



# Generalized Cesàro operators in weighted Banach spaces of analytic functions with sup-norms

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## Abstract

An investigation is made of the generalized Cesàro operators  $C_t$ , for  $t \in [0, 1]$ , when they act on the space  $H(\mathbb{D})$  of holomorphic functions on the open unit disc  $\mathbb{D}$ , on the Banach space  $H^\infty$  of bounded analytic functions and on the weighted Banach spaces  $H_v^\infty$  and  $H_v^0$  with their sup-norms. Of particular interest are the continuity, compactness, spectrum and point spectrum of  $C_t$  as well as their linear dynamics and mean ergodicity.

**Keywords** Generalized Cesàro operator · Weighted Banach spaces of analytic functions · Compact operator · Spectrum · Supercyclic · Mean ergodic · Power bounded

**Mathematics Subject Classification** Primary 46E15, 47B38; Secondary 46E10, 47A10, 47A16, 47A35

## 1 Introduction and preliminaries

The (discrete) generalized Cesàro operators  $C_t$ , for  $t \in [0, 1]$ , were first investigated by Rhaly [25, 26]. The action of  $C_t$  from the sequence space  $\omega := \mathbb{C}^{\mathbb{N}_0}$  into itself, with  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ , is given by

$$C_t x := \left( \frac{t^n x_0 + t^{n-1} x_1 + \dots + x_n}{n+1} \right)_{n \in \mathbb{N}_0}, \quad x = (x_n)_{n \in \mathbb{N}_0} \in \omega. \quad (1.1)$$

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For  $t = 0$  and with  $\varphi := (\frac{1}{n+1})_{n \in \mathbb{N}_0}$  note that  $C_0$  is the diagonal operator

$$D_\varphi x := \left( \frac{x_n}{n+1} \right)_{n \in \mathbb{N}_0}, \quad x = (x_n)_{n \in \mathbb{N}_0} \in \omega, \quad (1.2)$$

and, for  $t = 1$ , that  $C_1$  is the classical Cesàro averaging operator

$$C_1 x := \left( \frac{x_0 + x_1 + \cdots + x_n}{n+1} \right)_{n \in \mathbb{N}_0}, \quad x = (x_n)_{n \in \mathbb{N}_0} \in \omega. \quad (1.3)$$

The behaviour of  $C_t$  on various sequence spaces has been investigated by many authors. We refer the reader to [25–27], to the recent papers [28, 30, 31] and to the introduction of the papers [5, 13] and the references therein. The operator  $C_1$  was thoroughly investigated on weighted Banach spaces in [2]; see also [12]. Certain variants of the Cesàro operator  $C_1$  are considered in [9, 16].

Our aim is to investigate the operators  $C_t$ , for  $t \in [0, 1]$ , when they are suitably interpreted to act on the space  $H(\mathbb{D})$  of holomorphic functions on the open unit disc  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ , on the Banach space  $H^\infty$  of bounded analytic functions and on the weighted Banach spaces  $H_v^\infty$  and  $H_v^0$  with their sup-norms. The space  $H(\mathbb{D})$  is equipped with the topology  $\tau_c$  of uniform convergence on the compact subsets of  $\mathbb{D}$ . According to [21, §27.3(3)] the space  $H(\mathbb{D})$  is a Fréchet–Montel space. A family of norms generating  $\tau_c$  is given, for each  $0 < r < 1$ , by

$$q_r(f) := \sup_{|z| \leq r} |f(z)|, \quad f \in H(\mathbb{D}). \quad (1.4)$$

A *weight*  $v$  is a continuous, non-increasing function  $v: [0, 1] \rightarrow (0, \infty)$ . We extend  $v$  to  $\mathbb{D}$  by setting  $v(z) := v(|z|)$ , for  $z \in \mathbb{D}$ . Note that  $v(z) \leq v(0)$  for all  $z \in \mathbb{D}$ . Given a weight  $v$  on  $[0, 1)$ , we define the corresponding *weighted Banach spaces of analytic functions* on  $\mathbb{D}$  by

$$H_v^\infty := \{f \in H(\mathbb{D}) : \|f\|_{\infty, v} := \sup_{z \in \mathbb{D}} |f(z)|v(z) < \infty\},$$

and

$$H_v^0 := \{f \in H(\mathbb{D}) : \lim_{|z| \rightarrow 1^-} |f(z)|v(z) = 0\},$$

both endowed with the norm  $\|\cdot\|_{\infty, v}$ . Since  $\|f\|_{\infty, v} \leq v(0)\|f\|_\infty$  whenever  $f \in H^\infty$ , it is clear that  $H^\infty \subseteq H_v^\infty$  with a continuous inclusion. If  $v(z) = 1$  for all  $z \in \mathbb{D}$ , then  $H_v^\infty$  coincides with the space  $H^\infty$  of all bounded analytic functions on  $\mathbb{D}$  with the sup-norm  $\|\cdot\|_\infty$  and  $H_v^0$  reduces to  $\{0\}$ . Moreover,  $H_v^\infty \subseteq H(\mathbb{D})$  continuously. Indeed, fix  $0 < r < 1$ . Then  $\frac{1}{v(0)} \leq \frac{1}{v(z)} \leq \frac{1}{v(r)}$  for  $|z| \leq r$  and so (1.4) implies that

$$q_r(f) = \sup_{|z| \leq r} \frac{v(z)|f(z)|}{v(z)} \leq \frac{1}{v(r)} \sup_{|z| \leq r} v(z)|f(z)| \leq \frac{1}{v(r)} \|f\|_{\infty, v}, \quad f \in H_v^\infty.$$

We refer the reader to [10] for a recent survey of such types of weighted Banach spaces and operators between them.

Whenever necessary we will identify a function  $f \in H(\mathbb{D})$  with its sequence of Taylor coefficients  $\hat{f} := (\hat{f}(n))_{n \in \mathbb{N}_0}$  (i.e.,  $\hat{f}(n) := \frac{f^{(n)}(0)}{n!}$ , for  $n \in \mathbb{N}_0$ ), so that  $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ , for  $z \in \mathbb{D}$ . The linear map  $\Phi: H(\mathbb{D}) \rightarrow \omega$  is defined by

$$\Phi \left( f = \sum_{n=0}^{\infty} \hat{f}(n)z^n \right) := \hat{f}, \quad f \in H(\mathbb{D}).$$

It is injective (clearly) and continuous. Indeed, for each  $m \in \mathbb{N}_0$ ,

$$r_m(x) := \max_{0 \leq j \leq m} |x_j|, \quad x = (x_j)_{j \in \mathbb{N}_0} \in \omega,$$

is a continuous seminorm in  $\omega$ . Fix  $0 < r < 1$ , in which case

$$\begin{aligned} r_m(\Phi(f)) &= \max_{0 \leq j \leq m} |\hat{f}(j)| = \max_{0 \leq j \leq m} \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{j+1}} dz \right| \leq \max_{0 \leq j \leq m} \sup_{|z|=r} \frac{|f(z)|}{|z|^j} \\ &= \max_{0 \leq j \leq m} \frac{1}{r^j} q_r(f) = \frac{1}{r^m} q_r(f), \end{aligned}$$

for each  $f \in H(\mathbb{D})$  because  $\frac{1}{r^j} \leq \frac{1}{r^m}$  for all  $0 \leq j \leq m$ . Of course, the increasing sequence of seminorms  $\{r_m \mid m \in \mathbb{N}_0\}$  generates the topology of  $\omega$ .

We first provide an integral representation of the generalized Cesàro operators  $C_t$  defined on  $H(\mathbb{D})$ , for  $t \in [0, 1)$ . So, fix  $t \in [0, 1)$  and define  $C_t: H(\mathbb{D}) \rightarrow H(\mathbb{D})$  by  $C_t f(0) := f(0)$  and

$$C_t f(z) := \frac{1}{z} \int_0^z \frac{f(\xi)}{1-t\xi} d\xi, \quad z \in \mathbb{D} \setminus \{0\}, \quad (1.5)$$

for every  $f \in H(\mathbb{D})$ . It turns out that  $C_t$  is continuous on  $H(\mathbb{D})$ ; see Proposition 2.1. Moreover, the discrete Cesàro operator  $C_t: \omega \rightarrow \omega$ , when restricted to the subspace  $\Phi(H(\mathbb{D})) \subseteq \omega$  is transferred to  $H(\mathbb{D})$  as follows. For a fixed  $f \in H(\mathbb{D})$  we have  $f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$ , for  $\xi \in \mathbb{D}$ , with  $\hat{f} = (a_n)_{n \in \mathbb{N}_0}$  its sequence of Taylor coefficients. Since  $\frac{1}{1-t\xi} = \sum_{n=0}^{\infty} t^n \xi^n$ , for  $\xi \in \mathbb{D}$ , we can form the Cauchy product of the two series, thereby obtaining

$$\frac{f(\xi)}{1-t\xi} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n t^{n-k} a_k \right) \xi^n, \quad \xi \in \mathbb{D}.$$

Then (1.5) yields

$$z C_t f(z) = \int_0^z \sum_{n=0}^{\infty} \left( \sum_{k=0}^n t^{n-k} a_k \right) \xi^n d\xi = \sum_{n=0}^{\infty} \left( \frac{t^n a_0 + t^{n-1} a_1 + \cdots + a_n}{n+1} \right) z^{n+1}, \quad z \in \mathbb{D}.$$

The interchange of the infinite sum and the integral is permissible by uniform convergence of the series. This shows that  $C_t f \in H(\mathbb{D})$  also has the series representation

$$\begin{aligned} C_t f(z) &= \sum_{n=0}^{\infty} \left( \frac{t^n a_0 + t^{n-1} a_1 + \cdots + a_n}{n+1} \right) z^n \\ &= \sum_{n=0}^{\infty} \left( \frac{t^n \hat{f}(0) + t^{n-1} \hat{f}(1) + \cdots + \hat{f}(n)}{n+1} \right) z^n = \sum_{n=0}^{\infty} (C_t^{\omega}(\hat{f}))_n z^n, \end{aligned} \quad (1.6)$$

where the coefficients of the series are precisely as in (1.1). For the sake of clarity we will denote the discrete generalized Cesàro operator  $C_t: \omega \rightarrow \omega$  by  $C_t^{\omega}$  and reserve the notation  $C_t$  for the operator (1.5) acting in  $H(\mathbb{D})$ . Note that  $C_0^{\omega} = D_{\varphi}$  (see (1.2)). Moreover,  $C_0$  is given by  $C_0 f(z) = \frac{1}{z} \int_0^z f(\xi) d\xi$  for  $z \neq 0$  and  $C_0 f(0) = f(0)$ , which is the classical Hardy operator in  $H(\mathbb{D})$ .

The main results for  $C_t$  when acting in the Fréchet space  $H(\mathbb{D})$  occur in Proposition 2.1 (continuity), Proposition 3.3 (non-compactness), Proposition 3.7 (spectra) and Proposition 3.8 (linear dynamics and mean ergodicity). For the analogous information concerning  $C_t$  when acting in the weighted Banach spaces  $H_v^{\infty}$  and  $H_v^0$  see Proposition 2.4 and Corollary

2.5 (continuity), Proposition 2.7 (compactness), Proposition 2.8 (spectra) and Proposition 3.2 (linear dynamics and mean ergodicity).

We end this section by recalling a few definitions and some notation concerning locally convex spaces and operators between them. For further details about functional analysis and operator theory relevant to this paper see, for example, [15, 18, 20–22, 29].

Given locally convex Hausdorff spaces  $X, Y$  (briefly, lcHs) we denote by  $\mathcal{L}(X, Y)$  the space of all continuous linear operators from  $X$  into  $Y$ . If  $X = Y$ , then we simply write  $\mathcal{L}(X)$  for  $\mathcal{L}(X, X)$ . Equipped with the topology of pointwise convergence on  $X$  (i.e., the strong operator topology) the lcHs  $\mathcal{L}(X)$  is denoted by  $\mathcal{L}_s(X)$ . Equipped with the topology  $\tau_b$  of uniform convergence on the bounded subsets of  $X$  the lcHs  $\mathcal{L}(X)$  is denoted by  $\mathcal{L}_b(X)$ .

