1. \tag{4.37}$$

If we interpolate $f - p_n$ with respect to $\mathcal{P}_{m_n(\sigma)-1}$ on the point set

$$Z_n(\sigma) = \left\{\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,m_n(\sigma)}\right\} \subset E_\sigma,$$

then 0 is the polynomial of interpolation and $f - p_n$ can be written for $z \in E_R$ and $n \in \Lambda_1$ as

$$f(z) - p_n(z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{w_{m_n(\sigma)}(z)}{w_{m_n(\sigma)}(t)}\,\frac{f(t) - p_n(t)}{t - z}\,dt$$

with

$$w_{m_n(\sigma)}(t) = \prod_{i=1}^{m_n(\sigma)} (t - \zeta_{n,i}),\quad t \in \mathbb{C}.$$

Since $\mu_{\sigma,n}$ denotes the normalized counting measure of $Z_n(\sigma)$ with balayage measure $\widehat{\mu_{\sigma,n}}$ on Γ_σ, we obtain for $z \in \Gamma_r$, analogously to (4.29),

$$\begin{aligned}\frac{1}{m_n(\sigma)}\log|f(z) - p_n(z)| \leq\ & \max_{t\in\Gamma_R} U^{\widehat{\mu_{\sigma,n}}}(t) - U^{\widehat{\mu_{\sigma,n}}}(z)\\ & + \frac{1}{m_n(\sigma)}\left(\log\|f - p_n\|_{\Gamma_R} + c_2\right),\end{aligned} \tag{4.38}$$

where

$$c_2 = \log\left[\max_{z\in\Gamma_r}\max_{t\in\Gamma_R}\frac{1}{|t - z|}\right] + \log\frac{\text{length}(\Gamma_R)}{2\pi}.$$

Using (4.37) and (4.38), we can choose $n_3(\varepsilon) \geq n_2(\varepsilon)$ such that for $z \in \Gamma_r$ and $n \geq n_3(\varepsilon)$, $n \in \Lambda_1$,

$$\begin{aligned}\frac{1}{m_n(\sigma)}\log|f(z) - p_n(z)| \leq\ & \max_{t\in\Gamma_R} U^{\widehat{\mu_{\sigma,n}}}(t) - U^{\widehat{\mu_{\sigma,n}}}(z)\\ & + \frac{n}{m_n(\sigma)}\log\frac{R}{\rho(f)} + 2\varepsilon.\end{aligned}$$

With analogous arguments as in (4.31) we obtain for $z \in \Gamma_r$ and $n \geq n_3(\varepsilon)$, $n \in \Lambda_1$,

$$\begin{aligned}\frac{1}{m_n(\sigma)} & \log|f(z)-p_n(z)| \\ & \leq \max_{t\in\Gamma_R} (U^{\widehat{\mu_{\sigma,n}}} - U^{\mu_\sigma})(t) - (U^{\widehat{\mu_{\sigma,n}}}(z) - U^{\mu_\sigma}(z)) \\ & \quad + \frac{n}{m_n(\sigma)} \log \frac{r}{\rho(f)} + 2\varepsilon + \log\frac{r}{R}\left[1 - \frac{n}{m_n(\sigma)}\right].\end{aligned}$$

Because of (4.1), there exists $n_4(\varepsilon) \geq n_3(\varepsilon)$ such that for $n \geq n_4(\varepsilon), n \in \Lambda_1$,

$$\log\frac{r}{R}\left[1 - \frac{n}{m_n(\sigma)}\right] \leq \varepsilon,$$

and therefore for $z \in \Gamma_r$ and $n \in \Lambda_1, n \geq n_4(\varepsilon)$,

$$\begin{aligned}\frac{1}{m_n(\sigma)} & \log|f(z)-p_n(z)| \\ & \leq \max_{t\in\Gamma_R} (U^{\widehat{\mu_{\sigma,n}}} - U^{\mu_\sigma})(t) - (U^{\widehat{\mu_{\sigma,n}}}(z) - U^{\mu_\sigma}(z)) \\ & \quad + \frac{n}{m_n(\sigma)} \log \frac{r}{\rho(f)} + 3\varepsilon.\end{aligned}$$

