



Generalized Cesàro operators in weighted Banach spaces of analytic functions with sup-norms

Angela A. Albanese¹ · José Bonet² · Werner J. Ricker³

Received: 18 May 2023 / Accepted: 14 February 2024 / Published online: 12 March 2024
© The Author(s) 2024

Abstract

An investigation is made of the generalized Cesàro operators C_t , for $t \in [0, 1]$, when they act on the space $H(\mathbb{D})$ of holomorphic functions on the open unit disc \mathbb{D} , on the Banach space H^∞ of bounded analytic functions and on the weighted Banach spaces H_v^∞ and H_v^0 with their sup-norms. Of particular interest are the continuity, compactness, spectrum and point spectrum of C_t as well as their linear dynamics and mean ergodicity.

Keywords Generalized Cesàro operator · Weighted Banach spaces of analytic functions · Compact operator · Spectrum · Supercyclic · Mean ergodic · Power bounded

Mathematics Subject Classification Primary 46E15, 47B38; Secondary 46E10, 47A10, 47A16, 47A35

1 Introduction and preliminaries

The (discrete) generalized Cesàro operators C_t , for $t \in [0, 1]$, were first investigated by Rhaly [25, 26]. The action of C_t from the sequence space $\omega := \mathbb{C}^{\mathbb{N}_0}$ into itself, with $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, is given by

$$C_t x := \left(\frac{t^n x_0 + t^{n-1} x_1 + \dots + x_n}{n+1} \right)_{n \in \mathbb{N}_0}, \quad x = (x_n)_{n \in \mathbb{N}_0} \in \omega. \quad (1.1)$$

✉ Angela A. Albanese
angela.albanese@unisalento.it

José Bonet
jbonet@mat.upv.es

Werner J. Ricker
werner.ricker@ku.de

¹ Dipartimento di Matematica e Fisica “E. De Giorgi”, Università del Salento, C.P.193, 73100 Lecce, Italy

² Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de Valencia, Cubo F, Cuarta Planta, 46071 Valencia, Spain

³ Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt, 85072 Eichstätt, Germany

For $t = 0$ and with $\varphi := (\frac{1}{n+1})_{n \in \mathbb{N}_0}$ note that C_0 is the diagonal operator

$$D_\varphi x := \left(\frac{x_n}{n+1} \right)_{n \in \mathbb{N}_0}, \quad x = (x_n)_{n \in \mathbb{N}_0} \in \omega, \quad (1.2)$$

and, for $t = 1$, that C_1 is the classical Cesàro averaging operator

$$C_1 x := \left(\frac{x_0 + x_1 + \cdots + x_n}{n+1} \right)_{n \in \mathbb{N}_0}, \quad x = (x_n)_{n \in \mathbb{N}_0} \in \omega. \quad (1.3)$$

The behaviour of C_t on various sequence spaces has been investigated by many authors. We refer the reader to [25–27], to the recent papers [28, 30, 31] and to the introduction of the papers [5, 13] and the references therein. The operator C_1 was thoroughly investigated on weighted Banach spaces in [2]; see also [12]. Certain variants of the Cesàro operator C_1 are considered in [9, 16].

Our aim is to investigate the operators C_t , for $t \in [0, 1]$, when they are suitably interpreted to act on the space $H(\mathbb{D})$ of holomorphic functions on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, on the Banach space H^∞ of bounded analytic functions and on the weighted Banach spaces H_v^∞ and H_v^0 with their sup-norms. The space $H(\mathbb{D})$ is equipped with the topology τ_c of uniform convergence on the compact subsets of \mathbb{D} . According to [21, §27.3(3)] the space $H(\mathbb{D})$ is a Fréchet–Montel space. A family of norms generating τ_c is given, for each $0 < r < 1$, by

$$q_r(f) := \sup_{|z| \leq r} |f(z)|, \quad f \in H(\mathbb{D}). \quad (1.4)$$

A *weight* v is a continuous, non-increasing function $v : [0, 1) \rightarrow (0, \infty)$. We extend v to \mathbb{D} by setting $v(z) := v(|z|)$, for $z \in \mathbb{D}$. Note that $v(z) \leq v(0)$ for all $z \in \mathbb{D}$. Given a weight v on $[0, 1)$, we define the corresponding *weighted Banach spaces of analytic functions* on \mathbb{D} by

$$H_v^\infty := \{f \in H(\mathbb{D}) : \|f\|_{\infty, v} := \sup_{z \in \mathbb{D}} |f(z)|v(z) < \infty\},$$

and

$$H_v^0 := \{f \in H(\mathbb{D}) : \lim_{|z| \rightarrow 1^-} |f(z)|v(z) = 0\},$$

both endowed with the norm $\|\cdot\|_{\infty, v}$. Since $\|f\|_{\infty, v} \leq v(0)\|f\|_\infty$ whenever $f \in H^\infty$, it is clear that $H^\infty \subseteq H_v^\infty$ with a continuous inclusion. If $v(z) = 1$ for all $z \in \mathbb{D}$, then H_v^∞ coincides with the space H^∞ of all bounded analytic functions on \mathbb{D} with the sup-norm $\|\cdot\|_\infty$ and H_v^0 reduces to $\{0\}$. Moreover, $H_v^\infty \subseteq H(\mathbb{D})$ continuously. Indeed, fix $0 < r < 1$. Then $\frac{1}{v(0)} \leq \frac{1}{v(z)} \leq \frac{1}{v(r)}$ for $|z| \leq r$ and so (1.4) implies that

$$q_r(f) = \sup_{|z| \leq r} \frac{v(z)|f(z)|}{v(z)} \leq \frac{1}{v(r)} \sup_{|z| \leq r} v(z)|f(z)| \leq \frac{1}{v(r)} \|f\|_{\infty, v}, \quad f \in H_v^\infty.$$

We refer the reader to [10] for a recent survey of such types of weighted Banach spaces and operators between them.

Whenever necessary we will identify a function $f \in H(\mathbb{D})$ with its sequence of Taylor coefficients $\hat{f} := (\hat{f}(n))_{n \in \mathbb{N}_0}$ (i.e., $\hat{f}(n) := \frac{f^{(n)}(0)}{n!}$, for $n \in \mathbb{N}_0$), so that $f(z) = \sum_{n=0}^\infty \hat{f}(n)z^n$, for $z \in \mathbb{D}$. The linear map $\Phi : H(\mathbb{D}) \rightarrow \omega$ is defined by

$$\Phi \left(f = \sum_{n=0}^\infty \hat{f}(n)z^n \right) := \hat{f}, \quad f \in H(\mathbb{D}).$$

It is injective (clearly) and continuous. Indeed, for each $m \in \mathbb{N}_0$,

$$r_m(x) := \max_{0 \leq j \leq m} |x_j|, \quad x = (x_j)_{j \in \mathbb{N}_0} \in \omega,$$

is a continuous seminorm in ω . Fix $0 < r < 1$, in which case

$$\begin{aligned} r_m(\Phi(f)) &= \max_{0 \leq j \leq m} |\hat{f}(j)| = \max_{0 \leq j \leq m} \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{j+1}} dz \right| \leq \max_{0 \leq j \leq m} \sup_{|z|=r} \frac{|f(z)|}{|z|^j} \\ &= \max_{0 \leq j \leq m} \frac{1}{r^j} q_r(f) = \frac{1}{r^m} q_r(f), \end{aligned}$$

for each $f \in H(\mathbb{D})$ because $\frac{1}{r^j} \leq \frac{1}{r^m}$ for all $0 \leq j \leq m$. Of course, the increasing sequence of seminorms $\{r_m \mid m \in \mathbb{N}_0\}$ generates the topology of ω .