Let  $X$  be a lcHs space. The identity operator on  $X$  is denoted by  $I$ . The *transpose operator* of  $T \in \mathcal{L}(X)$  is denoted by  $T'$ ; it acts from the topological dual space  $X' := \mathcal{L}(X, \mathbb{C})$  of  $X$  into itself. Denote by  $X'_\sigma$  (resp., by  $X'_\beta$ ) the topological dual  $X'$  equipped with the weak\* topology  $\sigma(X', X)$  (resp., with the strong topology  $\beta(X', X)$ ); see [21, §21.2] for the definition. It is known that  $T' \in \mathcal{L}(X'_\sigma)$  and  $T' \in \mathcal{L}(X'_\beta)$ , [22, p. 134]. The bi-transpose operator  $(T')'$  of  $T$  is simply denoted by  $T''$  and belongs to  $\mathcal{L}(X'_\beta)'$ .

A linear map  $T: X \rightarrow Y$ , with  $X, Y$  lcHs', is called *compact* if there exists a neighbourhood  $\mathcal{U}$  of 0 in  $X$  such that  $T(\mathcal{U})$  is a relatively compact set in  $Y$ . It is routine to show that necessarily  $T \in \mathcal{L}(X, Y)$ . We recall the following well known result; see [20, Proposition 17.1.1], [22, §42.1(1)].

**Lemma 1.1** *Let  $X$  be a lcHs. The compact operators are a 2-sided ideal in  $\mathcal{L}(X)$ .*

Given a lcHs  $X$  and  $T \in \mathcal{L}(X)$ , the resolvent set  $\rho(T; X)$  of  $T$  consists of all  $\lambda \in \mathbb{C}$  such that  $R(\lambda, T) := (\lambda I - T)^{-1}$  exists in  $\mathcal{L}(X)$ . The set  $\sigma(T; X) := \mathbb{C} \setminus \rho(T; X)$  is called the *spectrum* of  $T$ . The *point spectrum*  $\sigma_{pt}(T; X)$  of  $T$  consists of all  $\lambda \in \mathbb{C}$  (also called an eigenvalue of  $T$ ) such that  $(\lambda I - T)$  is not injective. Some authors (eg. [29]) prefer the subset  $\rho^*(T; X)$  of  $\rho(T; X)$  consisting of all  $\lambda \in \mathbb{C}$  for which there exists  $\delta > 0$  such that the open disc  $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\} \subseteq \rho(T; X)$  and  $\{R(\mu, T) : \mu \in B(\lambda, \delta)\}$  is an equicontinuous subset of  $\mathcal{L}(X)$ . Define  $\sigma^*(T; X) := \mathbb{C} \setminus \rho^*(T; X)$ , which is a closed set with  $\sigma(T; X) \subseteq \sigma^*(T; X)$ . For the spectral theory of compact operators in lcHs' we refer to [15, 18], for linear dynamics to [6], [17] and for mean ergodic operators to [23], for example.

## 2 Continuity, compactness and spectrum of $C_t$

In this section we establish, for  $t \in [0, 1)$ , the continuity of  $C_t: H(\mathbb{D}) \rightarrow H(\mathbb{D})$  as well as the continuity of  $C_t$  from  $H^\infty$  (resp.,  $H_v^\infty$ ) into  $H^\infty$  (resp.,  $H_v^\infty$ ). The same is true for  $C_t: H_v^0 \rightarrow H_v^0$  whenever  $\lim_{r \rightarrow 1^-} v(r) = 0$ . It is also shown that the bi-transpose  $C_t''$  of  $C_t \in \mathcal{L}(H_v^\infty)$  is the generalized Cesàro operator  $C_t \in \mathcal{L}(H_v^\infty)$ , provided that  $\lim_{r \rightarrow 1^-} v(r) = 0$ . For such weights  $v$  it also turns out that both  $C_t \in \mathcal{L}(H_v^0)$  and  $C_t \in \mathcal{L}(H_v^\infty)$  are compact operators (cf. Proposition 2.7); their spectrum is identified in Proposition 2.8. Of particular interest are the standard weights  $v_\gamma(z) := (1 - |z|)^\gamma$ , for  $\gamma > 0$  and  $z \in \mathbb{D}$ .

**Proposition 2.1** *For every  $t \in [0, 1)$  the operator  $C_t: H(\mathbb{D}) \rightarrow H(\mathbb{D})$  is continuous. Moreover, the set  $\{C_t : t \in [0, 1)\}$  is equicontinuous in  $\mathcal{L}(H(\mathbb{D}))$ .*

**Proof** Fix  $f \in H(\mathbb{D})$ . Taking into account that  $C_t f(0) = f(0)$ , for all  $t \in [0, 1)$  and, for each  $r \in (0, 1)$ , that  $\sup_{|z| \leq r} |C_t f(z)| = \sup_{|z|=r} |C_t f(z)|$ , the formula (1.5) implies, for

each  $z \in \mathbb{D} \setminus \{0\}$ , that

$$\begin{aligned} |C_t f(z)| &= \frac{1}{|z|} \left| \int_0^z \frac{f(\xi)}{1-t\xi} d\xi \right| \leq \frac{1}{|z|} |z| \max_{\xi \in [0, z]} \frac{|f(\xi)|}{|1-t\xi|} \\ &\leq \frac{1}{1-|z|} \max_{|\xi| \leq |z|} |f(\xi)| = \frac{1}{1-|z|} \max_{|\xi|=|z|} |f(\xi)|, \end{aligned}$$

because  $|1-t\xi| \geq 1-t|\xi| \geq 1-|\xi| \geq 1-|z|$ , for all  $|\xi| \leq |z|$ . It follows from the previous inequality, for each  $r \in (0, 1)$ , that

$$q_r(C_t f) = \sup_{|z| \leq r} |C_t f(z)| \leq \frac{1}{1-r} \sup_{|\xi| \leq r} |f(\xi)| = \frac{1}{1-r} q_r(f); .$$

see (1.4). This implies the result.  $\square$

The following example will prove to be useful in the sequel.

**Example 2.2** Consider the constant function  $f_1(z) := 1$ , for every  $z \in \mathbb{D}$ , in which case  $C_t f_1(0) = f_1(0) = 1$  for every  $t \in [0, 1]$ . For  $t = 0$ , it was noted in Sect. 1 that  $C_0$  is the Hardy operator. In particular,  $C_0 f_1(z) = 1$ , for every  $z \in \mathbb{D}$ . For  $t \in (0, 1]$ , note that  $C_t f_1(0) = 1$  and

$$C_t f_1(z) = \frac{1}{z} \int_0^z \frac{d\xi}{1-t\xi} = -\frac{1}{tz} \log(1-tz), \quad z \in \mathbb{D} \setminus \{0\}.$$

For  $t = 1$  this shows, in particular, that  $C_1(H^\infty) \not\subset H^\infty$ , which is well known. For an investigation of the operator  $C_1$  acting in  $H^\infty$  we refer to [14].

Concerning  $t \in (0, 1)$ , recall the Taylor series expansion

$$-\log(1-z) = z \sum_{n=0}^{\infty} \frac{z^n}{n+1}, \quad z \in \mathbb{D},$$

from which it follows that

$$-\frac{\log(1-tz)}{tz} = \sum_{n=0}^{\infty} \frac{t^n}{n+1} z^n, \quad z \in \mathbb{D} \setminus \{0\},$$

with the series having radius of convergence  $\frac{1}{t} > 1$ . The claim is that  $\|C_t f_1\|_\infty = \sup_{|z|<1} |C_t f_1(z)| = -\frac{\log(1-t)}{t}$ . Indeed,  $C_t f_1$  is clearly holomorphic in  $B(0, \frac{1}{t}) := \{\xi \in \mathbb{C} : |\xi| < \frac{1}{t}\}$  hence, continuous in  $B(0, \frac{1}{t})$ , and satisfies  $C_t f_1(1) = -\frac{\log(1-t)}{t}$  with  $\lim_{r \rightarrow 1^-} C_t f_1(r) = C_t f_1(1)$ . On the other hand, for every  $z \in \mathbb{D} \setminus \{0\}$  and  $t \in (0, 1)$  we have that

$$|C_t f_1(z)| = \left| -\frac{\log(1-tz)}{tz} \right| \leq \sum_{n=0}^{\infty} \frac{t^n}{n+1} |z|^n \leq \sum_{n=0}^{\infty} \frac{t^n}{n+1} = -\frac{\log(1-t)}{t}.$$

This completes the proof of the claim. Observe that  $\|C_t f_1\|_\infty > 1$ . Indeed, define  $\gamma(t) = -\log(1-t) - t$ , for  $t \in [0, 1)$ . Then  $\gamma(0) = 0$ ,  $\lim_{t \rightarrow 1^-} \gamma(t) = \infty$  and  $\gamma'(t) = \frac{1}{1-t} - 1 = \frac{t}{1-t}$ , for  $t \in [0, 1)$ . Since  $\gamma'(t) > 0$ , for  $t \in (0, 1)$ , it follows that  $\gamma$  is strictly increasing and so  $\gamma(t) > 0$  for all  $t \in (0, 1)$ . This implies that  $\|C_t f_1\|_\infty = -\frac{\log(1-t)}{t} > 1$  for every

$t \in (0, 1)$ . On the other hand, for  $t \in (0, 1)$ , the inequality  $\sum_{n=0}^{\infty} t^n / (n + 1) < \sum_{n=0}^{\infty} t^n$  implies that  $-\frac{\log(1-t)}{t} < \frac{1}{1-t}$ . So, we have shown that  $\|C_0 f_1\|_{\infty} = 1$  and

$$1 < \|C_t f_1\|_{\infty} < \frac{1}{1-t}, \quad t \in (0, 1).$$

We now turn to the action of  $C_t$  in various Banach spaces. For  $t = 1$  it was noted above that  $C_1$  fails to act in  $H^\infty$ .

**Proposition 2.3** *For  $t \in [0, 1)$  the operator  $C_t: H^\infty \rightarrow H^\infty$  is continuous. Moreover,  $\|C_0\|_{H^\infty \rightarrow H^\infty} = 1$  and*

$$\|C_t\|_{H^\infty \rightarrow H^\infty} = -\frac{\log(1-t)}{t}, \quad t \in (0, 1).$$

**Proof** Let  $f \in H^\infty$  be fixed. Then

$$|C_0 f(z)| = \left| \frac{1}{z} \int_0^z f(\xi) d\xi \right| \leq \max_{|\xi| \leq |z|} |f(\xi)| \leq \|f\|_{\infty}.$$

This implies that  $\|C_0\|_{H^\infty \rightarrow H^\infty} \leq 1$ . On the other hand,  $C_0 f_1 = f_1$  and so we can conclude that  $\|C_0\|_{H^\infty \rightarrow H^\infty} = 1$ .