Using (4.35), we obtain for $z \in \Gamma_r$ and $n \geq n_4(\varepsilon), n \in \Lambda_1$,

$$\begin{aligned}\frac{1}{m_n(\sigma)} & \log|f(z)-p_n(z)| \\ & \leq c(\widehat{\mu_{\sigma,n}}; \Gamma_R) - (U^{\widehat{\mu_{\sigma,n}}}(z) - U^{\mu_\sigma}(z)) + \frac{n}{m_n(\sigma)} \log \frac{r}{\rho(f)} + 3\varepsilon,\end{aligned} \tag{4.39}$$

and for $z \in \Gamma_R$ and $n \geq n_4(\varepsilon), n \in \Lambda_1$, we can write with (4.37)

$$\frac{1}{m_n(\sigma)} \log|f(z)-p_n(z)| \leq \frac{n}{m_n(\sigma)} \log\frac{R}{\rho(f)} + 3\varepsilon. \tag{4.40}$$

Hence,

$$\frac{1}{m_n(\sigma)} \log|f(z)-p_n(z)|, \ z \in E_R \setminus \overline{E}_r,$$

is subharmonic and satisfies the boundary conditions (4.39) and (4.40) for $n \geq n_4(\varepsilon)$, $n \in \Lambda_1$. Consequently,

$$\frac{n}{m_n(\sigma)}(g_\Omega(z,\infty) - \log\rho(f)) + \phi(\widehat{\mu_{\sigma,n}}; z) + 3\varepsilon$$

is a harmonic majorant for the subharmonic function

$$\frac{1}{m_n(\sigma)} \log|f(z)-p_n(z)|, \ z \in E_R \setminus \overline{E}_r.$$

Therefore, we obtain for $z \in E_R \setminus \overline{E}_r$ and $n \geq n_4(\varepsilon), n \in \Lambda_1$,

$$\begin{aligned}\frac{1}{m_n(\sigma)} & \log|f(z)-p_n(z)| \\ & \leq \frac{n}{m_n(\sigma)}(g_\Omega(z,\infty) - \log\rho(f)) + \phi(\widehat{\mu_{\sigma,n}}; z) + 3\varepsilon.\end{aligned}$$

Hence, for $z \in D_{\tau^*,\tau}$ by (4.36)

$$\frac{1}{m_n(\sigma)} \log|f(z) - p_n(z)| \leq \frac{n}{m_n(\sigma)}(g_\Omega(z,\infty) - \log\rho(f)) - \varepsilon.$$

Since $\Gamma_{\tau_n} \subset D_{\tau^*,\tau}$ for all $n \geq n_4(\varepsilon), n \in \Lambda_1$, we get finally by (4.1)

$$\limsup_{n\in\Lambda_1, n\to\infty} \|f - p_n\|_{\Gamma_{\tau_n}}^{1/n} \leq \limsup_{n\in\Lambda_1, n\to\infty} \left(\frac{\tau_n}{\rho(f)} \left(e^{-\varepsilon}\right)^{\frac{m_n(\sigma)}{n}} \right) = \frac{\tau}{\rho(f)} e^{-\varepsilon},$$

and we have got a contradiction to the fact that $f - p_n$, $n \in \Lambda$, are near-circular at τ_- with associated sequence $\{\tau_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$, namely that

$$\liminf_{n\in\Lambda_1, n\to\infty} \|f - p_n\|_{\Gamma_{\tau_n}}^{1/n} \geq \liminf_{n\in\Lambda_1, n\to\infty} \min_{z\in\Gamma_{\tau_n}} |f(z) - p_n(z)|^{1/n} \geq \frac{\tau}{\rho(f)}.$$

Hence (4.3) holds and Lemma 4.3 is proven.

4.4. *Proof Lemma 4.4*

Since $f - p_n$, $n \in \Lambda$, are near-circular at σ_- with associated sequence $\{\sigma_n\}_{n\in\Lambda}$, connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$,

$$(f - p_n)(z) \neq 0\,, z \in \Gamma_{\sigma_n},\ n \in \Lambda.$$

Hence, $\gamma_n = (f - p_n)(\Gamma_{\sigma_n})$ is a closed, analytic curve and

$$\begin{aligned}\operatorname{Ind}_{\gamma_n}(0) &= \frac{1}{2\pi i}\int_{\gamma_n} \frac{dz}{z} = \frac{1}{2\pi i}\int_{\Gamma_{\sigma_n}} \frac{(f-p_n)'(t)}{(f-p_n)(t)} dt = m_n(\sigma_n) \\ &= n + o(n) \text{ as } n \in \Lambda, n \to \infty,\end{aligned}$$

using (4.2) of Lemma 4.3 in the last equality. Hence, Lemma 4.4 is proven.