We first provide an integral representation of the generalized Cesàro operators C_t defined on $H(\mathbb{D})$, for $t \in [0, 1)$. So, fix $t \in [0, 1)$ and define $C_t: H(\mathbb{D}) \rightarrow H(\mathbb{D})$ by $C_t f(0) := f(0)$ and

$$C_t f(z) := \frac{1}{z} \int_0^z \frac{f(\xi)}{1-t\xi} d\xi, \quad z \in \mathbb{D} \setminus \{0\}, \quad (1.5)$$

for every $f \in H(\mathbb{D})$. It turns out that C_t is continuous on $H(\mathbb{D})$; see Proposition 2.1. Moreover, the discrete Cesàro operator $C_t: \omega \rightarrow \omega$, when restricted to the subspace $\Phi(H(\mathbb{D})) \subseteq \omega$ is transferred to $H(\mathbb{D})$ as follows. For a fixed $f \in H(\mathbb{D})$ we have $f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$, for $\xi \in \mathbb{D}$, with $\hat{f} = (a_n)_{n \in \mathbb{N}_0}$ its sequence of Taylor coefficients. Since $\frac{1}{1-t\xi} = \sum_{n=0}^{\infty} t^n \xi^n$, for $\xi \in \mathbb{D}$, we can form the Cauchy product of the two series, thereby obtaining

$$\frac{f(\xi)}{1-t\xi} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n t^{n-k} a_k \right) \xi^n, \quad \xi \in \mathbb{D}.$$

Then (1.5) yields

$$z C_t f(z) = \int_0^z \sum_{n=0}^{\infty} \left(\sum_{k=0}^n t^{n-k} a_k \right) \xi^n d\xi = \sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \cdots + a_n}{n+1} \right) z^{n+1}, \quad z \in \mathbb{D}.$$

The interchange of the infinite sum and the integral is permissible by uniform convergence of the series. This shows that $C_t f \in H(\mathbb{D})$ also has the series representation

$$\begin{aligned} C_t f(z) &= \sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \cdots + a_n}{n+1} \right) z^n \\ &= \sum_{n=0}^{\infty} \left(\frac{t^n \hat{f}(0) + t^{n-1} \hat{f}(1) + \cdots + \hat{f}(n)}{n+1} \right) z^n = \sum_{n=0}^{\infty} (C_t^\omega(\hat{f}))_n z^n, \end{aligned} \quad (1.6)$$

where the coefficients of the series are precisely as in (1.1). For the sake of clarity we will denote the discrete generalized Cesàro operator $C_t: \omega \rightarrow \omega$ by C_t^ω and reserve the notation C_t for the operator (1.5) acting in $H(\mathbb{D})$. Note that $C_0^\omega = D_\varphi$ (see (1.2)). Moreover, C_0 is given by $C_0 f(z) = \frac{1}{z} \int_0^z f(\xi) d\xi$ for $z \neq 0$ and $C_0 f(0) = f(0)$, which is the classical Hardy operator in $H(\mathbb{D})$.

The main results for C_t when acting in the Fréchet space $H(\mathbb{D})$ occur in Proposition 2.1 (continuity), Proposition 3.3 (non-compactness), Proposition 3.7 (spectra) and Proposition 3.8 (linear dynamics and mean ergodicity). For the analogous information concerning C_t when acting in the weighted Banach spaces H_v^∞ and H_v^0 see Proposition 2.4 and Corollary

2.5 (continuity), Proposition 2.7 (compactness), Proposition 2.8 (spectra) and Proposition 3.2 (linear dynamics and mean ergodicity).

We end this section by recalling a few definitions and some notation concerning locally convex spaces and operators between them. For further details about functional analysis and operator theory relevant to this paper see, for example, [15, 18, 20–22, 29].

Given locally convex Hausdorff spaces X, Y (briefly, lchS) we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X into Y . If $X = Y$, then we simply write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. Equipped with the topology of pointwise convergence on X (i.e., the strong operator topology) the lchS $\mathcal{L}(X)$ is denoted by $\mathcal{L}_s(X)$. Equipped with the topology τ_b of uniform convergence on the bounded subsets of X the lchS $\mathcal{L}(X)$ is denoted by $\mathcal{L}_b(X)$.

Let X be a lchS space. The identity operator on X is denoted by I . The *transpose operator* of $T \in \mathcal{L}(X)$ is denoted by T' ; it acts from the topological dual space $X' := \mathcal{L}(X, \mathbb{C})$ of X into itself. Denote by X'_σ (resp., by X'_β) the topological dual X' equipped with the weak* topology $\sigma(X', X)$ (resp., with the strong topology $\beta(X', X)$); see [21, §21.2] for the definition. It is known that $T' \in \mathcal{L}(X'_\sigma)$ and $T' \in \mathcal{L}(X'_\beta)$, [22, p. 134]. The bi-transpose operator $(T')'$ of T is simply denoted by T'' and belongs to $\mathcal{L}((X'_\beta)'_\beta)$.

A linear map $T: X \rightarrow Y$, with X, Y lchS, is called *compact* if there exists a neighbourhood \mathcal{U} of 0 in X such that $T(\mathcal{U})$ is a relatively compact set in Y . It is routine to show that necessarily $T \in \mathcal{L}(X, Y)$. We recall the following well known result; see [20, Proposition 17.1.1], [22, §42.1(1)].

Lemma 1.1 *Let X be a lchS. The compact operators are a 2-sided ideal in $\mathcal{L}(X)$.*

Given a lchS X and $T \in \mathcal{L}(X)$, the resolvent set $\rho(T; X)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The set $\sigma(T; X) := \mathbb{C} \setminus \rho(T; X)$ is called the *spectrum* of T . The *point spectrum* $\sigma_{pt}(T; X)$ of T consists of all $\lambda \in \mathbb{C}$ (also called an eigenvalue of T) such that $(\lambda I - T)$ is not injective. Some authors (eg. [29]) prefer the subset $\rho^*(T; X)$ of $\rho(T; X)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that the open disc $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\} \subseteq \rho(T; X)$ and $\{R(\mu, T) : \mu \in B(\lambda, \delta)\}$ is an equicontinuous subset of $\mathcal{L}(X)$. Define $\sigma^*(T; X) := \mathbb{C} \setminus \rho^*(T; X)$, which is a closed set with $\sigma(T; X) \subseteq \sigma^*(T; X)$. For the spectral theory of compact operators in lchS we refer to [15, 18], for linear dynamics to [6], [17] and for mean ergodic operators to [23], for example.

2 Continuity, compactness and spectrum of C_t

In this section we establish, for $t \in [0, 1)$, the continuity of $C_t: H(\mathbb{D}) \rightarrow H(\mathbb{D})$ as well as the continuity of C_t from H^∞ (resp., H_v^∞) into H^∞ (resp., H_v^∞). The same is true for $C_t: H_v^0 \rightarrow H_v^0$ whenever $\lim_{r \rightarrow 1^-} v(r) = 0$. It is also shown that the bi-transpose C_t'' of $C_t \in \mathcal{L}(H_v^0)$ is the generalized Cesàro operator $C_t \in \mathcal{L}(H_v^\infty)$, provided that $\lim_{r \rightarrow 1^-} v(r) = 0$. For such weights v it also turns out that both $C_t \in \mathcal{L}(H_v^0)$ and $C_t \in \mathcal{L}(H_v^\infty)$ are compact operators (cf. Proposition 2.7); their spectrum is identified in Proposition 2.8. Of particular interest are the standard weights $v_\gamma(z) := (1 - |z|)^\gamma$, for $\gamma > 0$ and $z \in \mathbb{D}$.

Proposition 2.1 *For every $t \in [0, 1)$ the operator $C_t: H(\mathbb{D}) \rightarrow H(\mathbb{D})$ is continuous. Moreover, the set $\{C_t : t \in [0, 1)\}$ is equicontinuous in $\mathcal{L}(H(\mathbb{D}))$.*

Proof Fix $f \in H(\mathbb{D})$. Taking into account that $C_t f(0) = f(0)$, for all $t \in [0, 1)$ and, for each $r \in (0, 1)$, that $\sup_{|z| \leq r} |C_t f(z)| = \sup_{|z| = r} |C_t f(z)|$, the formula (1.5) implies, for

each $z \in \mathbb{D} \setminus \{0\}$, that

$$\begin{aligned} |C_t f(z)| &= \frac{1}{|z|} \left| \int_0^z \frac{f(\xi)}{1-t\xi} d\xi \right| \leq \frac{1}{|z|} |z| \max_{\xi \in [0, z]} \frac{|f(\xi)|}{|1-t\xi|} \\ &\leq \frac{1}{1-|z|} \max_{|\xi| \leq |z|} |f(\xi)| = \frac{1}{1-|z|} \max_{|\xi|=|z|} |f(\xi)|, \end{aligned}$$

because $|1-t\xi| \geq 1-t|\xi| \geq 1-|\xi| \geq 1-|z|$, for all $|\xi| \leq |z|$. It follows from the previous inequality, for each $r \in (0, 1)$, that

$$q_r(C_t f) = \sup_{|z| \leq r} |C_t f(z)| \leq \frac{1}{1-r} \sup_{|\xi| \leq r} |f(\xi)| = \frac{1}{1-r} q_r(f); .$$

see (1.4). This implies the result. \square

The following example will prove to be useful in the sequel.