Now let  $t \in (0, 1)$ . Then, for the parametrization  $\xi := sz$ , for  $s \in (0, 1)$ , it follows from  $|1 - stz| \geq 1 - |stz| \geq 1 - st$  that

$$\begin{aligned} |C_t f(z)| &= \left| \frac{1}{z} \int_0^z \frac{f(\xi)}{1 - t\xi} d\xi \right| = \left| \int_0^1 \frac{f(sz)}{1 - stz} ds \right| \leq \max_{|\xi| \leq |z|} |f(\xi)| \int_0^1 \frac{ds}{1 - st|z|} \\ &\leq \|f\|_{\infty} \int_0^1 \frac{ds}{1 - st} = -\frac{\log(1-t)}{t} \|f\|_{\infty}. \end{aligned}$$

So,  $C_t \in \mathcal{L}(H^\infty)$  with  $\|C_t\|_{H^\infty \rightarrow H^\infty} \leq -\frac{\log(1-t)}{t}$ . But,  $\|C_t f_1\|_{\infty} = -\frac{\log(1-t)}{t}$ . Accordingly,  $\|C_t\|_{H^\infty \rightarrow H^\infty} = -\frac{\log(1-t)}{t}$ .  $\square$

**Proposition 2.4** *Let  $v$  be a weight function on  $[0, 1]$ . For each  $t \in [0, 1)$  the operator  $C_t: H_v^\infty \rightarrow H_v^\infty$  is continuous. Moreover,  $\|C_0\|_{H_v^\infty \rightarrow H_v^\infty} = 1$  and*

$$1 \leq \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \leq -\frac{\log(1-t)}{t}, \quad t \in (0, 1).$$

**Proof** Recall that  $C_t f(0) := f(0)$  for each  $f \in H(\mathbb{D})$  and  $t \in [0, 1]$ . Fix  $t \in (0, 1)$ . Given  $f \in H_v^\infty$  and  $z \in \mathbb{D} \setminus \{0\}$ , observe that

$$\begin{aligned} v(z)|C_t f(z)| &= \frac{v(z)}{|z|} \left| \int_0^z \frac{f(\xi)}{1 - t\xi} d\xi \right| = v(z) \left| \int_0^1 \frac{f(sz)}{1 - stz} ds \right| \\ &\leq v(z) \int_0^1 \frac{|f(sz)|}{|1 - stz|} ds \leq \int_0^1 \frac{v(sz)|f(sz)|}{|1 - stz|} ds \\ &\leq \|f\|_{\infty, v} \int_0^1 \frac{ds}{|1 - stz|} \leq \|f\|_{\infty, v} \int_0^1 \frac{ds}{1 - st|z|} \\ &= -\frac{\log(1 - t|z|)}{t|z|} \|f\|_{\infty, v}, \end{aligned}$$

where we used that  $v(sz) = v(s|z|) \geq v(|z|) = v(z)$ , for  $s \in (0, 1)$ , as  $v$  is non-increasing on  $(0, 1)$  and that  $|1 - sz| \geq 1 - st|z|$ , for  $s \in (0, 1)$ . According to the calculations in Example 2.2 we can conclude that

$$\|C_t f\|_{\infty, v} = \sup_{z \in \mathbb{D}} |C_t f(z)|v(z) \leq \|f\|_{\infty, v} \sup_{z \in \mathbb{D}} \left[ -\frac{\log(1 - t|z|)}{t|z|} \right] = -\frac{\log(1 - t)}{t} \|f\|_{\infty, v}.$$

This implies that  $C_t \in \mathcal{L}(H_v^\infty)$  and  $\|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \leq -\frac{\log(1-t)}{t}$ .

For  $t = 0$  observe that

$$|C_0 f(z)| \leq \int_0^1 |f(sz)|ds \leq \max_{|\xi| \leq |z|} |f(\xi)| = \frac{1}{v(z)} \max_{|\xi|=|z|} |f(\xi)|v(\xi) \leq \frac{1}{v(z)} \|f\|_{\infty, v},$$

as  $v(\xi) = v(z)$  whenever  $|\xi| = |z|$  with  $\xi \in \mathbb{D}$ . This shows that  $\|C_0\|_{H_v^\infty \rightarrow H_v^\infty} \leq 1$ . Since  $C_0 f_1 = f_1$ , it follows that actually  $\|C_0\|_{H_v^\infty \rightarrow H_v^\infty} = 1$ .

It remains to show that  $\|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \geq 1$  for  $t \in (0, 1)$ . To this end, fix  $t \in (0, 1)$  and consider the function  $g_0(z) := \frac{1}{1-tz} = \sum_{n=0}^{\infty} t^n z^n$ , for  $z \in \mathbb{D}$ . Then  $\|g_0\|_\infty = \frac{1}{1-t}$  and so  $g_0 \in H^\infty \subseteq H_v^\infty$ . Moreover, for every  $z \in \mathbb{D} \setminus \{0\}$ , it is the case that

$$C_t g_0(z) = \frac{1}{z} \int_0^z \frac{d\xi}{(1-t\xi)^2} = \frac{1}{z} \left[ \frac{1}{t(1-t\xi)} \right]_0^z = \frac{1}{tz} \left[ \frac{1}{1-tz} - 1 \right] = \frac{1}{1-tz} = g_0(z).$$

It follows that  $\|g_0\|_{\infty, v} = \|C_t g_0\|_{\infty, v} \leq \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \|g_0\|_{\infty, v}$  which implies that  $\|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \geq 1$ .  $\square$

**Corollary 2.5** *Let  $v$  be a weight function on  $[0, 1)$  satisfying  $\lim_{r \rightarrow 1^-} v(r) = 0$ . For each  $t \in [0, 1)$  the operator  $C_t: H_v^0 \rightarrow H_v^0$  is continuous and satisfies  $\|C_t\|_{H_v^0 \rightarrow H_v^0} = \|C_t\|_{H_v^\infty \rightarrow H_v^\infty}$ .*

**Proof** By Proposition 2.4 and the fact that  $H_v^0$  is a closed subspace of  $H_v^\infty$ , to obtain the result it suffices to establish that  $C_t(H_v^0) \subseteq H_v^0$ . To this effect, observe that  $H^\infty \subseteq H_v^0$  and that  $H^\infty$  is dense in  $H_v^0$ , as the space of polynomials is dense in  $H_v^0$ ; see Section 1 of [11] and also [7]. Proposition 2.3 implies that  $C_t(H^\infty) \subseteq H^\infty \subseteq H_v^0$ . Since  $C_t$  acts continuously on  $H_v^\infty$ , it follows that

$$C_t(H_v^0) = C_t(\overline{H^\infty}) \subseteq \overline{C_t(H^\infty)} \subseteq H_v^0.$$

Moreover,  $\lim_{r \rightarrow 1^-} v(r) = 0$  implies that  $H_v^\infty$  is canonically isometric to the bidual of  $H_v^0$ , [8, Example 2.1], and that the bi-transpose  $C_t'': H_v^\infty \rightarrow H_v^\infty$  of  $C_t: H_v^0 \rightarrow H_v^0$  coincides with  $C_t: H_v^\infty \rightarrow H_v^\infty$  (see Lemma 2.6 below), from which the identity  $\|C_t\|_{H_v^0 \rightarrow H_v^0} = \|C_t\|_{H_v^\infty \rightarrow H_v^\infty}$  follows.  $\square$

**Lemma 2.6** *Let  $v$  be a weight function on  $[0, 1)$  satisfying  $\lim_{r \rightarrow 1^-} v(r) = 0$ . For each  $t \in [0, 1)$ , the bi-transpose  $C_t'': H_v^\infty \rightarrow H_v^\infty$  of  $C_t: H_v^0 \rightarrow H_v^0$  coincides with  $C_t: H_v^\infty \rightarrow H_v^\infty$ .*

**Proof** By Proposition 2.3 and Corollary 2.5, together with the fact that  $H_v^\infty$  is canonically isometric to the bidual of  $H_v^0$ , both of the operators  $C_t''$  and  $C_t$  act continuously on  $H_v^\infty$ .

To show that the bi-transpose  $C_t'': H_v^\infty \rightarrow H_v^\infty$  of  $C_t: H_v^0 \rightarrow H_v^0$  coincides with  $C_t: H_v^\infty \rightarrow H_v^\infty$  we proceed via several steps.

*First step* Given  $f \in H(\mathbb{D})$ , its Taylor polynomials  $p_k(z) = \sum_{j=0}^k \hat{f}(j)z^j$ ,  $z \in \mathbb{D}$ , for  $k \in \mathbb{N}_0$ , converge to  $f$  uniformly on compact subsets of  $\mathbb{D}$ . That is,  $p_k \rightarrow f$  in  $(H(\mathbb{D}), \tau_c)$  as  $k \rightarrow \infty$ . Accordingly, the averages of  $(p_k)_{k \in \mathbb{N}_0}$ , that is, the Cesàro means  $f_n(z) := \frac{1}{n+1} \sum_{j=0}^n p_j(z)$ , for  $z \in \mathbb{D}$  and  $n \in \mathbb{N}_0$ , also converge to  $f$  in  $(H(\mathbb{D}), \tau_c)$  as  $n \rightarrow \infty$ .

*Second step* Lemma 1.1 in [7] implies, for every  $f \in H_v^\infty$  and  $n \in \mathbb{N}_0$ , that  $\|f_n\|_{\infty,v} \leq \|f\|_{\infty,v}$ , where  $f_n$  is the  $n$ -th Cesàro mean of  $f$ , as defined in the *First step*. Denote by  $U_v$  the closed unit ball of  $(H_v^\infty, \|\cdot\|_{\infty,v})$ . Then, for any given  $f \in U_v$ , its sequence of Cesàro means satisfies  $(f_n)_{n \in \mathbb{N}_0} \subseteq U_v$  and  $f_n \rightarrow f$  in  $(H(\mathbb{D}), \tau_c)$  as  $n \rightarrow \infty$ .

*Third step* With the topology of uniform convergence on the compact subsets of  $U_v$  denoted by  $\tau_c$ , let  $X := \{F \in (H_v^\infty)': F|_{U_v} \text{ is } \tau_c\text{-continuous}\}$  be endowed with the norm  $\|F\| := \sup\{|F(f)| : f \in U_v\}$ . Then [8, Theorem 1.1(a)] ensures that  $(X, \|\cdot\|)$  is a Banach space and that the evaluation map  $\Psi: H_v^\infty \rightarrow X'$  defined by  $(\Psi(f))(F) := \langle f, F \rangle$ , for  $F \in X$  and  $f \in H_v^\infty$ , is an isometric isomorphism onto  $X'$  (where  $X'$  is the dual Banach space of  $(X, \|\cdot\|)$ ). Moreover, by [8, Theorem 1.1(b) and Example 2.1] the restriction map  $R: X \rightarrow (H_v^0)'$  given by  $F \mapsto F|_{H_v^0}$ , is also a surjective isometric isomorphism. Therefore, the spaces  $H_v^\infty$  and  $(H_v^0)''$  are isometrically isomorphic, that is,  $X$  and  $(H_v^0)'$  are isometrically isomorphic and hence, also  $H_v^\infty$  and  $(H_v^0)''$  are isometrically isomorphic.

It is easy to see, since the Banach space  $X$  above is the predual of  $H_v^\infty$ , that the evaluation map  $\delta_z \in X$ , for every  $z \in \mathbb{D}$ , where  $\delta_z: f \mapsto f(z)$ , for  $f \in H_v^\infty$ , satisfies  $|\langle f, \delta_z \rangle| \leq \|f\|_{\infty,v}/v(z)$ . In particular, the linear span  $L$  of the set  $\{\delta_z : z \in \mathbb{D}\}$  separates the points of  $H_v^\infty = X'$  and hence,  $L$  is dense in  $X$ . Therefore, the pointwise convergence topology  $\tau_p$  on  $H_v^\infty$  is Hausdorff and coarser than the  $w^*$ -topology  $\sigma(H_v^\infty, X)$ .

*Fourth step* The closed unit ball  $U_v$  of  $H_v^\infty$  is a  $\tau_c$ -compact set by Montel's theorem, as it is  $\tau_c$ -bounded and closed. On the other hand,  $U_v$  is also  $\sigma(H_v^\infty, X)$ -compact by the Alaoglu-Bourbaki theorem. Since  $\tau_p|_{U_v}$  is coarser than  $\tau_c|_{U_v}$  and Hausdorff, we can conclude that  $\tau_p|_{U_v} = \tau_c|_{U_v}$ . In the same way, it follows that  $\tau_p|_{U_v} = \sigma(H_v^\infty, X)|_{U_v}$ . Accordingly,  $\tau_p|_{U_v} = \tau_c|_{U_v} = \sigma(H_v^\infty, X)|_{U_v}$ .