5. Proof of the main theorem

Let us choose ρ, τ and β such that

$$1 < \rho < \sigma < \tau \leq \beta < \rho(f) < \infty.$$

Then according to Corollary 3.3, there exists $\Lambda \in \Lambda(\rho, \sigma, \tau)$, i.e., the functions $f - p_n$, $n \in \Lambda$, are near-circular at ρ_-, at σ_- and at τ_- with associated sequences $\{\rho_n\}_{n\in\Lambda}$, $\{\sigma_n\}_{n\in\Lambda}$, $\{\tau_n\}_{n\in\Lambda}$ connected with the sequence $\{\varepsilon_n\}_{n\in\Lambda}$.

Since $\Lambda \in \Lambda(\rho, \tau)$ and $\Lambda \in \Lambda(\sigma, \tau)$, we obtain by Lemma 4.3

$$m_n(\rho) = n + o(n) \text{ as } n \in \Lambda, n \to \infty, \tag{5.1}$$

$$m_n(\sigma) = n + o(n) \text{ as } n \in \Lambda, n \to \infty, \tag{5.2}$$

$$\widehat{\mu_{\sigma,n}} \xrightarrow{*} \mu_\sigma \text{ as } n \in \Lambda, n \to \infty.$$

For the balayage measures $\widehat{\mu_{\sigma,n|_E}}$ of $\mu_{\sigma,n|_E}$ onto ∂E we have

$$U^{\widehat{\mu_{\sigma,n|_E}}}(z) = U^{\mu_{\sigma,n|_E}}(z), \quad z \in \overline{\mathbb{C}} \setminus E,$$

and for the balayage measure $\widehat{\mu_{\sigma,n|_\Omega}}$ of $\mu_{\sigma,n|_\Omega}$ onto the boundary $\partial\Omega = \partial E$ we get

$$U^{\widehat{\mu_{\sigma,n|_\Omega}}}(z) = U^{\mu_{\sigma,n|_\Omega}}(z) + \frac{1}{m_n(\sigma)} \sum_{\zeta\in Z_n(\sigma)\cap\Omega} g_\Omega(\zeta,\infty), \quad z \in \overline{\mathbb{C}} \setminus E,$$

(cf. [13], Chapter II, Theorem 4.7). Let us define for $n \in \Lambda$

$$S_n(\sigma) := \frac{1}{m_n(\sigma)} \sum_{\zeta\in Z_n(\sigma)\cap\Omega} g_\Omega(\zeta,\infty),$$

then

$$\begin{aligned} S_n(\sigma) &= \frac{1}{m_n(\sigma)} \left(\sum_{\zeta\in Z_n(\sigma)\cap(E_\rho\setminus E)} g_\Omega(\zeta,\infty) + \sum_{\zeta\in Z_n(\sigma)\cap(\Omega\setminus E_\rho)} g_\Omega(\zeta,\infty) \right) \\ &\le \frac{1}{m_n(\sigma)} \left((m_n(\rho)\log\rho + (m_n(\sigma) - m_n(\rho))\log\sigma \right). \end{aligned} \tag{5.3}$$

Let $0 < \delta < 1$, then because of (5.1) and (5.2) there exists $n_0(\delta)$ such that for $n \ge n_0(\delta)$, $n \in \Lambda$,

$$n - \delta n \le m_n(\rho) \le n + \delta n, \tag{5.4}$$

$$n - \delta n \le m_n(\sigma) \le n + \delta n. \tag{5.5}$$

Hence, inserting (5.4) and (5.5) in (5.3), we obtain for $n \in \Lambda$, $n \ge n_0(\delta)$,

$$\begin{aligned} S_n(\sigma) &\le \frac{1}{m_n(\sigma)} \left((n+\delta n)\log\rho + (n+\delta n - (n-\delta n))\log\sigma \right) \\ &\le \frac{1+\delta}{1-\delta} (\log\rho + 2\delta\log\sigma) \\ &=: T(\rho,\sigma;\delta). \end{aligned} \tag{5.6}$$

Now, we consider a sequence

$$\{\rho^i\}_{i\in\mathbb{N}}, \ 1 < \rho^{i+1} < \rho^i < \sigma, \ \lim_{i\to\infty} \rho^i = 1.$$

According to Corollary 3.3, there exists $\Lambda^i \in \Lambda(\rho^i,\sigma,\tau)$.