Example 2.2 Consider the constant function $f_1(z) := 1$, for every $z \in \mathbb{D}$, in which case $C_t f_1(0) = f_1(0) = 1$ for every $t \in [0, 1]$. For $t = 0$, it was noted in Sect. 1 that C_0 is the Hardy operator. In particular, $C_0 f_1(z) = 1$, for every $z \in \mathbb{D}$. For $t \in (0, 1]$, note that $C_t f_1(0) = 1$ and

$$C_t f_1(z) = \frac{1}{z} \int_0^z \frac{d\xi}{1-t\xi} = -\frac{1}{tz} \log(1-tz), \quad z \in \mathbb{D} \setminus \{0\}.$$

For $t = 1$ this shows, in particular, that $C_1(H^\infty) \not\subset H^\infty$, which is well known. For an investigation of the operator C_1 acting in H^∞ we refer to [14].

Concerning $t \in (0, 1)$, recall the Taylor series expansion

$$-\log(1-z) = z \sum_{n=0}^{\infty} \frac{z^n}{n+1}, \quad z \in \mathbb{D},$$

from which it follows that

$$-\frac{\log(1-tz)}{tz} = \sum_{n=0}^{\infty} \frac{t^n}{n+1} z^n, \quad z \in \mathbb{D} \setminus \{0\},$$

with the series having radius of convergence $\frac{1}{t} > 1$. The claim is that $\|C_t f_1\|_\infty = \sup_{|z| < 1} |C_t f_1(z)| = -\frac{\log(1-t)}{t}$. Indeed, $C_t f_1$ is clearly holomorphic in $B(0, \frac{1}{t}) := \{\xi \in \mathbb{C} : |\xi| < \frac{1}{t}\}$ hence, continuous in $B(0, \frac{1}{t})$, and satisfies $C_t f_1(1) = -\frac{\log(1-t)}{t}$ with $\lim_{r \rightarrow 1^-} C_t f_1(r) = C_t f_1(1)$. On the other hand, for every $z \in \mathbb{D} \setminus \{0\}$ and $t \in (0, 1)$ we have that

$$|C_t f_1(z)| = \left| -\frac{\log(1-tz)}{tz} \right| \leq \sum_{n=0}^{\infty} \frac{t^n}{n+1} |z|^n \leq \sum_{n=0}^{\infty} \frac{t^n}{n+1} = -\frac{\log(1-t)}{t}.$$

This completes the proof of the claim. Observe that $\|C_t f_1\|_\infty > 1$. Indeed, define $\gamma(t) = -\log(1-t) - t$, for $t \in [0, 1)$. Then $\gamma(0) = 0$, $\lim_{t \rightarrow 1^-} \gamma(t) = \infty$ and $\gamma'(t) = \frac{1}{1-t} - 1 = \frac{t}{1-t}$, for $t \in [0, 1)$. Since $\gamma'(t) > 0$, for $t \in (0, 1)$, it follows that γ is strictly increasing and so $\gamma(t) > 0$ for all $t \in (0, 1)$. This implies that $\|C_t f_1\|_\infty = -\frac{\log(1-t)}{t} > 1$ for every

$t \in (0, 1)$. On the other hand, for $t \in (0, 1)$, the inequality $\sum_{n=0}^{\infty} t^n / (n+1) < \sum_{n=0}^{\infty} t^n$ implies that $-\frac{\log(1-t)}{t} < \frac{1}{1-t}$. So, we have shown that $\|C_0 f_1\|_{\infty} = 1$ and

$$1 < \|C_t f_1\|_{\infty} < \frac{1}{1-t}, \quad t \in (0, 1).$$

We now turn to the action of C_t in various Banach spaces. For $t = 1$ it was noted above that C_1 fails to act in H^{∞} .

Proposition 2.3 *For $t \in [0, 1)$ the operator $C_t: H^{\infty} \rightarrow H^{\infty}$ is continuous. Moreover, $\|C_0\|_{H^{\infty} \rightarrow H^{\infty}} = 1$ and*

$$\|C_t\|_{H^{\infty} \rightarrow H^{\infty}} = -\frac{\log(1-t)}{t}, \quad t \in (0, 1).$$

Proof Let $f \in H^{\infty}$ be fixed. Then

$$|C_0 f(z)| = \left| \frac{1}{z} \int_0^z f(\xi) d\xi \right| \leq \max_{|\xi| \leq |z|} |f(\xi)| \leq \|f\|_{\infty}.$$

This implies that $\|C_0\|_{H^{\infty} \rightarrow H^{\infty}} \leq 1$. On the other hand, $C_0 f_1 = f_1$ and so we can conclude that $\|C_0\|_{H^{\infty} \rightarrow H^{\infty}} = 1$.

Now let $t \in (0, 1)$. Then, for the parametrization $\xi := sz$, for $s \in (0, 1)$, it follows from $|1 - stz| \geq 1 - |stz| \geq 1 - st$ that

$$\begin{aligned} |C_t f(z)| &= \left| \frac{1}{z} \int_0^z \frac{f(\xi)}{1 - t\xi} d\xi \right| = \left| \int_0^1 \frac{f(sz)}{1 - stz} ds \right| \leq \max_{|\xi| \leq |z|} |f(\xi)| \int_0^1 \frac{ds}{1 - st|z|} \\ &\leq \|f\|_{\infty} \int_0^1 \frac{ds}{1 - st} = -\frac{\log(1-t)}{t} \|f\|_{\infty}. \end{aligned}$$

So, $C_t \in \mathcal{L}(H^{\infty})$ with $\|C_t\|_{H^{\infty} \rightarrow H^{\infty}} \leq -\frac{\log(1-t)}{t}$. But, $\|C_t f_1\|_{\infty} = -\frac{\log(1-t)}{t}$. Accordingly, $\|C_t\|_{H^{\infty} \rightarrow H^{\infty}} = -\frac{\log(1-t)}{t}$. \square

Proposition 2.4 *Let v be a weight function on $[0, 1)$. For each $t \in [0, 1)$ the operator $C_t: H_v^{\infty} \rightarrow H_v^{\infty}$ is continuous. Moreover, $\|C_0\|_{H_v^{\infty} \rightarrow H_v^{\infty}} = 1$ and*

$$1 \leq \|C_t\|_{H_v^{\infty} \rightarrow H_v^{\infty}} \leq -\frac{\log(1-t)}{t}, \quad t \in (0, 1).$$

Proof Recall that $C_t f(0) := f(0)$ for each $f \in H(\mathbb{D})$ and $t \in [0, 1]$. Fix $t \in (0, 1)$. Given $f \in H_v^{\infty}$ and $z \in \mathbb{D} \setminus \{0\}$, observe that

$$\begin{aligned} v(z)|C_t f(z)| &= \frac{v(z)}{|z|} \left| \int_0^z \frac{f(\xi)}{1 - t\xi} d\xi \right| = v(z) \left| \int_0^1 \frac{f(sz)}{1 - stz} ds \right| \\ &\leq v(z) \int_0^1 \frac{|f(sz)|}{|1 - stz|} ds \leq \int_0^1 \frac{v(sz)|f(sz)|}{|1 - stz|} ds \\ &\leq \|f\|_{\infty, v} \int_0^1 \frac{ds}{|1 - stz|} \leq \|f\|_{\infty, v} \int_0^1 \frac{ds}{1 - st|z|} \\ &= -\frac{\log(1 - t|z|)}{t|z|} \|f\|_{\infty, v}, \end{aligned}$$

where we used that $v(sz) = v(s|z|) \geq v(|z|) = v(z)$, for $s \in (0, 1)$, as v is non-increasing on $(0, 1)$ and that $|1 - stz| \geq 1 - st|z|$, for $s \in (0, 1)$. According to the calculations in Example 2.2 we can conclude that

$$\|C_t f\|_{\infty, v} = \sup_{z \in \mathbb{D}} |C_t f(z)| v(z) \leq \|f\|_{\infty, v} \sup_{z \in \mathbb{D}} \left[-\frac{\log(1 - t|z|)}{t|z|} \right] = -\frac{\log(1 - t)}{t} \|f\|_{\infty, v}.$$

This implies that $C_t \in \mathcal{L}(H_v^\infty)$ and $\|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \leq -\frac{\log(1-t)}{t}$.