We are now ready to prove that  $(C_t)'' = C_t$ . To show this, it suffices to establish that  $(C_t)''f = C_t f$  for every  $f \in U_v$ .

So, fix  $f \in U_v$ . With  $(f_n)_{n \in \mathbb{N}_0}$  as in the *First step* it follows from there that  $f_n \rightarrow f$  in  $(H(\mathbb{D}), \tau_c)$  as  $n \rightarrow \infty$  and, by the *Second step*, that  $(f_n)_{n \in \mathbb{N}_0} \subseteq U_v$ . This implies that  $C_t f_n \rightarrow C_t f$  in  $(H(\mathbb{D}), \tau_c)$  as  $n \rightarrow \infty$ . Since  $C_t \in \mathcal{L}(H_v^\infty)$  and  $f \in U_v$ , it is clear that  $C_t f \in H_v^\infty$ . On the other hand, by the *Fourth step* the sequence  $(f_n)_{n \in \mathbb{N}_0}$  also converges to  $f$  in  $(H_v^\infty, \sigma(H_v^\infty, X)) = (H_v^\infty, \sigma(H_v^\infty, (H_v^0)'))$ . Since  $(C_t)'' : ((H_v^0)'', \sigma((H_v^0)'', (H_v^0)')) \rightarrow ((H_v^0)'', \sigma((H_v^0)'', (H_v^0)'))$  is continuous, [20, §8.6], that is,  $(C_t)'' : (H_v^\infty, \sigma(H_v^\infty, X)) \rightarrow (H_v^\infty, \sigma(H_v^\infty, X))$  is continuous, it follows that  $(C_t)'' f_n \rightarrow (C_t)'' f$  in  $(H_v^\infty, \sigma(H_v^\infty, X))$  as  $n \rightarrow \infty$ . Now,  $(f_n)_{n \in \mathbb{N}_0} \subset H^\infty \subseteq H_v^0$ , as each  $f_n$  is a polynomial, and  $(C_t)'' f_n = C_t f_n$  for every  $n \in \mathbb{N}_0$ . Moreover, the sequence  $C_t f_n \rightarrow (C_t)'' f$  in  $(H(\mathbb{D}), \tau_p)$  as  $n \rightarrow \infty$ . Thus,  $(C_t)'' f = C_t f$  as desired.  $\square$

**Proposition 2.7** *Let  $v$  be a weight function satisfying  $\lim_{r \rightarrow 1^-} v(r) = 0$ . For each  $t \in [0, 1)$ , both of the operators  $C_t: H_v^\infty \rightarrow H_v^\infty$  and  $C_t: H_v^0 \rightarrow H_v^0$  are compact.*

**Proof** Fix  $t \in [0, 1)$ . Since  $H_v^0$  is a closed subspace of  $H_v^\infty$  and  $C_t(H_v^0) \subseteq H_v^0$  (cf. Corollary 2.5), it suffices to show that  $C_t: H_v^\infty \rightarrow H_v^\infty$  is compact. First we establish the following Claim:

(\*) Let the sequence  $(f_n)_{n \in \mathbb{N}} \subset H_v^\infty$  satisfy  $\|f_n\|_{\infty,v} \leq 1$  for every  $n \in \mathbb{N}$  and  $f_n \rightarrow 0$  in  $(H(\mathbb{D}), \tau_c)$  for  $n \rightarrow \infty$ . Then  $C_t f_n \rightarrow 0$  in  $H_v^\infty$ .

To prove the Claim, let  $(f_n)_{n \in \mathbb{N}} \subset H_v^\infty$  be a sequence as in (\*). Fix  $\varepsilon > 0$  and select  $\delta \in (0, \beta)$ , where  $\beta := \min\{1, \frac{\varepsilon(1-t)}{2}, \frac{\varepsilon(1-t)}{2v(0)}\}$ . Since  $\{\xi \in \mathbb{C} \mid |\xi| \leq (1 - \delta)\}$  is a compact subset of  $\mathbb{D}$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\max_{|\xi| \leq 1-\delta} |f_n(\xi)| < \delta, \quad n \geq n_0.$$

Recall that  $C_t f_n(0) = f_n(0)$  for every  $n \in \mathbb{N}$ . For  $z \in \mathbb{D} \setminus \{0\}$  we have seen previously that

$$v(z)|C_t f_n(z)| = v(z) \left| \int_0^1 \frac{f_n(sz)}{1-stz} ds \right| \leq v(z) \int_0^{1-\delta} \frac{|f_n(sz)|}{|1-stz|} ds + v(z) \int_{1-\delta}^1 \frac{|f_n(sz)|}{|1-stz|} ds.$$

Denote the first (resp., second) summand in the right-side of the previous inequality by  $(A_n)$  (resp., by  $(B_n)$ ). Using the facts that  $|1-stz| \geq 1-st|z| \geq \max\{1-s, 1-t, 1-|z|\}$ , for all  $s, t \in [0, 1)$  and  $z \in \mathbb{D}$ , and that  $v$  is non-increasing on  $[0, 1)$  it follows, for every  $n \geq n_0$ , that  $\int_0^{1-\delta} |f_n(sz)| ds \leq (1-\delta) \max_{|\xi| \leq (1-\delta)} |f_n(\xi)|$  (as  $|sz| \leq (1-\delta)$  for all  $s \in [0, 1-\delta]$ ) and hence, that

$$(A_n) \leq \frac{v(0)(1-\delta)}{1-t} \max_{|\xi| \leq 1-\delta} |f_n(\xi)| < \frac{\varepsilon}{2}.$$

On the other hand, for every  $n \geq n_0$ , we have (as  $\|f_n\|_{\infty, v} = \sup_{\xi \in \mathbb{D}} v(\xi) |f_n(\xi)| \leq 1$ ) that

$$(B_n) = \int_{1-\delta}^1 \frac{v(z)}{v(sz)} \frac{v(sz)|f_n(sz)|}{|1-stz|} ds \leq \int_{1-\delta}^1 \frac{\|f_n\|_{\infty, v}}{1-t} ds \leq \frac{\delta}{1-t} < \frac{\varepsilon}{2}.$$

It follows that  $\|C_t f_n\|_{\infty, v} < \varepsilon$  for every  $n \geq n_0$ . That is,  $C_t f_n \rightarrow 0$  in  $H_v^\infty$  for  $n \rightarrow \infty$  and so  $(*)$  is proved.

The compactness of  $C_t \in \mathcal{L}(H_v^\infty)$  can be deduced from  $(*)$  as follows. Let  $(f_n)_{n \in \mathbb{N}} \subset H_v^\infty$  be any bounded sequence. There is no loss of generality in assuming that  $\|f_n\|_{\infty, v} \leq 1$  for all  $n \in \mathbb{N}$ . To establish the compactness of  $C_t \in \mathcal{L}(H_v^\infty)$  we need to show that  $(C_t f_n)_{n \in \mathbb{N}}$  has a convergent subsequence in  $H_v^\infty$ .

Since  $H_v^\infty \subseteq H(\mathbb{D})$  continuously, the sequence  $(f_n)_{n \in \mathbb{N}}$  is also bounded in the Fréchet–Montel space  $H(\mathbb{D})$ . Hence, there is a subsequence  $g_j := f_{n_j}$ , for  $j \in \mathbb{N}$ , of  $(f_n)_{n \in \mathbb{N}}$  and  $f \in H(\mathbb{D})$  such that  $g_j \rightarrow f$  in  $H(\mathbb{D})$  with respect to  $\tau_c$ . In particular,  $g_j \rightarrow f$  pointwise on  $\mathbb{D}$ . Since  $v(z)|g_j(z)| = v(z)|f_{n_j}(z)| \leq 1$  for all  $z \in \mathbb{D}$  and  $j \in \mathbb{N}$ , letting  $j \rightarrow \infty$  it follows that  $v(z)|f(z)| \leq 1$  for all  $z \in \mathbb{D}$ , that is,  $f \in H_v^\infty$  with  $\|f\|_{\infty, v} \leq 1$ . Let  $h_j := \frac{1}{2}(g_j - f)$ , for  $j \in \mathbb{N}$ . Then  $\|h_j\|_{\infty, v} \leq 1$ , for  $j \in \mathbb{N}$ , and  $h_j \rightarrow 0$  in  $H(\mathbb{D})$  with respect to  $\tau_c$ . Condition  $(*)$  implies that  $C_t h_j \rightarrow 0$  in  $H_v^\infty$  from which it follows that  $C_t f_{n_j} = C_t g_j = C_t(g_j - f) + C_t f = 2C_t h_j + C_t f \rightarrow C_t f$  in  $H_v^\infty$ , as desired.  $\square$

**Proposition 2.8** *Let  $v$  be a weight function on  $[0, 1)$  satisfying  $\lim_{r \rightarrow 1^-} v(r) = 0$ . For each  $t \in [0, 1)$  the spectra of  $C_t \in \mathcal{L}(H_v^\infty)$  and of  $C_t \in \mathcal{L}(H_v^0)$  are given by*

$$\sigma_{pt}(C_t; H_v^\infty) = \sigma_{pt}(C_t; H_v^0) = \left\{ \frac{1}{m+1} : m \in \mathbb{N}_0 \right\}, \quad (2.1)$$

and

$$\sigma(C_t; H_v^\infty) = \sigma(C_t; H_v^0) = \left\{ \frac{1}{m+1} : m \in \mathbb{N}_0 \right\} \cup \{0\}. \quad (2.2)$$

**Proof** Let  $t \in [0, 1)$  be fixed. By [13, Lemma 3.6] we know that the point spectrum of the operator  $C_t^\omega \in \mathcal{L}(\omega)$  is given by  $\sigma_{pt}(C_t^\omega; \omega) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$  and, for each  $m \in \mathbb{N}_0$ , that the corresponding eigenspace  $\text{Ker}(\frac{1}{m+1} I - C_t^\omega)$  is 1-dimensional and is generated by an eigenvector  $x^{[m]} = (x_n^{[m]})_{n \in \mathbb{N}_0} \in \ell^1$ . Since  $H_v^0 \subseteq H_v^\infty \subseteq H(\mathbb{D})$  with continuous inclusions and  $\Phi: H(\mathbb{D}) \rightarrow \omega$  (cf. Sect. 1) is a continuous embedding, this implies that  $\sigma_{pt}(C_t; H_v^0) \subseteq \sigma_{pt}(C_t; H_v^\infty) \subseteq \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$ . Indeed, let  $f \in H(\mathbb{D}) \setminus \{0\}$  and  $\lambda \in \mathbb{C}$  satisfy  $C_t f = \lambda f$ . Then  $\lambda f(z) = \sum_{n=0}^{\infty} (\widehat{\lambda f})(n) z^n = \sum_{n=0}^{\infty} \lambda \widehat{f}(n) z^n$  and, by (1.6), we have that  $(C_t f)(z) = \sum_{n=0}^{\infty} (C_t^\omega \widehat{f})_n z^n$ . It follows that  $C_t^\omega \widehat{f} = \lambda \widehat{f}$  in  $\omega$  with  $\widehat{f} \neq 0$  and so  $\lambda \in \sigma_{pt}(C_t^\omega; \omega) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$ .

To conclude the proof, it remains to show that  $\{\frac{1}{m+1} : m \in \mathbb{N}_0\} \subseteq \sigma_{pt}(C_t; H_v^0)$ . To establish this recall, for each  $m \in \mathbb{N}_0$ , that the eigenvector  $x^{[m]} \in \ell^1$  and hence, the function  $g_m(z) := \sum_{n=0}^{\infty} (x^{[m]})_n z^n$  belongs to  $H_v^0$  because  $0 \leq v(z)|g_m(z)| \leq v(z)\|x^{[m]}\|_{\ell^1}$  for  $z \in \mathbb{D}$  and  $\lim_{r \rightarrow 1^-} v(r) = 0$ . Moreover, according to (1.5) and (1.6) we have, for each  $z \in \mathbb{D}$ , that

$$C_t g_m(z) = \sum_{n=0}^{\infty} (C_t x^{[m]})_n z^n = \sum_{n=0}^{\infty} \left( \frac{1}{m+1} x^{[m]} \right)_n z^n = \frac{1}{m+1} \sum_{n=0}^{\infty} (x^{[m]})_n z^n = \frac{1}{m+1} g_m(z).$$

Thus  $g_m$  is an eigenvector of  $C_t \in \mathcal{L}(H_v^0)$  corresponding to the eigenvalue  $\frac{1}{m+1}$ .