Then we replace

$$1 < \rho < \sigma \ \text{ by } \ 1 < \rho^i < \sigma \ \text{ and } \ \Lambda \text{ by } \ \Lambda^i$$

and the parameter δ with properties (5.4) and (5.5) by

$$\delta^i := \frac{\log\rho^i}{2\log\sigma}.$$

Then $0 < \delta^i < 1$ and $\lim_{i\to\infty}\delta^i = 0$, and we can choose $n_i(\delta^i) \in \Lambda^i$, $i \in \mathbb{N}$, such that and

$$n - \delta^i n \le m_n(\rho^i) \le n + \delta^i n, \quad n \in \Lambda^i, n \ge n_i(\delta^i), \tag{5.7}$$

$$n - \delta^i n \le m_n(\sigma) \le n + \delta^i n, \quad n \in \Lambda^i, n \ge n_i(\delta^i), \tag{5.8}$$

$$\sigma - \frac{1}{\delta^i} \le \sigma_n^i \le \sigma, \quad n \in \Lambda^i, n \ge n_i(\delta^i), \tag{5.9}$$

$$\tau - \frac{1}{\delta^i} \le \tau_n^i \le \tau, \quad n \in \Lambda^i, n \ge n_i(\delta^i), \tag{5.10}$$

$$0 < \varepsilon_{n_i(\delta^i)}^i < \frac{1}{\delta^i}. \tag{5.11}$$

Moreover, we can arrange $n_i(\delta^i)$ such that $n_i(\delta^i) < n_{i+1}(\delta^{i+1})$, $i \in \mathbb{N}$. Define

$$\widetilde{\Lambda} := \{n_i(\delta^i)\}_{i \in \mathbb{N}}$$

and

$$\sigma_{n_i(\delta^i)} := \sigma_{n_i(\delta^i)}^i, \quad \tau_{n_i(\delta^i)} := \tau_{n_i(\delta^i)}^i, \quad \varepsilon_{n_i(\delta^i)} := \varepsilon_{n_i(\delta^i)}^i. \tag{5.12}$$

Since $n_i(\delta^i) \in \Lambda^i$, $i \in \mathbb{N}$,

$$\| f - p_{n_i(\delta_i)} \|_E^{1/n_i(\delta^i)} < \frac{1}{\rho(f)} e^{\varepsilon_{n_i(\delta^i)}}, \tag{5.13}$$

then (5.9)–(5.13) imply that $f - p_n$, $n \in \widetilde{\Lambda}$, are near circular at σ_- and at τ_- with associated sequence $\{\sigma_n\}_{n\in\widetilde{\Lambda}}$, $\{\tau_n\}_{n\in\widetilde{\Lambda}}$ connected with $\{\varepsilon_n\}_{n\in\widetilde{\Lambda}}$. Hence, by Lemma 4.3 the properties (4.1)–(4.3) hold, where Λ is replaced by $\widetilde{\Lambda}$, i.e., (2.1) and (2.2) of the Main Theorem are proved for $\Lambda = \widetilde{\Lambda}$ and

$$\widehat{\mu_{\sigma,n}} \xrightarrow{*} \mu_\sigma \text{ as } n \in \widetilde{\Lambda}, n \to \infty. \tag{5.14}$$

Next, because of

$$\operatorname{supp}(\widehat{\mu_{\sigma,n|_E}} + \mu_{\sigma,n|_\Omega}) \subset \overline{E_\sigma},$$

there exists by Helly's Theorem a subset $\Lambda_1 \subset \widetilde{\Lambda}$ and a Borel measure $\nu \in \mathcal{M}(\overline{E}_\sigma)$ such that

$$\widehat{\mu_{\sigma,n|_E}} + \mu_{\sigma,n|_\Omega} \xrightarrow[n\in\Lambda_1, n\to\infty]{*} \nu. \tag{5.15}$$