For $t = 0$ observe that

$$|C_0 f(z)| \leq \int_0^1 |f(sz)| ds \leq \max_{|\xi| \leq |z|} |f(\xi)| = \frac{1}{v(z)} \max_{|\xi| \leq |z|} |f(\xi)| v(\xi) \leq \frac{1}{v(z)} \|f\|_{\infty, v},$$

as $v(\xi) = v(z)$ whenever $|\xi| = |z|$ with $\xi \in \mathbb{D}$. This shows that $\|C_0\|_{H_v^\infty \rightarrow H_v^\infty} \leq 1$. Since $C_0 f_1 = f_1$, it follows that actually $\|C_0\|_{H_v^\infty \rightarrow H_v^\infty} = 1$.

It remains to show that $\|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \geq 1$ for $t \in (0, 1)$. To this end, fix $t \in (0, 1)$ and consider the function $g_0(z) := \frac{1}{1-tz} = \sum_{n=0}^{\infty} t^n z^n$, for $z \in \mathbb{D}$. Then $\|g_0\|_\infty = \frac{1}{1-t}$ and so $g_0 \in H^\infty \subseteq H_v^\infty$. Moreover, for every $z \in \mathbb{D} \setminus \{0\}$, it is the case that

$$C_t g_0(z) = \frac{1}{z} \int_0^z \frac{d\xi}{(1-t\xi)^2} = \frac{1}{z} \left[\frac{1}{t(1-t\xi)} \right]_0^z = \frac{1}{tz} \left[\frac{1}{1-tz} - 1 \right] = \frac{1}{1-tz} = g_0(z).$$

It follows that $\|g_0\|_{\infty, v} = \|C_t g_0\|_{\infty, v} \leq \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \|g_0\|_{\infty, v}$ which implies that $\|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \geq 1$. \square

Corollary 2.5 *Let v be a weight function on $[0, 1)$ satisfying $\lim_{r \rightarrow 1^-} v(r) = 0$. For each $t \in [0, 1)$ the operator $C_t: H_v^0 \rightarrow H_v^0$ is continuous and satisfies $\|C_t\|_{H_v^0 \rightarrow H_v^0} = \|C_t\|_{H_v^\infty \rightarrow H_v^\infty}$.*

Proof By Proposition 2.4 and the fact that H_v^0 is a closed subspace of H_v^∞ , to obtain the result it suffices to establish that $C_t(H_v^0) \subseteq H_v^0$. To this effect, observe that $H^\infty \subseteq H_v^0$ and that H^∞ is dense in H_v^0 , as the space of polynomials is dense in H_v^0 ; see Section 1 of [11] and also [7]. Proposition 2.3 implies that $C_t(H^\infty) \subseteq H^\infty \subseteq H_v^0$. Since C_t acts continuously on H_v^∞ , it follows that

$$C_t(H_v^0) = C_t(\overline{H^\infty}) \subseteq \overline{C_t(H^\infty)} \subseteq H_v^0.$$

Moreover, $\lim_{r \rightarrow 1^-} v(r) = 0$ implies that H_v^∞ is canonically isometric to the bidual of H_v^0 , [8, Example 2.1], and that the bi-transpose $C_t'': H_v^\infty \rightarrow H_v^\infty$ of $C_t: H_v^0 \rightarrow H_v^0$ coincides with $C_t: H_v^\infty \rightarrow H_v^\infty$ (see Lemma 2.6 below), from which the identity $\|C_t\|_{H_v^0 \rightarrow H_v^0} = \|C_t\|_{H_v^\infty \rightarrow H_v^\infty}$ follows. \square

Lemma 2.6 *Let v be a weight function on $[0, 1)$ satisfying $\lim_{r \rightarrow 1^-} v(r) = 0$. For each $t \in [0, 1)$, the bi-transpose $C_t'': H_v^\infty \rightarrow H_v^\infty$ of $C_t: H_v^0 \rightarrow H_v^0$ coincides with $C_t: H_v^\infty \rightarrow H_v^\infty$.*

Proof By Proposition 2.3 and Corollary 2.5, together with the fact that H_v^∞ is canonically isometric to the bidual of H_v^0 , both of the operators C_t'' and C_t act continuously on H_v^∞ .

To show that the bi-transpose $C_t'': H_v^\infty \rightarrow H_v^\infty$ of $C_t: H_v^0 \rightarrow H_v^0$ coincides with $C_t: H_v^\infty \rightarrow H_v^\infty$ we proceed via several steps.

First step Given $f \in H(\mathbb{D})$, its Taylor polynomials $p_k(z) = \sum_{j=0}^k \hat{f}(j) z^j$, $z \in \mathbb{D}$, for $k \in \mathbb{N}_0$, converge to f uniformly on compact subsets of \mathbb{D} . That is, $p_k \rightarrow f$ in $(H(\mathbb{D}), \tau_c)$ as $k \rightarrow \infty$. Accordingly, the averages of $(p_k)_{k \in \mathbb{N}_0}$, that is, the Cesàro means $f_n(z) := \frac{1}{n+1} \sum_{j=0}^n p_j(z)$, for $z \in \mathbb{D}$ and $n \in \mathbb{N}_0$, also converge to f in $(H(\mathbb{D}), \tau_c)$ as $n \rightarrow \infty$.

Second step Lemma 1.1 in [7] implies, for every $f \in H_v^\infty$ and $n \in \mathbb{N}_0$, that $\|f_n\|_{\infty, v} \leq \|f\|_{\infty, v}$, where f_n is the n -th Cesàro mean of f , as defined in the *First step*. Denote by U_v the closed unit ball of $(H_v^\infty, \|\cdot\|_{\infty, v})$. Then, for any given $f \in U_v$, its sequence of Cesàro means satisfies $(f_n)_{n \in \mathbb{N}_0} \subseteq U_v$ and $f_n \rightarrow f$ in $(H(\mathbb{D}), \tau_c)$ as $n \rightarrow \infty$.

Third step With the topology of uniform convergence on the compact subsets of U_v denoted by τ_c , let $X := \{F \in (H_v^\infty)^\circ : F|_{U_v} \text{ is } \tau_c\text{-continuous}\}$ be endowed with the norm $\|F\| := \sup\{|F(f)| : f \in U_v\}$. Then [8, Theorem 1.1(a)] ensures that $(X, \|\cdot\|)$ is a Banach space and that the evaluation map $\Psi : H_v^\infty \rightarrow X'$ defined by $(\Psi(f))(F) := \langle f, F \rangle$, for $F \in X$ and $f \in H_v^\infty$, is an isometric isomorphism onto X' (where X' is the dual Banach space of $(X, \|\cdot\|)$). Moreover, by [8, Theorem 1.1(b) and Example 2.1] the restriction map $R : X \rightarrow (H_v^0)^\circ$ given by $F \mapsto F|_{H_v^0}$, is also a surjective isometric isomorphism. Therefore, the spaces H_v^∞ and $(H_v^0)^\circ$ are isometrically isomorphic, that is, X and $(H_v^0)^\circ$ are isometrically isomorphic and hence, also H_v^∞ and $(H_v^0)^\circ$ are isometrically isomorphic.

It is easy to see, since the Banach space X above is the predual of H_v^∞ , that the evaluation map $\delta_z \in X$, for every $z \in \mathbb{D}$, where $\delta_z : f \mapsto f(z)$, for $f \in H_v^\infty$, satisfies $|\langle f, \delta_z \rangle| \leq \|f\|_{\infty, v/v(z)}$. In particular, the linear span L of the set $\{\delta_z : z \in \mathbb{D}\}$ separates the points of $H_v^\infty = X'$ and hence, L is dense in X . Therefore, the pointwise convergence topology τ_p on H_v^∞ is Hausdorff and coarser than the w^* -topology $\sigma(H_v^\infty, X)$.