The validity of  $\sigma(C_t; H_v^0) = \sigma(C_t; H_v^\infty) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\} \cup \{0\}$  follows from the fact that  $C_t$  is a compact operator on both spaces.  $\square$

We now investigate the norm of  $C_t$  on  $H_v^\infty$  for the standard weights  $v_\gamma(z) := (1 - |z|)^\gamma$ , for  $\gamma > 0$  and  $z \in \mathbb{D}$ , which satisfy  $\lim_{r \rightarrow 1^-} v_\gamma(r) = 0$ .

**Proposition 2.9** *Let  $t \in (0, 1)$  and  $\gamma > 0$ .*

- (i) *The operator norm  $\|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} = 1$ , for every  $\gamma \geq 1$ .*
- (ii) *For each  $\gamma \in (0, 1)$ , the inequality  $\|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} \leq \min\{-\frac{\log(1-t)}{t}, \frac{1}{\gamma}\}$  is valid.*

**Proof** We adapt the arguments given for the Cesàro operator  $C_1$  in the proof of [2, Theorem 2.3].

Let  $\gamma > 0$  and  $t \in (0, 1)$  be fixed. For  $f \in H_{v_\gamma}^\infty$  with  $\|f\|_{\infty, v_\gamma} = 1$  we have

$$\begin{aligned} |C_t f(z)| &= \frac{1}{|z|} \left| \int_0^1 \frac{f(sz)}{1 - stz} ds \right| \leq \int_0^1 \frac{|f(sz)|}{1 - st|z|} ds \\ &\leq \int_0^1 \frac{|f(sz)|}{1 - s|z|} ds \leq \int_0^1 \frac{ds}{(1 - s|z|)^{\gamma+1}} = \frac{1}{(1 - |z|)^\gamma} \frac{1 - (1 - |z|)^\gamma}{\gamma|z|}, \end{aligned}$$

as  $z \in \mathbb{D}$  implies that  $1 - st|z| \geq 1 - s|z|$ , for  $s \in (0, 1)$ . Accordingly,

$$v_\gamma(z)|C_t f(z)| = (1 - |z|)^\gamma |C_t f(z)| \leq \frac{1 - (1 - |z|)^\gamma}{\gamma|z|}, \quad z \neq 0,$$

and hence,

$$\|C_t f\|_{\infty, v_\gamma} \leq \frac{1}{\gamma} \sup_{z \in \mathbb{D}} \frac{1 - (1 - |z|)^\gamma}{|z|}.$$

Define  $\phi(s) := \frac{1 - (1 - s)^\gamma}{s}$  for  $s \in (0, 1]$  and  $\phi(0) = \gamma$ , in which case  $\phi$  is continuous. So, the previous inequality yields  $\|C_t f\|_{\infty, v_\gamma} \leq \frac{M_\gamma}{\gamma}$ , for all  $\|f\|_{\infty, v_\gamma} \leq 1$ , that is,  $\|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} \leq \frac{M_\gamma}{\gamma}$ , where  $M_\gamma := \sup_{s \in [0, 1]} \phi(s)$ . Proposition 2.4 yields that  $1 \leq \|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} \leq -\frac{\log(1-t)}{t}$  for  $t \in (0, 1)$ . On page 101 of [2] it is shown that  $\frac{M_\gamma}{\gamma} \leq 1$  whenever  $\gamma \geq 1$  and that  $M_\gamma \leq 1$  for all  $\gamma \in (0, 1)$ . The proof of both parts (i) and (ii) follows immediately.  $\square$

**Remark 2.10** For each  $\gamma > 0$  let  $v_\gamma(z) = (1 - |z|)^\gamma$ , for  $z \in \mathbb{D}$ . Proposition 2.9 implies that  $\sup_{0 \leq t < 1} \|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} < \infty$ . Moreover, if  $\gamma \geq 1$ , then  $\|C_t^n\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} = 1$  for every  $n \in \mathbb{N}$ ; see case (i) in the proof of [2, Theorem 2.3] together with the fact that  $1 \in \sigma_{pt}(C_t, H_{v_\gamma}^\infty)$  by Proposition 2.8.

Let  $n \in \mathbb{N}$  be fixed. Consider the weight  $v(z) = (\log \frac{e}{1-|z|})^{-n}$ , for  $z \in \mathbb{D}$ , which satisfies  $v(0) = 1$  and  $\lim_{|z| \rightarrow 1^-} v(z) = 0$ .

The function  $f(z) := [\log(1-z)]^n \in H(\mathbb{D})$  belongs to  $H_v^\infty$ . Indeed, for each  $z \in \mathbb{D}$ , we have that

$$|\log(1-z)| = \left| -\sum_{n=1}^{\infty} \frac{z^n}{n} \right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n} = -\log(1-|z|)$$

and hence, that  $|f(z)| = |\log(1-z)|^n \leq (-\log(1-|z|))^n$ . Since  $v$  is given by  $v(z) = (1-\log(1-|z|))^{-n}$  and  $\lim_{|z| \rightarrow 1^-} \frac{-\log(1-|z|)}{1-\log(1-|z|)} = 1$ , it follows that  $\|f\|_{\infty, v} < \infty$  and so  $f \in H_v^\infty$ . On the other hand,

$$C_1 f(z) = \frac{1}{z} \int_0^z \frac{(\log(1-\xi))^n}{1-\xi} d\xi = -\frac{1}{(n+1)z} (\log(1-z))^{n+1}, \quad z \in \mathbb{D}.$$

Accordingly,  $C_1 f \notin H_v^\infty$  since

$$\begin{aligned} \lim_{s \rightarrow z^-} v(s) |(C_1 f)(s)| &= \frac{1}{n+1} \lim_{s \rightarrow 1^-} \left| \frac{(\log(1-s))^{n+1}}{s(1-\log(1-s))^n} \right| \\ &= \frac{1}{n+1} \lim_{s \rightarrow 1^-} \left| \left( \frac{\log(1-s)}{1-\log(1-s)} \right)^n \frac{\log(1-s)}{s} \right| = \infty. \end{aligned}$$

This implies that the Cesàro operator  $C_1$  is not well-defined on  $H_v^\infty$ , that is,  $C_1(H_v^\infty) \not\subseteq H_v^\infty$ . But, by Proposition 2.4 the generalized Cesàro operator  $C_t \in \mathcal{L}(H_v^\infty)$  for every  $t \in [0, 1)$ . At this point, the following question arises: *Is  $\sup_{t \in [0, 1)} \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} < \infty$  for this particular  $v$ ?* Our next two results show that the answer is negative for certain weights  $v$ , which includes  $v(z) = \left(\log \frac{e}{1-|z|}\right)^{-n}$  for  $z \in \mathbb{D}$ .

**Proposition 2.11** *Let  $v$  be a weight function on  $[0, 1)$  such that  $\sup_{t \in [0, 1)} \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} < \infty$ . Then  $C_1 \in \mathcal{L}(H_v^\infty)$ .*

**Proof** Proposition 2.1 implies that  $\{C_t : t \in [0, 1)\}$  is equicontinuous in  $\mathcal{L}(H(\mathbb{D}))$ . The claim is that  $\lim_{t \rightarrow 1^-} C_t f(z) = C_1 f(z)$ , for every  $f \in H(\mathbb{D})$  and  $z \in \mathbb{D}$ .

To prove this claim fix  $f \in H(\mathbb{D})$  and  $z \in \mathbb{D} \setminus \{0\}$ . Recall, for  $t \in [0, 1)$ , that

$$C_t f(z) = \frac{1}{z} \int_0^z \frac{f(\xi)}{1-t\xi} d\xi = \int_0^1 \frac{f(sz)}{1-stz} ds$$

and

$$C_1 f(z) = \frac{1}{z} \int_0^z \frac{f(\xi)}{1-\xi} d\xi = \int_0^1 \frac{f(sz)}{1-sz} ds.$$

Moreover, for each  $z \in \mathbb{D} \setminus \{0\}$ , we have (as  $|1-stz| \geq (1-|z|)$ ) that

$$\left| \frac{f(sz)}{1-stz} \right| \leq \frac{|f(sz)|}{1-|z|} \leq \frac{1}{1-|z|} \max_{|\xi| \leq |z|} |f(\xi)|, \quad s \in [0, 1],$$

and that  $\lim_{t \rightarrow 1^-} \frac{f(sz)}{1-stz} = \frac{f(sz)}{1-sz}$  for every  $s \in [0, 1]$ . So, we can apply the dominated convergence theorem to conclude that  $\lim_{t \rightarrow 1^-} C_t f(z) = C_1 f(z)$  for  $z \in \mathbb{D} \setminus \{0\}$ . For  $z = 0$  we have  $C_t f(0) = f(0) = C_1 f(0)$  for each  $f \in H(\mathbb{D})$  and  $t \in [0, 1)$ . So, for each  $f \in H(\mathbb{D})$ , we can conclude that  $C_t f \rightarrow C_1 f$  pointwise on  $\mathbb{D}$  for  $t \rightarrow 1^-$ . The claim is thereby established.

We now show that  $C_t f \rightarrow C_1 f$  in  $H(\mathbb{D})$  as  $t \rightarrow 1^-$  for every  $f \in H_v^\infty$ . The assumption  $\sup_{t \in [0, 1)} \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} < \infty$  implies that there exists  $M > 0$  satisfying  $\|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \leq M$  for every  $t \in [0, 1)$ . Therefore,

$$\sup_{z \in \mathbb{D}} |C_t f(z)| v(z) \leq M \|f\|_{\infty, v}, \quad f \in H_v^\infty, \quad t \in [0, 1). \quad (2.3)$$

Fix  $f \in H_v^\infty$ . Then  $\{C_t f : t \in [0, 1)\}$  is a bounded set in  $H(\mathbb{D})$ . Indeed, given  $r \in (0, 1)$  and  $t \in [0, 1)$  we have (as  $v(r) \leq v(z)$  for all  $|z| \leq r$ ) that

$$q_r(C_t f) = \sup_{|z| \leq r} |C_t f(z)| = \max_{|z|=r} |C_t f(z)| \leq \frac{M}{v(r)} \|f\|_{\infty, v}.$$

So, the set  $\{C_t f : t \in [0, 1)\}$  is bounded in the Fréchet–Montel space  $H(\mathbb{D})$  and hence, it is relatively compact in  $H(\mathbb{D})$ . Since  $C_t f \rightarrow C_1 f$  pointwise on  $\mathbb{D}$  for  $t \rightarrow 1^-$ , it follows that  $C_t f \rightarrow C_1 f$  with respect to  $\tau_c$ , that is, in the Fréchet space  $H(\mathbb{D})$ , for  $t \rightarrow 1^-$ . In particular,  $C_1 f \in H(\mathbb{D})$ .

Since  $H_v^\infty \subseteq H(\mathbb{D})$  and  $C_t h \rightarrow C_1 h$  pointwise on  $\mathbb{D}$  as  $t \rightarrow 1^-$ , for every  $h \in H(\mathbb{D})$ , letting  $t \rightarrow 1^-$  in (2.3) it follows that

$$|C_1 f(z)| v(z) \leq M \|f\|_{\infty, v}, \quad z \in \mathbb{D},$$

that is,  $\|C_1 f\|_{\infty, v} \leq M \|f\|_{\infty, v}$ . But,  $f \in H_v^\infty$  is arbitrary and so  $C_1 \in \mathcal{L}(H_v^\infty)$ .  $\square$

**Proposition 2.12** *For each  $n \in \mathbb{N}$ , let  $v(z) = (\log(\frac{e}{1-|z|}))^{-n}$  for  $z \in \mathbb{D}$ . Then  $\sup_{t \in [0, 1)} \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} = \infty$ .*

**Proof** Apply Proposition 2.11 and the discussion prior it.  $\square$

### 3 Linear dynamics and mean ergodicity of $C_t$

The aim of this section is to investigate the mean ergodicity and the linear dynamics of the operators  $C_t$ , for  $t \in [0, 1)$ , acting on  $H(\mathbb{D})$ ,  $H_v^\infty$  and  $H_v^0$ .