Because of (5.7) and (5.8), we apply (5.6) and get for $n = n_i(\delta^i) \in \widetilde{\Lambda}$

$$S_n(\sigma) = S_{n_i(\delta^i)}(\sigma) \le T(\rho^i, \sigma; \delta^i) = \frac{2(1+\delta^i)}{(1-\delta^i)} \log \rho^i$$

and

$$\lim_{i\to\infty} T(\rho^i, \sigma; \delta^i) = 0.$$

Let K be compact in Ω, then

$$\gamma := \min_{z\in K} g_\Omega(z, \infty) > 0$$

and

$$T(\rho^i, \sigma; \delta^i) \ge S_n(\sigma) \ge \gamma\, \mu_{\sigma,n}(K), \quad n = n_i(\delta^i) \in \widetilde{\Lambda},$$

or

$$\mu_{\sigma,n}(K) \le \frac{1}{\gamma}\, T(\rho^i, \sigma; \delta^i), \quad n = n_i(\delta^i) \in \widetilde{\Lambda}.$$

Then

$$\lim_{i\to\infty} \mu_{\sigma,n_i(\delta^i)}(K) \le \frac{1}{\gamma} \lim_{i\to\infty} T(\rho^i, \sigma; \delta^i) = 0,$$

and consequently, $\mathrm{supp}(\nu) \subset E$ for the Borel measure ν in (5.15). By (5.14) and (5.15) we obtain for $z \in \overline{\mathbb{C}} \setminus \overline{E_\sigma}$

$$\begin{aligned} U^\nu(z) &= \lim_{n\in\Lambda_1, n\to\infty} \left(U^{\widehat{\mu_{\sigma,n|E}}}(z) + U^{\mu_{\sigma,n|\Omega}}(z) \right) \\ &= \lim_{n\in\Lambda_1, n\to\infty} \left(U^{\mu_{\sigma,n|E}}(z) + U^{\mu_{\sigma,n|\Omega}}(z) \right) \\ &= \lim_{n\in\Lambda_1, n\to\infty} U^{\mu_{\sigma,n}}(z) \\ &= U^{\mu_\sigma}(z). \end{aligned} \tag{5.16}$$

Moreover, for $z \in \overline{\mathbb{C}} \setminus \overline{E_\sigma}$

$$U^{\mu_\sigma}(z) = -g_{\Omega(\sigma)}(z,\infty) - \log \mathrm{cap}\, E - \log \sigma = -g_\Omega(z,\infty) - \log \mathrm{cap}\, E,$$

where $g_{\Omega(\sigma)}(z,\infty)$ is the Green's function of $\Omega(\sigma) = \Omega \setminus \overline{E_\sigma}$ with pole at ∞, and we obtain by (5.16)

$$U^\nu(z) = -g_\Omega(z,\infty) - \log \mathrm{cap}\, E = U^{\mu_E}(z), \ z \in \overline{\mathbb{C}} \setminus \overline{E_\sigma}.$$

Now, $U^\nu(z)$ and $U^{\mu_E}(z)$ are harmonic functions on $\mathbb{C} \setminus E$, coinciding on $\mathbb{C} \setminus \overline{E_\sigma}$ and the identity principle for harmonic functions implies

$$U^\nu(z) = U^{\mu_E}(z), \ z \in \overline{\mathbb{C}} \setminus E.$$

Then by Carleson's Unicity Theorem, implicitly contained in [7] and extended by Cornea [8] (cf. [13], Chapter II, Theorem 4.13), we obtain

$$\nu = \mu_E. \tag{5.17}$$

Since (5.17) holds for any weak* limit point ν of

$$\left\{ \widehat{\mu_{\sigma,n|E}} + \mu_{\sigma,n|\Omega} \right\}_{n\in\widetilde{\Lambda}},$$

we have got

$$\widehat{\mu_{\sigma,n|E}} + \mu_{\sigma,n|\Omega} \xrightarrow[n\in\widetilde{\Lambda}, n\to\infty]{*} \mu_E.$$

Hence if we set $\Lambda = \widetilde{\Lambda}$, then $\Lambda \in \Lambda(\sigma,\tau)$ and $\Lambda(\sigma,\tau) \subset \Lambda(\sigma)$ imply together with Lemma 4.4 that Λ satisfies the properties (2.1)–(2.4) and the Main Theorem is proven.

Data availability

Data will be made available on request.

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