Fourth step The closed unit ball U_v of H_v^∞ is a τ_c -compact set by Montel's theorem, as it is τ_c -bounded and closed. On the other hand, U_v is also $\sigma(H_v^\infty, X)$ -compact by the Alaoglu-Bourbaki theorem. Since $\tau_p|_{U_v}$ is coarser than $\tau_c|_{U_v}$ and Hausdorff, we can conclude that $\tau_p|_{U_v} = \tau_c|_{U_v}$. In the same way, it follows that $\tau_p|_{U_v} = \sigma(H_v^\infty, X)|_{U_v}$. Accordingly, $\tau_p|_{U_v} = \tau_c|_{U_v} = \sigma(H_v^\infty, X)|_{U_v}$.

We are now ready to prove that $(C_t)'' = C_t$. To show this, it suffices to establish that $(C_t)''f = C_t f$ for every $f \in U_v$.

So, fix $f \in U_v$. With $(f_n)_{n \in \mathbb{N}_0}$ as in the *First step* it follows from there that $f_n \rightarrow f$ in $(H(\mathbb{D}), \tau_c)$ as $n \rightarrow \infty$ and, by the *Second step*, that $(f_n)_{n \in \mathbb{N}_0} \subseteq U_v$. This implies that $C_t f_n \rightarrow C_t f$ in $(H(\mathbb{D}), \tau_c)$ as $n \rightarrow \infty$. Since $C_t \in \mathcal{L}(H_v^\infty)$ and $f \in U_v$, it is clear that $C_t f \in H_v^\infty$. On the other hand, by the *Fourth step* the sequence $(f_n)_{n \in \mathbb{N}_0}$ also converges to f in $(H_v^\infty, \sigma(H_v^\infty, X)) = (H_v^\infty, \sigma(H_v^\infty, (H_v^0)^\circ))$. Since $(C_t)'' : ((H_v^0)^\circ, \sigma((H_v^0)^\circ, (H_v^0)^\circ)) \rightarrow ((H_v^0)^\circ, \sigma((H_v^0)^\circ, (H_v^0)^\circ))$ is continuous, [20, §8.6], that is, $(C_t)'' : (H_v^\infty, \sigma(H_v^\infty, X)) \rightarrow (H_v^\infty, \sigma(H_v^\infty, X))$ is continuous, it follows that $(C_t)''f_n \rightarrow (C_t)''f$ in $(H_v^\infty, \sigma(H_v^\infty, X))$ as $n \rightarrow \infty$. Now, $(f_n)_{n \in \mathbb{N}_0} \subset H^\infty \subseteq H_v^0$, as each f_n is a polynomial, and $(C_t)''f_n = C_t f_n$ for every $n \in \mathbb{N}_0$. Moreover, the sequence $C_t f_n \rightarrow (C_t)''f$ in $(H(\mathbb{D}), \tau_p)$ as $n \rightarrow \infty$. Thus, $(C_t)''f = C_t f$ as desired. \square

Proposition 2.7 *Let v be a weight function satisfying $\lim_{r \rightarrow 1^-} v(r) = 0$. For each $t \in [0, 1)$, both of the operators $C_t : H_v^\infty \rightarrow H_v^\infty$ and $C_t : H_v^0 \rightarrow H_v^0$ are compact.*

Proof Fix $t \in [0, 1)$. Since H_v^0 is a closed subspace of H_v^∞ and $C_t(H_v^0) \subseteq H_v^0$ (cf. Corollary 2.5), it suffices to show that $C_t : H_v^\infty \rightarrow H_v^\infty$ is compact. First we establish the following Claim:

(*) Let the sequence $(f_n)_{n \in \mathbb{N}} \subset H_v^\infty$ satisfy $\|f_n\|_{\infty, v} \leq 1$ for every $n \in \mathbb{N}$ and $f_n \rightarrow 0$ in $(H(\mathbb{D}), \tau_c)$ for $n \rightarrow \infty$. Then $C_t f_n \rightarrow 0$ in H_v^∞ .

To prove the Claim, let $(f_n)_{n \in \mathbb{N}} \subset H_v^\infty$ be a sequence as in (*). Fix $\varepsilon > 0$ and select $\delta \in (0, \beta)$, where $\beta := \min\{1, \frac{\varepsilon(1-t)}{2}, \frac{\varepsilon(1-t)}{2v(0)}\}$. Since $\{\xi \in \mathbb{C} : |\xi| \leq (1-\delta)\}$ is a compact subset of \mathbb{D} , there exists $n_0 \in \mathbb{N}$ such that

$$\max_{|\xi| \leq 1-\delta} |f_n(\xi)| < \delta, \quad n \geq n_0.$$

Recall that $C_t f_n(0) = f_n(0)$ for every $n \in \mathbb{N}$. For $z \in \mathbb{D} \setminus \{0\}$ we have seen previously that

$$v(z)|C_t f_n(z)| = v(z) \left| \int_0^1 \frac{f_n(sz)}{1-stz} ds \right| \leq v(z) \int_0^{1-\delta} \frac{|f_n(sz)|}{|1-stz|} ds + v(z) \int_{1-\delta}^1 \frac{|f_n(sz)|}{|1-stz|} ds.$$

Denote the first (resp., second) summand in the right-side of the previous inequality by (A_n) (resp., by (B_n)). Using the facts that $|1-stz| \geq 1-st|z| \geq \max\{1-s, 1-t, 1-|z|\}$, for all $s, t \in [0, 1]$ and $z \in \mathbb{D}$, and that v is non-increasing on $[0, 1]$ it follows, for every $n \geq n_0$, that $\int_0^{1-\delta} |f_n(sz)| ds \leq (1-\delta) \max_{|\xi| \leq (1-\delta)} |f_n(\xi)|$ (as $|sz| \leq (1-\delta)$ for all $s \in [0, 1-\delta]$) and hence, that

$$(A_n) \leq \frac{v(0)(1-\delta)}{1-t} \max_{|\xi| \leq 1-\delta} |f_n(\xi)| < \frac{\varepsilon}{2}.$$

On the other hand, for every $n \geq n_0$, we have (as $\|f_n\|_{\infty, v} = \sup_{\xi \in \mathbb{D}} v(\xi)|f_n(\xi)| \leq 1$) that

$$(B_n) = \int_{1-\delta}^1 \frac{v(z)}{v(sz)} \frac{|f_n(sz)|}{|1-stz|} ds \leq \int_{1-\delta}^1 \frac{\|f_n\|_{\infty, v}}{1-t} ds \leq \frac{\delta}{1-t} < \frac{\varepsilon}{2}.$$

It follows that $\|C_t f_n\|_{\infty, v} < \varepsilon$ for every $n \geq n_0$. That is, $C_t f_n \rightarrow 0$ in H_v^∞ for $n \rightarrow \infty$ and so (*) is proved.

The compactness of $C_t \in \mathcal{L}(H_v^\infty)$ can be deduced from (*) as follows. Let $(f_n)_{n \in \mathbb{N}} \subset H_v^\infty$ be any bounded sequence. There is no loss of generality in assuming that $\|f_n\|_{\infty, v} \leq 1$ for all $n \in \mathbb{N}$. To establish the compactness of $C_t \in \mathcal{L}(H_v^\infty)$ we need to show that $(C_t f_n)_{n \in \mathbb{N}}$ has a convergent subsequence in H_v^∞ .