An operator  $T \in \mathcal{L}(X)$ , with  $X$  a lcHs, is called *power bounded* if  $\{T^n : n \in \mathbb{N}_0\}$  is an equicontinuous subset of  $\mathcal{L}(X)$ . For a Banach space  $X$ , this means that  $\sup_{n \in \mathbb{N}_0} \|T^n\|_{X \rightarrow X} < \infty$ . Given  $T \in \mathcal{L}(X)$ , the averages

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^n T^m, \quad n \in \mathbb{N},$$

are usually called the Cesàro means of  $T$ . The operator  $T$  is said to be *mean ergodic* (resp., *uniformly mean ergodic*) if  $(T_{[n]})_{n \in \mathbb{N}}$  is a convergent sequence in  $\mathcal{L}_s(X)$  (resp., in  $\mathcal{L}_b(X)$ ). It is routine to check that  $\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n} T_{[n-1]}$ , for  $n \geq 2$ , and hence,  $\tau_s\text{-}\lim_{n \rightarrow \infty} \frac{T^n}{n} = 0$  whenever  $T$  is mean ergodic. Every power bounded operator on a Fréchet–Montel space  $X$  is necessarily uniformly mean ergodic, [1, Proposition 2.8]. Concerning the linear dynamics of  $T \in \mathcal{L}(X)$ , with  $X$  a lcHs, the operator  $T$  is called *supercyclic* if, for some  $z \in X$ , the projective orbit  $\{\lambda T^n z : \lambda \in \mathbb{C}, n \in \mathbb{N}_0\}$  is dense in  $X$ . Since the closure of the linear span of a projective orbit is separable, if such a supercyclic operator  $T \in \mathcal{L}(X)$  exists, then  $X$  is necessarily separable.

Observe that the space  $H_v^\infty$  is never separable, [24, Theorem 1.1]. Therefore, every operator  $T \in \mathcal{L}(H_v^\infty)$  is clearly not supercyclic. However, the spaces  $H(\mathbb{D})$ , [21, Theorem 27.2.5],

and  $H_v^0$ , [24, Theorem 1.1], for every weight  $v$  are always separable. Hence, the problem of supercyclicity for non-zero operators  $T \in \mathcal{L}(H(\mathbb{D}))$  and  $T \in \mathcal{L}(H_v^0)$  arises.

The following result, [5, Theorem 6.4], is stated here for Banach spaces.

**Theorem 3.1** *Let  $X$  be a Banach space and let  $T \in \mathcal{L}(X)$  be a compact operator such that  $1 \in \sigma(T; X)$  with  $\sigma(T; X) \setminus \{1\} \subseteq \overline{B(0, \delta)}$  for some  $\delta \in (0, 1)$  and satisfying  $\text{Ker}(I - T) \cap \text{Im}(I - T) = \{0\}$ . Then  $T$  is power bounded and uniformly mean ergodic.*

A consequence of the previous theorem is the following result.

**Proposition 3.2** *Let  $v$  be a weight function on  $[0, 1)$  satisfying  $\lim_{r \rightarrow 1^-} v(r) = 0$ . For each  $t \in [0, 1)$  both of the operators  $C_t \in \mathcal{L}(H_v^\infty)$  and  $C_t \in \mathcal{L}(H_v^0)$  are power bounded, uniformly mean ergodic and fail to be supercyclic.*

**Proof** Fix  $t \in [0, 1)$ . It was already noted that  $C_t \in \mathcal{L}(H_v^\infty)$  cannot be supercyclic. The operator  $C_t$  is a compact operator on both  $H_v^\infty$  and on  $H_v^0$  (cf. Proposition 2.7). Therefore, the compact transpose operators  $C'_t \in \mathcal{L}((H_v^\infty)')$  and  $C'_t \in \mathcal{L}((H_v^0)')$  have the same non-zero eigenvalues as  $C_t$  (see, e.g., [15, Theorem 9.10-2(2)]). In view of Proposition 2.8 it follows that  $\sigma_{pt}(C'_t; (H_v^\infty)') = \sigma_{pt}(C'_t; (H_v^0)') = \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$ . We can apply [6, Proposition 1.26] to conclude that  $C_t$  is not supercyclic on  $H_v^0$ .

By Proposition 2.8 and its proof (as  $x^{[0]} = (t^n)_{n \in \mathbb{N}_0}$ ) we have that  $\text{Ker}(I - C_t) = \text{span}\{g_0\}$ , with  $g_0(z) = \sum_{n=0}^{\infty} t^n z^n$ , for  $z \in \mathbb{D}$ . On the other hand,  $\text{Im}(I - C_t)$  is a closed subspace of  $H_v^\infty$  (resp., of  $H_v^0$ ), as  $C_t$  is compact in  $H_v^\infty$  (resp., in  $H_v^0$ ), and  $\text{Im}(I - C_t) \subseteq \{g \in H_v^\infty : g(0) = 0\}$  (resp.,  $\subseteq \{g \in H_v^0 : g(0) = 0\}$ ), because  $C_t f(0) = f(0)$  for each  $f \in H_v^\infty$  (resp., each  $f \in H_v^0$ ). Moreover, [15, Theorem 9.10.1] implies that  $\text{codim } \text{Im}(I - C_t) = \dim \text{Ker}(I - C_t) = 1$ . Accordingly, both  $\text{Im}(I - C_t)$  and  $\{g \in H_v^\infty : g(0) = 0\} = \text{Ker}(\delta_0)$  are hyperplanes, where  $\delta_0 \in (H_v^\infty)'$  is the linear evaluation functional  $f \mapsto f(0)$ , for  $f \in H_v^\infty$ . It follows that necessarily  $\text{Im}(I - C_t) = \{g \in H_v^\infty : g(0) = 0\}$ .

Let  $h \in \text{Im}(I - C_t) \cap \text{Ker}(I - C_t)$ . Then  $h(0) = 0$  and there exists  $\lambda \in \mathbb{C}$  such that  $h = \lambda g_0$ . This yields that  $0 = h(0) = \lambda g_0(0) = \lambda$ . Hence,  $h = 0$ . So,  $\text{Im}(I - C_t) \cap \text{Ker}(I - C_t) = \{0\}$ .

Proposition 2.8 implies that  $1 \in \sigma(C_t; H_v^\infty) = \sigma(C_t; H_v^0) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\} \cup \{0\}$ . Consequently, for  $\delta = \frac{1}{2}$ , all the assumptions of Theorem 3.1 are satisfied. So, we can conclude that  $C_t$  is power bounded and uniformly mean ergodic on both  $H_v^\infty$  and on  $H_v^0$ .  $\square$

In contrast to the compactness of  $C_t$  acting in the Banach spaces  $H_v^\infty$  and  $H_v^0$  (cf. Proposition 2.7) the situation for the Fréchet space  $H(\mathbb{D})$  is different.

**Proposition 3.3** *For each  $t \in [0, 1)$  the operator  $C_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  is an isomorphism and, hence, it is not compact.*

**Proof** Fix  $t \in [0, 1)$ . Consider the operator  $T_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ , for  $f \in H(\mathbb{D})$ , given by

$$T_t f(z) := (1 - tz)(z f(z))' = (1 - tz)(f(z) + zf'(z)), \quad z \in \mathbb{D}.$$

Then  $T_t$  is clearly well-defined. Moreover, its graph is closed. Indeed, for a given sequence  $(f_n)_{n \in \mathbb{N}} \subset H(\mathbb{D})$ , suppose that  $f_n \rightarrow f$  in  $H(\mathbb{D})$  and  $T_t f_n \rightarrow g$  in  $H(\mathbb{D})$ . Since multiplication operators (by elements from  $H(\mathbb{D})$ ) and the differentiation operator are continuous on  $H(\mathbb{D})$  and the evaluation functionals at points of  $\mathbb{D}$  belong to  $H(\mathbb{D})'$ , it follows that  $f'_n \rightarrow f'$  in  $H(\mathbb{D})$  and hence,  $T_t f_n = (1 - tz)(f_n + zf'_n) \rightarrow (1 - tz)(f + zf') = T_t f$  in  $H(\mathbb{D})$ . Accordingly,  $g = T_t f$ . Since  $H(\mathbb{D})$  is a Fréchet space, the closed graph theorem, [20, Corollary 5.4.3], implies that  $T_t \in \mathcal{L}(H(\mathbb{D}))$ .

Finally, it is routine to verify that  $C_t \circ T_t = T_t \circ C_t = I$ . So, the inverse operator  $C_t^{-1} = T_t \in \mathcal{L}(H(\mathbb{D}))$  exists and hence,  $C_t$  is a bi-continuous isomorphism of  $H(\mathbb{D})$  onto itself. In particular,  $C_t$  cannot be compact.  $\square$

Let  $\Lambda := \{\frac{1}{n+1} : n \in \mathbb{N}_0\}$  and  $\Lambda_0 := \Lambda \cup \{0\}$ . We recall from [4, Lemma 2.7] the following lemma, which is an extension of a result of Rhoades [27].

**Lemma 3.4** *For every  $\mu \in \mathbb{C} \setminus \Lambda_0$  there exist  $\delta = \delta_\mu > 0$  and constants  $d_\delta, D_\delta > 0$  such that  $\overline{B(\mu, \delta)} \cap \Lambda_0 = \emptyset$  and*

$$\frac{d_\delta}{n^{\alpha(v)}} \leq \prod_{k=1}^n \left| 1 - \frac{1}{kv} \right| \leq \frac{D_\delta}{n^{\alpha(v)}}, \quad \forall n \in \mathbb{N}, v \in B(\mu, \delta), \quad (3.1)$$

where  $\alpha(v) := \operatorname{Re}(\frac{1}{v})$ .

**Remark 3.5** As a direct application of Lemma 3.4 we obtain, for every  $\mu \in \mathbb{C} \setminus \Lambda_0$ , that there exist  $\delta > 0$  and  $d_\delta, D_\delta > 0$  such that  $\overline{B(\mu, \delta)} \cap \Lambda_0 = \emptyset$  and, for every  $v \in B(\mu, \delta)$  and  $n \in \mathbb{N}_0$ , we have that

$$d_\delta D_\delta^{-1} \left( \frac{n-h}{n+1} \right)^{\alpha(v)} \leq \prod_{j=n-h+1}^{n+1} \left| 1 - \frac{1}{jv} \right| \leq D_\delta d_\delta^{-1} \left( \frac{n-h}{n+1} \right)^{\alpha(v)}, \quad (3.2)$$

for all  $h \in \{1, \dots, n-1\}$ , where  $\alpha(v) = \operatorname{Re}(\frac{1}{v})$ .

For each  $k \in \mathbb{N}$  with  $k \geq 2$  define  $r_k := (1 - \frac{1}{k})$ . Define the norms  $\|\cdot\|_k$  and  $\|\|\cdot\|\|_k$  on  $H(\mathbb{D})$  by

$$\|f\|_k := \sum_{n=0}^{\infty} |\hat{f}(n)| r_k^n, \quad f = \sum_{n=0}^{\infty} \hat{f}(n) z^n,$$

and

$$\|\|\cdot\|\|_k := \sup_{n \in \mathbb{N}_0} |\hat{f}(n)| r_k^n \quad f = \sum_{n=0}^{\infty} \hat{f}(n) z^n.$$

**Lemma 3.6** *Each of the sequences  $\{\|\cdot\|_k\}_{k \geq 2}$  and  $\{\|\|\cdot\|\|_k\}_{k \geq 2}$  is a fundamental system of norms for  $(H(\mathbb{D}), \tau_c)$ .*

**Proof** Given  $r \in (0, 1)$  choose any  $k \geq 2$  such that  $0 < r < (1 - \frac{1}{k})$ . Then, for every  $f \in H(\mathbb{D})$ , we have

$$q_r(f) = \sup_{|z|=r} \left| \sum_{n=0}^{\infty} \hat{f}(n) z^n \right| \leq \sum_{n=0}^{\infty} |\hat{f}(n)| r^n \leq \sum_{n=0}^{\infty} |\hat{f}(n)| \left( 1 - \frac{1}{k} \right)^n = \|f\|_k.$$

On the other hand, given  $k \geq 2$ , let  $r_k := (1 - \frac{1}{k}) < (1 - \frac{1}{k+1}) := r_{k+1}$ . By the Cauchy inequalities, for  $n \in \mathbb{N}_0$ , we have

$$|\hat{f}(n)| \leq \frac{1}{r_{k+1}^n} \max_{|z|=r_{k+1}} |f(z)| = \frac{1}{r_{k+1}^n} q_{r_{k+1}}(f), \quad f \in H(\mathbb{D}),$$

and hence,

$$\|f\|_{r_k} = \sum_{n=0}^{\infty} |\hat{f}(n)| r_k^n \leq q_{r_{k+1}}(f) \sum_{n=0}^{\infty} \left( \frac{r_k}{r_{k+1}} \right)^n = c q_{r_{k+1}}(f), \quad f \in H(\mathbb{D}),$$

with  $c = \frac{1}{1 - \frac{r_k}{r_{k+1}}} = k^2 > 0$  as  $\frac{r_k}{r_{k+1}} < 1$ , which is independent of  $f$ .

So, the systems  $\{q_r\}_{r \in (0,1)}$  and  $\{\|\cdot\|_k\}_{k \geq 2}$  are equivalent on  $H(\mathbb{D})$ .

Observe, for every  $k \geq 2$ , that

$$|||f|||_k = \sup_{n \in \mathbb{N}_0} |\hat{f}(n)|r_k^n \leq \sum_{n=0}^{\infty} |\hat{f}(n)|r_k^n = \|f\|_k, \quad f \in H(\mathbb{D}),$$

and that

$$\begin{aligned} \|f\|_k &= \sum_{n=0}^{\infty} |\hat{f}(n)|r_k^n = \sum_{n=0}^{\infty} |\hat{f}(n)| \left(\frac{r_k}{r_{k+1}}\right)^n r_{k+1}^n \\ &\leq \sup_{n \in \mathbb{N}_0} |\hat{f}(n)|r_{k+1}^n \sum_{n=0}^{\infty} \left(\frac{r_k}{r_{k+1}}\right)^n = k^2 |||f|||_{k+1}, \end{aligned}$$

for  $f \in H(\mathbb{D})$ , where  $\sum_{n=0}^{\infty} \left(\frac{r_k}{r_{k+1}}\right)^n = k^2$ . Therefore, the systems  $\{\|\cdot\|_k\}_{k \geq 2}$  and  $\{||| \cdot |||_k\}_{k \geq 2}$  are equivalent.  $\square$

**Proposition 3.7** *For each  $t \in [0, 1)$  the spectra of the operator  $C_t \in \mathcal{L}(H(\mathbb{D}))$  are given by*

$$\sigma_{pt}(C_t; H(\mathbb{D})) = \sigma(C_t; H(\mathbb{D})) = \Lambda \quad (3.3)$$

and

$$\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0. \quad (3.4)$$

**Proof** Let  $t \in [0, 1)$  be fixed. For any weight function  $v$  on  $[0, 1)$  satisfying  $\lim_{r \rightarrow 1^-} v(r) = 0$ , we have  $H_v^\infty \subseteq H(\mathbb{D})$  continuously and  $\Phi: H(\mathbb{D}) \rightarrow \omega$  is a continuous imbedding. Accordingly,  $\sigma_{pt}(C_t; H_v^\infty) \subseteq \sigma_{pt}(C_t; H(\mathbb{D})) \subseteq \Lambda$ ; see the proof of Proposition 2.8. Since  $\sigma_{pt}(C_t; H_v^\infty) = \Lambda$  (cf. Proposition 2.8) and  $\sigma_{pt}(C_t^\omega; \omega) = \Lambda$  [5, Theorem 3.7], it follows that  $\sigma_{pt}(C_t; H(\mathbb{D})) = \Lambda$ . Moreover, in view of Proposition 2.8 above and Theorem 3.7 in [5], the eigenspace corresponding to each eigenvalue  $\frac{1}{n+1} \in \Lambda$  is 1-dimensional. By Proposition 3.3, the operator  $C_t: H(\mathbb{D}) \rightarrow H(\mathbb{D})$  is a bi-continuous isomorphism and so  $0 \notin \sigma(C_t; H(\mathbb{D}))$ .

The claim is that  $\mathbb{C} \setminus \Lambda_0 \subseteq \rho(C_t; H(\mathbb{D}))$ . To establish this claim, fix  $v \in \mathbb{C} \setminus \Lambda_0$ . Given  $g(z) = \sum_{n=0}^{\infty} c_n z^n \in H(\mathbb{D})$ , consider the identity

$$(C_t - vI)f(z) = g(z), \quad z \in \mathbb{D}, \quad (3.5)$$

where  $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$  is to be determined. It follows from (1.6) that  $C_t f(z) = \sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \dots + a_n}{n+1}\right) z^n$  from which the identity  $(C_t - vI)f(z) = \sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \dots + a_n}{n+1} - v a_n\right) z^n$  is clear. So, (3.5) is satisfied if and only if

$$\sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \dots + a_n}{n+1} - v a_n\right) z^n = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{D},$$

that is, if and only if

$$\frac{t^n a_0 + t^{n-1} a_1 + \dots + a_n}{n+1} - v a_n = c_n, \quad n \in \mathbb{N}_0.$$

In view of this we can argue, as in the proof of [5, Lemma 3.6], to show that if a function  $f \in H(\mathbb{D})$  exists which satisfies the identity (3.5), then the Taylor coefficients  $(a_n)_{n \in \mathbb{N}_0}$  of

$f$  must verify the following equalities

$$\begin{aligned} a_0 &= \frac{c_0}{1 - \nu} \\ a_n &= \frac{c_n}{(\frac{1}{n+1} - \nu)} + \sum_{h=1}^n (-1)^h \frac{\nu^{h-1} t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{j\nu})} \\ &=: A_n + B_n, \quad n \geq 1. \end{aligned} \quad (3.6)$$

Observe, for each  $n \geq 1$  and  $h \in \{1, \dots, n\}$ , that

$$(-1)^h \prod_{j=n-h+1}^{n+1} \left( \frac{1}{j} - \nu \right) = - \prod_{j=n-h+1}^{n+1} \left( \nu - \frac{1}{j} \right) = -\nu^{h+1} \prod_{j=n-h+1}^{n+1} \left( 1 - \frac{1}{j\nu} \right)$$

and so

$$B_n = - \sum_{h=1}^n \frac{\nu^{h-1} t^h c_{n-h}}{\nu^{h+1} (n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{j\nu})} = - \frac{1}{\nu^2} \sum_{h=1}^n \frac{t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{j\nu})}.$$

Accordingly, to verify the claim we need to prove that the power series  $\sum_{n=0}^{\infty} a_n z^n$  is convergent in  $\mathbb{D}$ , with  $(a_n)_{n \in \mathbb{N}_0}$  defined according to (3.6). First, observe that the series  $g(z) = \sum_{n=0}^{\infty} c_n z^n$  is convergent in  $\mathbb{D}$  and satisfies

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \limsup_{n \rightarrow \infty} \sqrt[n]{\frac{|c_n|}{|\frac{1}{n+1} - \nu|}} = \limsup_{n \rightarrow \infty} \sqrt[n]{|A_n|}.$$

Therefore, the series  $\sum_{n=1}^{\infty} A_n z^n$  has the same radius of convergence as the series  $\sum_{n=0}^{\infty} c_n z^n$  and hence, it converges in  $H(\mathbb{D})$ . Accordingly,  $f_1(z) := \sum_{n=1}^{\infty} A_n z^n$ , for  $z \in \mathbb{D}$ , belongs to  $H(\mathbb{D})$ . On the other hand, the series

$$\begin{aligned} \sum_{n=1}^{\infty} B_n z^n &= - \frac{1}{\nu^2} \sum_{n=1}^{\infty} \sum_{h=1}^n \frac{t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{j\nu})} \\ &= - \frac{1}{\nu^2} \sum_{h=1}^{\infty} t^h z^h \sum_{n=h}^{\infty} \frac{c_{n-h} z^{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{j\nu})}, \quad z \in \mathbb{D}. \end{aligned}$$

To establish the convergence of the series  $\sum_{n=1}^{\infty} B_n z^n$  in  $H(\mathbb{D})$ , fix  $z \in \mathbb{D} \setminus \{0\}$  and  $r \in (|z|, 1)$ . Recall, for every  $n \in \mathbb{N}_0$ , that the Taylor coefficients of  $g$  satisfy (as  $\frac{1}{r} > 1$ )

$$|c_n| = \left| \frac{g^{(n)}(0)}{n!} \right| = \left| \frac{1}{2\pi i} \int_{|\xi|=r} \frac{g(\xi)}{\xi^{n+1}} d\xi \right| \leq \frac{1}{r^n} \max_{|\xi|=r} |g(\xi)| \leq \frac{C}{r^{n+1}}$$

where  $C := \max_{|\xi|=r} |g(\xi)|$ . Therefore, setting  $\alpha := \alpha(\nu) = \operatorname{Re}(\frac{1}{\nu})$  and  $d := d_{\delta}$  and  $D := D_{\delta}$  for a suitable  $\delta > 0$  (cf. Remark 3.5), we obtain via (3.1) and (3.2) that

$$\begin{aligned} &\sum_{h=1}^{\infty} t^h |z|^h \sum_{n=h}^{\infty} \frac{|c_{n-h}| |z|^{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{j\nu}|} \\ &\leq C \sum_{h=1}^{\infty} t^h |z|^{h-1} \left( \frac{|z|}{r} d^{-1} (h+1)^{-\alpha-1} + \sum_{n=h+1}^{\infty} \left( \frac{|z|}{r} \right)^{n-h+1} D d^{-1} \left( \frac{n+1}{n-h} \right)^{\alpha} \right) \\ &= C d^{-1} \sum_{h=1}^{\infty} t^h (h+1)^{-\alpha-1} |z|^h + C D d^{-1} \sum_{h=1}^{\infty} t^h |z|^{h-1} \sum_{n=h+1}^{\infty} \left( \frac{|z|}{r} \right)^{n-h+1} \left( \frac{n+1}{n-h} \right)^{\alpha} \end{aligned}$$

$$\leq Cd^{-1} \left( \sum_{h=1}^{\infty} t^h (h+1)^{-\alpha-1} |z|^h + D \sum_{h=1}^{\infty} t^h |z|^{h-1} \max\{1, (2+h)^\alpha\} \sum_{n=h+1}^{\infty} \left(\frac{|z|}{r}\right)^{n-h+1} \right),$$

which is finite after observing that if  $\alpha \leq 0$ , then  $\left(\frac{n+1}{n-h}\right)^\alpha = \left(\frac{n-h}{n+1}\right)^{-\alpha} \leq 1$  for every  $h \in \mathbb{N}$  and every  $n \geq h+1$ , whereas if  $\alpha > 0$ , then  $\left(\frac{n+1}{n-h}\right)^\alpha = (1 + \frac{h+1}{n-h})^\alpha \leq (2+h)^\alpha$ . This implies that the series  $\sum_{n=1}^{\infty} B^n z^n$  converges in  $H(\mathbb{D})$ . Accordingly,  $f_2(z) := \sum_{n=1}^{\infty} B_n z^n$ , for  $z \in \mathbb{D}$ , belongs to  $H(\mathbb{D})$ .