Since $H_v^\infty \subseteq H(\mathbb{D})$ continuously, the sequence $(f_n)_{n \in \mathbb{N}}$ is also bounded in the Fréchet–Montel space $H(\mathbb{D})$. Hence, there is a subsequence $g_j := f_{n_j}$, for $j \in \mathbb{N}$, of $(f_n)_{n \in \mathbb{N}}$ and $f \in H(\mathbb{D})$ such that $g_j \rightarrow f$ in $H(\mathbb{D})$ with respect to τ_c . In particular, $g_j \rightarrow f$ pointwise on \mathbb{D} . Since $v(z)|g_j(z)| = v(z)|f_{n_j}(z)| \leq 1$ for all $z \in \mathbb{D}$ and $j \in \mathbb{N}$, letting $j \rightarrow \infty$ it follows that $v(z)|f(z)| \leq 1$ for all $z \in \mathbb{D}$, that is, $f \in H_v^\infty$ with $\|f\|_{\infty, v} \leq 1$. Let $h_j := \frac{1}{2}(g_j - f)$, for $j \in \mathbb{N}$. Then $\|h_j\|_{\infty, v} \leq 1$, for $j \in \mathbb{N}$, and $h_j \rightarrow 0$ in $H(\mathbb{D})$ with respect to τ_c . Condition (*) implies that $C_t h_j \rightarrow 0$ in H_v^∞ from which it follows that $C_t f_{n_j} = C_t g_j = C_t(g_j - f) + C_t f = 2C_t h_j + C_t f \rightarrow C_t f$ in H_v^∞ , as desired. \square

Proposition 2.8 *Let v be a weight function on $[0, 1]$ satisfying $\lim_{r \rightarrow 1^-} v(r) = 0$. For each $t \in [0, 1]$ the spectra of $C_t \in \mathcal{L}(H_v^\infty)$ and of $C_t \in \mathcal{L}(H_v^0)$ are given by*

$$\sigma_{pt}(C_t; H_v^\infty) = \sigma_{pt}(C_t; H_v^0) = \left\{ \frac{1}{m+1} : m \in \mathbb{N}_0 \right\}, \quad (2.1)$$

and

$$\sigma(C_t; H_v^\infty) = \sigma(C_t; H_v^0) = \left\{ \frac{1}{m+1} : m \in \mathbb{N}_0 \right\} \cup \{0\}. \quad (2.2)$$

Proof Let $t \in [0, 1]$ be fixed. By [13, Lemma 3.6] we know that the point spectrum of the operator $C_t^\omega \in \mathcal{L}(\omega)$ is given by $\sigma_{pt}(C_t^\omega; \omega) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$ and, for each $m \in \mathbb{N}_0$, that the corresponding eigenspace $\text{Ker}(\frac{1}{m+1}I - C_t^\omega)$ is 1-dimensional and is generated by an eigenvector $x^{[m]} = (x_n^{[m]})_{n \in \mathbb{N}_0} \in \ell^1$. Since $H_v^0 \subseteq H_v^\infty \subseteq H(\mathbb{D})$ with continuous inclusions and $\Phi: H(\mathbb{D}) \rightarrow \omega$ (cf. Sect. 1) is a continuous embedding, this implies that $\sigma_{pt}(C_t; H_v^0) \subseteq \sigma_{pt}(C_t; H_v^\infty) \subseteq \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$. Indeed, let $f \in H(\mathbb{D}) \setminus \{0\}$ and $\lambda \in \mathbb{C}$ satisfy $C_t f = \lambda f$. Then $\lambda f(z) = \sum_{n=0}^\infty \widehat{(\lambda f)}(n) z^n = \sum_{n=0}^\infty \lambda \hat{f}(n) z^n$ and, by (1.6), we have that $(C_t f)(z) = \sum_{n=0}^\infty (C_t^\omega \hat{f})(n) z^n$. It follows that $C_t^\omega \hat{f} = \lambda \hat{f}$ in ω with $\hat{f} \neq 0$ and so $\lambda \in \sigma_{pt}(C_t^\omega; \omega) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$.

To conclude the proof, it remains to show that $\{\frac{1}{m+1} : m \in \mathbb{N}_0\} \subseteq \sigma_{pt}(C_t; H_v^0)$. To establish this recall, for each $m \in \mathbb{N}_0$, that the eigenvector $x^{[m]} \in \ell^1$ and hence, the function $g_m(z) := \sum_{n=0}^{\infty} (x^{[m]})_n z^n$ belongs to H_v^0 because $0 \leq v(z)|g_m(z)| \leq v(z)\|x^{[m]}\|_{\ell^1}$ for $z \in \mathbb{D}$ and $\lim_{r \rightarrow 1^-} v(r) = 0$. Moreover, according to (1.5) and (1.6) we have, for each $z \in \mathbb{D}$, that

$$C_t g_m(z) = \sum_{n=0}^{\infty} (C_t^\omega x^{[m]})_n z^n = \sum_{n=0}^{\infty} \left(\frac{1}{m+1} x^{[m]}\right)_n z^n = \frac{1}{m+1} \sum_{n=0}^{\infty} (x^{[m]})_n z^n = \frac{1}{m+1} g_m(z).$$

Thus g_m is an eigenvector of $C_t \in \mathcal{L}(H_v^0)$ corresponding to the eigenvalue $\frac{1}{m+1}$.

The validity of $\sigma(C_t; H_v^0) = \sigma(C_t; H_v^\infty) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\} \cup \{0\}$ follows from the fact that C_t is a compact operator on both spaces. \square

We now investigate the norm of C_t on H_v^∞ for the standard weights $v_\gamma(z) := (1 - |z|)^\gamma$, for $\gamma > 0$ and $z \in \mathbb{D}$, which satisfy $\lim_{r \rightarrow 1^-} v_\gamma(r) = 0$.

Proposition 2.9 *Let $t \in (0, 1)$ and $\gamma > 0$.*

- (i) *The operator norm $\|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} = 1$, for every $\gamma \geq 1$.*
- (ii) *For each $\gamma \in (0, 1)$, the inequality $\|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} \leq \min\{-\frac{\log(1-t)}{t}, \frac{1}{\gamma}\}$ is valid.*

Proof We adapt the arguments given for the Cesàro operator C_1 in the proof of [2, Theorem 2.3].

Let $\gamma > 0$ and $t \in (0, 1)$ be fixed. For $f \in H_{v_\gamma}^\infty$ with $\|f\|_{\infty, v_\gamma} = 1$ we have

$$\begin{aligned} |C_t f(z)| &= \frac{1}{|z|} \left| \int_0^1 \frac{f(sz)}{1-stz} ds \right| \leq \int_0^1 \frac{|f(sz)|}{1-st|z|} ds \\ &\leq \int_0^1 \frac{|f(sz)|}{1-s|z|} ds \leq \int_0^1 \frac{ds}{(1-s|z|)^{\gamma+1}} = \frac{1}{(1-|z|)^\gamma} \frac{1-(1-|z|)^\gamma}{\gamma|z|}, \end{aligned}$$

as $z \in \mathbb{D}$ implies that $1-st|z| \geq 1-s|z|$, for $s \in (0, 1)$. Accordingly,

$$v_\gamma(z)|C_t f(z)| = (1-|z|)^\gamma |C_t f(z)| \leq \frac{1-(1-|z|)^\gamma}{\gamma|z|}, \quad z \neq 0,$$

and hence,

$$\|C_t f\|_{\infty, v_\gamma} \leq \frac{1}{\gamma} \sup_{z \in \mathbb{D}} \frac{1-(1-|z|)^\gamma}{|z|}.$$

Define $\phi(s) := \frac{1-(1-s)^\gamma}{s}$ for $s \in (0, 1]$ and $\phi(0) = \gamma$, in which case ϕ is continuous. So, the previous inequality yields $\|C_t f\|_{\infty, v_\gamma} \leq \frac{M_\gamma}{\gamma}$, for all $\|f\|_{\infty, v_\gamma} \leq 1$, that is, $\|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} \leq \frac{M_\gamma}{\gamma}$, where $M_\gamma := \sup_{s \in [0, 1]} \phi(s)$. Proposition 2.4 yields that $1 \leq \|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} \leq -\frac{\log(1-t)}{t}$ for $t \in (0, 1)$. On page 101 of [2] it is shown that $\frac{M_\gamma}{\gamma} \leq 1$ whenever $\gamma \geq 1$ and that $M_\gamma \leq 1$ for all $\gamma \in (0, 1)$. The proof of both parts (i) and (ii) follows immediately. \square

Remark 2.10 For each $\gamma > 0$ let $v_\gamma(z) = (1 - |z|)^\gamma$, for $z \in \mathbb{D}$. Proposition 2.9 implies that $\sup_{0 \leq t < 1} \|C_t\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} < \infty$. Moreover, if $\gamma \geq 1$, then $\|C_t^n\|_{H_{v_\gamma}^\infty \rightarrow H_{v_\gamma}^\infty} = 1$ for every $n \in \mathbb{N}$; see case (i) in the proof of [2, Theorem 2.3] together with the fact that $1 \in \sigma_{pt}(C_t, H_{v_\gamma}^\infty)$ by Proposition 2.8.