Set  $f(z) := \frac{c_0}{1-v} + f_1(z) + f_2(z)$ , for  $z \in \mathbb{D}$ . Then  $f \in H(\mathbb{D})$ . Moreover, the arguments above imply that  $f$  satisfies (3.5). The identities (3.6) imply that  $f$  is the unique solution of (3.5). Accordingly, the inverse operator  $(C_t - vI)^{-1} : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  exists. In particular,  $(C_t - vI)^{-1} \in \mathcal{L}(H(\mathbb{D}))$  as it is the inverse of a continuous linear operator on a Fréchet space.

Since  $v \in \mathbb{C} \setminus \Lambda_0$  is arbitrary and  $0 \in \rho(C_t; H(\mathbb{D}))$ , we can conclude that  $\sigma(C_t; H(\mathbb{D})) = \Lambda$ .

It remains to show that  $\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0$ . To establish this, fix  $\mu \in \mathbb{C} \setminus \Lambda_0$  and observe, by Lemma 3.4, that there exist  $\delta > 0$  and constants  $d_\delta, D_\delta > 0$  such that  $\overline{B(\mu, \delta)} \cap \Lambda_0 = \emptyset$  and the inequalities (3.1) and (3.2) are satisfied. We will show that  $B(\mu, \delta) \subset \rho(C_t; H(\mathbb{D}))$  and that the set  $\{(C_t - vI)^{-1} : v \in \overline{B(\mu, \delta)}\}$  is equicontinuous in  $\mathcal{L}(H(\mathbb{D}))$ . To see this, first observe that the function  $v \in \overline{B(\mu, \delta)} \mapsto \operatorname{Re}(\frac{1}{v}) \in \mathbb{R}$  is continuous and hence,  $\alpha_0 := \max_{v \in \overline{B(\mu, \delta)}} \{\operatorname{Re}(\frac{1}{v})\}$  exists. For the sake of simplicity of notation set  $d := d_\delta$  and  $D := D_\delta$ .

Let  $v \in B(\mu, r)$ , where  $r := \frac{1}{2}d(\Lambda_0, \overline{B(\mu, \delta)}) > 0$  has the property that  $|v - \frac{1}{j}| > r$  for all  $j \in \mathbb{N}$ . It was proved above, for any fixed  $g(z) = \sum_{n=0}^{\infty} c_n z^n \in H(\mathbb{D})$ , that

$$(C_t - vI)^{-1} g(z) = \frac{c_0}{1-v} + \sum_{n=1}^{\infty} \left( \frac{c_n}{\frac{1}{n+1} - v} - \frac{1}{v^2} \sum_{h=1}^n \frac{(-1)^h t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{jv})} \right) z^n,$$

for each  $z \in \mathbb{D}$ . So, for  $k \geq 2$  fixed, consider the norm  $\|\cdot\|_k$  in  $H(\mathbb{D})$ . Then we have, via (3.6), that

$$\begin{aligned} & \| (C_t - vI)^{-1} g \|_k \\ & \leq \frac{|c_0|}{|1-v|} + \sum_{n=1}^{\infty} \left| \frac{c_n}{\frac{1}{n+1} - v} - \frac{1}{v^2} \sum_{h=1}^n \frac{(-1)^h t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{jv})} \right| \left(1 - \frac{1}{k}\right)^n \\ & \leq \left( \frac{1}{r} \sum_{n=0}^{\infty} |c_n| \left(1 - \frac{1}{k}\right)^n \right) + \frac{1}{|v|^2} \sum_{n=1}^{\infty} \sum_{h=1}^n \frac{t^h |c_{n-h}|}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^n \\ & = \frac{1}{r} \|g\|_k + \frac{1}{|v|^2} \sum_{h=1}^{\infty} t^h \left(1 - \frac{1}{k}\right)^h \sum_{n=h}^{\infty} \frac{|c_{n-h}|}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^{n-h}. \end{aligned}$$

Moreover, (3.1) and (3.2) with  $\alpha(v) = \operatorname{Re}(\frac{1}{v}) \leq \alpha_0$  imply, for each  $h \in \mathbb{N}$ , that

$$\begin{aligned} & \sum_{n=h}^{\infty} \frac{|c_{n-h}|}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^{n-h} = \sum_{l=0}^{\infty} \frac{|c_l|}{(l+h+1) \prod_{j=l+1}^{l+h+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^l \\ & = \frac{|c_0|}{(h+1) \prod_{j=1}^{h+1} |1 - \frac{1}{jv}|} + \sum_{l=1}^{\infty} \frac{|c_l|}{(l+h+1) \prod_{j=l+1}^{l+h+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^l \\ & \leq d^{-1} |c_0| (h+1)^{\alpha(v)-1} + d^{-1} D \sum_{l=1}^{\infty} \frac{|c_l|}{l+h+1} \left(\frac{l+h+1}{l}\right)^{\alpha(v)} \left(1 - \frac{1}{k}\right)^l \end{aligned}$$

$$\begin{aligned} &\leq d^{-1}|c_0|(h+1)^{\alpha_0-1} + d^{-1}D \sum_{l=1}^{\infty} \frac{|c_l|}{l+h+1} \left(\frac{l+h+1}{l}\right)^{\alpha_0} \left(1-\frac{1}{k}\right)^l \\ &\leq \max\{d^{-1}, d^{-1}D\} (2+h)^{\alpha_0} \sum_{l=0}^{\infty} |c_l| \left(1-\frac{1}{k}\right)^l = K(2+h)^{\alpha_0} \|g\|_k, \end{aligned}$$

with  $K := \max\{d^{-1}, d^{-1}D\}$ , and hence, since  $|\nu| > r$  for all  $\nu \in B(\mu, \delta)$ , that

$$\begin{aligned} &\frac{1}{|\nu|^2} \sum_{h=1}^{\infty} t^h \left(1-\frac{1}{k}\right)^h \sum_{n=h}^{\infty} \frac{|c_{n-h}|}{(n+1) \prod_{j=n-h+1}^{n+1} |1-\frac{1}{j\nu}|} \left(1-\frac{1}{k}\right)^{n-h} \\ &\leq \frac{K}{r^2} \|g\|_k \sum_{h=1}^{\infty} t^h \left(1-\frac{1}{k}\right)^h (2+h)^{\alpha_0} = K' \|g\|_k, \end{aligned}$$

with  $K' = \frac{K}{r^2} \sum_{h=1}^{\infty} t^h \left(1-\frac{1}{k}\right)^h (2+h)^{\alpha_0} < \infty$ , by the ratio test, for instance.

We have established, for every  $\nu \in B(\mu, \delta)$ , that

$$\|(C_t - \nu I)^{-1} g\|_k \leq \left(\frac{1}{r} + K'\right) \|g\|_k.$$

Since  $g \in H(\mathbb{D})$  and  $k \geq 2$  are arbitrary, this shows that the set  $\{(C_t - \nu I)^{-1} : \nu \in B(\mu, \delta)\}$  is equicontinuous. Hence,  $\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0$ .  $\square$

**Proposition 3.8** *For each  $t \in [0, 1)$  the operator  $C_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  is power bounded, uniformly mean ergodic but, it fails to be supercyclic. Moreover,  $(I - C_t)(H(\mathbb{D}))$  is the closed subspace of  $H(\mathbb{D})$  given by*

$$(I - C_t)(H(\mathbb{D})) = \{g \in H(\mathbb{D}) : g(0) = 0\} \quad (3.7)$$

and we have the decomposition

$$H(\mathbb{D}) = \text{Ker}(I - C_t) \oplus (I - C_t)(H(\mathbb{D})). \quad (3.8)$$

**Proof** Fix  $t \in [0, 1)$ . We first prove that  $C_t$  is power bounded. Once this is established,  $C_t$  is necessarily uniform mean ergodic because  $H(\mathbb{D})$  is a Fréchet- Montel space (see [1, Proposition 2.8]).

Given  $k \geq 2$  we have, for every  $f \in H(\mathbb{D})$  and with  $r_k := (1 - \frac{1}{k})$ , that

$$\begin{aligned} |||C_t f|||_k &= \sup_{n \in \mathbb{N}_0} \left| \frac{1}{n+1} \sum_{j=0}^n t^{n-j} \hat{f}(j) \right| r_k \leq \sup_{n \in \mathbb{N}_0} \frac{1}{n+1} \sum_{j=0}^n |\hat{f}(j)| r_k^n \\ &\leq \sup_{n \in \mathbb{N}_0} \frac{1}{n+1} \sum_{j=0}^n |\hat{f}(j)| r_k^j \leq \sup_{j \in \mathbb{N}_0} |\hat{f}(j)| r_k^j = |||f|||_k, \end{aligned}$$

because  $r_k^n \leq r_k^j$  for all  $j \in \{0, 1, \dots, n\}$ . It follows, for every  $n \in \mathbb{N}$ , that

$$|||C_t^n f|||_k \leq |||f|||_k, \quad f \in H(\mathbb{D}).$$

Since  $k \geq 2$  is arbitrary, the operator  $C_t \in \mathcal{L}(H(\mathbb{D}))$  is indeed power bounded.

To establish that  $C_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  is not supercyclic, note that the continuous embedding  $\Phi : H(\mathbb{D}) \rightarrow \omega$  has dense range. The operator  $C_t^\omega \in \mathcal{L}(\omega)$  satisfies  $\Phi \circ C_t = C_t^\omega \circ \Phi$  as an identity in  $\mathcal{L}(H(\mathbb{D}), \omega)$ , which implies if  $C_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  is supercyclic, then also

$C_t^\omega: \omega \rightarrow \omega$  must be supercyclic as  $\Phi \circ C_t^n = \Phi \circ C_t \circ C_t^{n-1} = C_t^\omega \circ \Phi \circ C_t^{n-1} = \cdots = (C_t^\omega)^n \circ \Phi$ , for all  $n \in \mathbb{N}$ , and  $\Phi(H(\mathbb{D}))$  is dense in  $\omega$ . A contradiction with [5, Theorem 6.1].

To establish (3.7) note that  $(I - C_t)(H(\mathbb{D})) \subseteq \{g \in H(\mathbb{D}) : g(0) = 0\}$  because  $C_t f(0) = f(0)$  for every  $f \in H(\mathbb{D})$ . To show the reverse inclusion, let  $g \in H(\mathbb{D})$  satisfy  $g(0) = 0$ . Then  $h(z) := zg'(z) + g(z)$ , for  $z \in \mathbb{D}$ , is holomorphic and  $h(0) = 0$ . Accordingly, also  $z \mapsto \frac{h(z)}{z}$ , for  $z \in \mathbb{D} \setminus \{0\}$ , and taking the value  $h'(0)$  at  $z = 0$  is holomorphic in  $\mathbb{D}$ . Define  $f \in H(\mathbb{D})$  by

$$f(z) := \frac{1}{tz - 1} \int_0^z (1 - t\xi) \frac{h(\xi)}{\xi} d\xi, \quad z \in \mathbb{D},$$

and note that  $f(0) = 0$ . Direct calculation reveals that

$$\frac{f(z)}{1 - tz} - (zf(z))' = h(z) = (zg(z))', \quad z \in \mathbb{D},$$

from which it follows that

$$\int_0^z \frac{f(\xi)}{1 - t\xi} d\xi - zf(z) = zg(z), \quad z \in \mathbb{D}.$$

Since  $f(0) = 0$ , we can conclude that

$$\frac{1}{z} \int_0^z \frac{f(\xi)}{1 - t\xi} d\xi - f(z) = g(z), \quad z \in \mathbb{D},$$

that is,  $(C_t - I)f = g$  and so  $g \in (I - C_t)(H(\mathbb{D}))$ . Hence, (3.7) is valid.

To show the validity of (3.8) it suffices to repeat the argument given in the proof of Proposition 3.2.  $\square$

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