Let $n \in \mathbb{N}$ be fixed. Consider the weight $v(z) = (\log \frac{e}{1-|z|})^{-n}$, for $z \in \mathbb{D}$, which satisfies $v(0) = 1$ and $\lim_{|z| \rightarrow 1^-} v(z) = 0$.

The function $f(z) := [\log(1-z)]^n \in H(\mathbb{D})$ belongs to H_v^∞ . Indeed, for each $z \in \mathbb{D}$, we have that

$$|\log(1-z)| = \left| -\sum_{n=1}^{\infty} \frac{z^n}{n} \right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n} = -\log(1-|z|)$$

and hence, that $|f(z)| = |\log(1-z)|^n \leq (-\log(1-|z|))^n$. Since v is given by $v(z) = (1 - \log(1-|z|))^{-n}$ and $\lim_{|z| \rightarrow 1^-} \frac{-\log(1-|z|)}{1-\log(1-|z|)} = 1$, it follows that $\|f\|_{\infty, v} < \infty$ and so $f \in H_v^\infty$. On the other hand,

$$C_1 f(z) = \frac{1}{z} \int_0^z \frac{(\log(1-\xi))^n}{1-\xi} d\xi = -\frac{1}{(n+1)z} (\log(1-z))^{n+1}, \quad z \in \mathbb{D}.$$

Accordingly, $C_1 f \notin H_v^\infty$ since

$$\begin{aligned} \lim_{s \rightarrow s^-} v(s) |(C_1 f)(s)| &= \frac{1}{n+1} \lim_{s \rightarrow 1^-} \left| \frac{(\log(1-s))^{n+1}}{s(1-\log(1-s))^n} \right| \\ &= \frac{1}{n+1} \lim_{s \rightarrow 1^-} \left| \left(\frac{\log(1-s)}{1-\log(1-s)} \right)^n \frac{\log(1-s)}{s} \right| = \infty. \end{aligned}$$

This implies that the Cesàro operator C_1 is not well-defined on H_v^∞ , that is, $C_1(H_v^\infty) \not\subseteq H_v^\infty$. But, by Proposition 2.4 the generalized Cesàro operator $C_t \in \mathcal{L}(H_v^\infty)$ for every $t \in [0, 1)$. At this point, the following question arises: Is $\sup_{t \in [0, 1)} \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} < \infty$ for this particular v ? Our next two results show that the answer is negative for certain weights v , which includes $v(z) = \left(\log \frac{e}{1-|z|}\right)^{-n}$ for $z \in \mathbb{D}$.

Proposition 2.11 *Let v be a weight function on $[0, 1)$ such that $\sup_{t \in [0, 1)} \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} < \infty$. Then $C_1 \in \mathcal{L}(H_v^\infty)$.*

Proof Proposition 2.1 implies that $\{C_t : t \in [0, 1)\}$ is equicontinuous in $\mathcal{L}(H(\mathbb{D}))$. The claim is that $\lim_{t \rightarrow 1^-} C_t f(z) = C_1 f(z)$, for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

To prove this claim fix $f \in H(\mathbb{D})$ and $z \in \mathbb{D} \setminus \{0\}$. Recall, for $t \in [0, 1)$, that

$$C_t f(z) = \frac{1}{z} \int_0^z \frac{f(\xi)}{1-t\xi} d\xi = \int_0^1 \frac{f(sz)}{1-stz} ds$$

and

$$C_1 f(z) = \frac{1}{z} \int_0^z \frac{f(\xi)}{1-\xi} d\xi = \int_0^1 \frac{f(sz)}{1-sz} ds.$$

Moreover, for each $z \in \mathbb{D} \setminus \{0\}$, we have (as $|1-stz| \geq (1-|z|)$) that

$$\left| \frac{f(sz)}{1-stz} \right| \leq \frac{|f(sz)|}{1-|z|} \leq \frac{1}{1-|z|} \max_{|\xi| \leq |z|} |f(\xi)|, \quad s \in [0, 1],$$

and that $\lim_{t \rightarrow 1^-} \frac{f(sz)}{1-stz} = \frac{f(sz)}{1-sz}$ for every $s \in [0, 1]$. So, we can apply the dominated convergence theorem to conclude that $\lim_{t \rightarrow 1^-} C_t f(z) = C_1 f(z)$ for $z \in \mathbb{D} \setminus \{0\}$. For $z = 0$ we have $C_t f(0) = f(0) = C_1 f(0)$ for each $f \in H(\mathbb{D})$ and $t \in [0, 1)$. So, for each $f \in H(\mathbb{D})$, we can conclude that $C_t f \rightarrow C_1 f$ pointwise on \mathbb{D} for $t \rightarrow 1^-$. The claim is thereby established.

We now show that $C_t f \rightarrow C_1 f$ in $H(\mathbb{D})$ as $t \rightarrow 1^-$ for every $f \in H_v^\infty$. The assumption $\sup_{t \in [0,1)} \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} < \infty$ implies that there exists $M > 0$ satisfying $\|C_t\|_{H_v^\infty \rightarrow H_v^\infty} \leq M$ for every $t \in [0, 1)$. Therefore,

$$\sup_{z \in \mathbb{D}} |C_t f(z)| v(z) \leq M \|f\|_{\infty, v}, \quad f \in H_v^\infty, \quad t \in [0, 1). \quad (2.3)$$

Fix $f \in H_v^\infty$. Then $\{C_t f : t \in [0, 1)\}$ is a bounded set in $H(\mathbb{D})$. Indeed, given $r \in (0, 1)$ and $t \in [0, 1)$ we have (as $v(r) \leq v(z)$ for all $|z| \leq r$) that

$$q_r(C_t f) = \sup_{|z| \leq r} |C_t f(z)| = \max_{|z|=r} |C_t f(z)| \leq \frac{M}{v(r)} \|f\|_{\infty, v}.$$

So, the set $\{C_t f : t \in [0, 1)\}$ is bounded in the Fréchet–Montel space $H(\mathbb{D})$ and hence, it is relatively compact in $H(\mathbb{D})$. Since $C_t f \rightarrow C_1 f$ pointwise on \mathbb{D} for $t \rightarrow 1^-$, it follows that $C_t f \rightarrow C_1 f$ with respect to τ_c , that is, in the Fréchet space $H(\mathbb{D})$, for $t \rightarrow 1^-$. In particular, $C_1 f \in H(\mathbb{D})$.

Since $H_v^\infty \subseteq H(\mathbb{D})$ and $C_t h \rightarrow C_1 h$ pointwise on \mathbb{D} as $t \rightarrow 1^-$, for every $h \in H(\mathbb{D})$, letting $t \rightarrow 1^-$ in (2.3) it follows that

$$|C_1 f(z)| v(z) \leq M \|f\|_{\infty, v}, \quad z \in \mathbb{D},$$

that is, $\|C_1 f\|_{\infty, v} \leq M \|f\|_{\infty, v}$. But, $f \in H_v^\infty$ is arbitrary and so $C_1 \in \mathcal{L}(H_v^\infty)$. \square

Proposition 2.12 *For each $n \in \mathbb{N}$, let $v(z) = (\log(\frac{e}{1-|z|}))^{-n}$ for $z \in \mathbb{D}$. Then $\sup_{t \in [0,1)} \|C_t\|_{H_v^\infty \rightarrow H_v^\infty} = \infty$.*

Proof Apply Proposition 2.11 and the discussion prior it. \square

3 Linear dynamics and mean ergodicity of C_t

The aim of this section is to investigate the mean ergodicity and the linear dynamics of the operators C_t , for $t \in [0, 1)$, acting on $H(\mathbb{D})$, H_v^∞ and H_v^0 .

An operator $T \in \mathcal{L}(X)$, with X a lchS, is called *power bounded* if $\{T^n : n \in \mathbb{N}_0\}$ is an equicontinuous subset of $\mathcal{L}(X)$. For a Banach space X , this means that $\sup_{n \in \mathbb{N}_0} \|T^n\|_{X \rightarrow X} < \infty$. Given $T \in \mathcal{L}(X)$, the averages

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^n T^m, \quad n \in \mathbb{N},$$

are usually called the Cesàro means of T . The operator T is said to be *mean ergodic* (resp., *uniformly mean ergodic*) if $(T_{[n]})_{n \in \mathbb{N}}$ is a convergent sequence in $\mathcal{L}_s(X)$ (resp., in $\mathcal{L}_b(X)$). It is routine to check that $\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n} T_{[n-1]}$, for $n \geq 2$, and hence, $\tau_s\text{-}\lim_{n \rightarrow \infty} \frac{T^n}{n} = 0$ whenever T is mean ergodic. Every power bounded operator on a Fréchet–Montel space X is necessarily uniformly mean ergodic, [1, Proposition 2.8]. Concerning the linear dynamics of $T \in \mathcal{L}(X)$, with X a lchS, the operator T is called *supercyclic* if, for some $z \in X$, the projective orbit $\{\lambda T^n z : \lambda \in \mathbb{C}, n \in \mathbb{N}_0\}$ is dense in X . Since the closure of the linear span of a projective orbit is separable, if such a supercyclic operator $T \in \mathcal{L}(X)$ exists, then X is necessarily separable.

Observe that the space H_v^∞ is never separable, [24, Theorem 1.1]. Therefore, every operator $T \in \mathcal{L}(H_v^\infty)$ is clearly not supercyclic. However, the spaces $H(\mathbb{D})$, [21, Theorem 27.2.5],

and H_v^0 , [24, Theorem 1.1], for every weight v are always separable. Hence, the problem of supercyclicity for non-zero operators $T \in \mathcal{L}(H(\mathbb{D}))$ and $T \in \mathcal{L}(H_v^0)$ arises.

The following result, [5, Theorem 6.4], is stated here for Banach spaces.

Theorem 3.1 *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be a compact operator such that $1 \in \sigma(T; X)$ with $\sigma(T; X) \setminus \{1\} \subseteq \overline{B(0, \delta)}$ for some $\delta \in (0, 1)$ and satisfying $\text{Ker}(I - T) \cap \text{Im}(I - T) = \{0\}$. Then T is power bounded and uniformly mean ergodic.*

A consequence of the previous theorem is the following result.

Proposition 3.2 *Let v be a weight function on $[0, 1)$ satisfying $\lim_{r \rightarrow 1^-} v(r) = 0$. For each $t \in [0, 1)$ both of the operators $C_t \in \mathcal{L}(H_v^\infty)$ and $C_t \in \mathcal{L}(H_v^0)$ are power bounded, uniformly mean ergodic and fail to be supercyclic.*

Proof Fix $t \in [0, 1)$. It was already noted that $C_t \in \mathcal{L}(H_v^\infty)$ cannot be supercyclic. The operator C_t is a compact operator on both H_v^∞ and on H_v^0 (cf. Proposition 2.7). Therefore, the compact transpose operators $C_t' \in \mathcal{L}((H_v^\infty)')$ and $C_t' \in \mathcal{L}((H_v^0)')$ have the same non-zero eigenvalues as C_t (see, e.g., [15, Theorem 9.10-2(2)]). In view of Proposition 2.8 it follows that $\sigma_{pt}(C_t'; (H_v^\infty)') = \sigma_{pt}(C_t'; (H_v^0)') = \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$. We can apply [6, Proposition 1.26] to conclude that C_t is not supercyclic on H_v^0 .

By Proposition 2.8 and its proof (as $x^{[0]} = (t^n)_{n \in \mathbb{N}_0}$) we have that $\text{Ker}(I - C_t) = \text{span}\{g_0\}$, with $g_0(z) = \sum_{n=0}^\infty t^n z^n$, for $z \in \mathbb{D}$. On the other hand, $\text{Im}(I - C_t)$ is a closed subspace of H_v^∞ (resp., of H_v^0), as C_t is compact in H_v^∞ (resp., in H_v^0), and $\text{Im}(I - C_t) \subseteq \{g \in H_v^\infty : g(0) = 0\}$ (resp., $\subseteq \{g \in H_v^0 : g(0) = 0\}$), because $C_t f(0) = f(0)$ for each $f \in H_v^\infty$ (resp., each $f \in H_v^0$). Moreover, [15, Theorem 9.10.1] implies that $\text{codim Im}(I - C_t) = \dim \text{Ker}(I - C_t) = 1$. Accordingly, both $\text{Im}(I - C_t)$ and $\{g \in H_v^\infty : g(0) = 0\} = \text{Ker}(\delta_0)$ are hyperplanes, where $\delta_0 \in (H_v^\infty)'$ is the linear evaluation functional $f \mapsto f(0)$, for $f \in H_v^\infty$. It follows that necessarily $\text{Im}(I - C_t) = \{g \in H_v^\infty : g(0) = 0\}$.

Let $h \in \text{Im}(I - C_t) \cap \text{Ker}(I - C_t)$. Then $h(0) = 0$ and there exists $\lambda \in \mathbb{C}$ such that $h = \lambda g_0$. This yields that $0 = h(0) = \lambda g_0(0) = \lambda$. Hence, $h = 0$. So, $\text{Im}(I - C_t) \cap \text{Ker}(I - C_t) = \{0\}$.

Proposition 2.8 implies that $1 \in \sigma(C_t; H_v^\infty) = \sigma(C_t; H_v^0) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\} \cup \{0\}$. Consequently, for $\delta = \frac{1}{2}$, all the assumptions of Theorem 3.1 are satisfied. So, we can conclude that C_t is power bounded and uniformly mean ergodic on both H_v^∞ and on H_v^0 . \square

In contrast to the compactness of C_t acting in the Banach spaces H_v^∞ and H_v^0 (cf. Proposition 2.7) the situation for the Fréchet space $H(\mathbb{D})$ is different.

Proposition 3.3 *For each $t \in [0, 1)$ the operator $C_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ is an isomorphism and, hence, it is not compact.*

Proof Fix $t \in [0, 1)$. Consider the operator $T_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$, for $f \in H(\mathbb{D})$, given by

$$T_t f(z) := (1 - tz)(zf(z))' = (1 - tz)(f(z) + zf'(z)), \quad z \in \mathbb{D}.$$

Then T_t is clearly well-defined. Moreover, its graph is closed. Indeed, for a given sequence $(f_n)_{n \in \mathbb{N}} \subset H(\mathbb{D})$, suppose that $f_n \rightarrow f$ in $H(\mathbb{D})$ and $T_t f_n \rightarrow g$ in $H(\mathbb{D})$. Since multiplication operators (by elements from $H(\mathbb{D})$) and the differentiation operator are continuous on $H(\mathbb{D})$ and the evaluation functionals at points of \mathbb{D} belong to $H(\mathbb{D})'$, it follows that $f_n' \rightarrow f'$ in $H(\mathbb{D})$ and hence, $T_t f_n = (1 - tz)(f_n + zf_n') \rightarrow (1 - tz)(f + zf') = T_t f$ in $H(\mathbb{D})$. Accordingly, $g = T_t f$. Since $H(\mathbb{D})$ is a Fréchet space, the closed graph theorem, [20, Corollary 5.4.3], implies that $T_t \in \mathcal{L}(H(\mathbb{D}))$.

Finally, it is routine to verify that $C_t \circ T_t = T_t \circ C_t = I$. So, the inverse operator $C_t^{-1} = T_t \in \mathcal{L}(H(\mathbb{D}))$ exists and hence, C_t is a bi-continuous isomorphism of $H(\mathbb{D})$ onto itself. In particular, C_t cannot be compact. \square

Let $\Lambda := \{\frac{1}{n+1} : n \in \mathbb{N}_0\}$ and $\Lambda_0 := \Lambda \cup \{0\}$. We recall from [4, Lemma 2.7] the following lemma, which is an extension of a result of Rhoades [27].

Lemma 3.4 *For every $\mu \in \mathbb{C} \setminus \Lambda_0$ there exist $\delta = \delta_\mu > 0$ and constants $d_\delta, D_\delta > 0$ such that $\overline{B(\mu, \delta)} \cap \Lambda_0 = \emptyset$ and*

$$\frac{d_\delta}{n^{\alpha(v)}} \leq \prod_{k=1}^n \left| 1 - \frac{1}{kv} \right| \leq \frac{D_\delta}{n^{\alpha(v)}}, \quad \forall n \in \mathbb{N}, v \in B(\mu, \delta), \quad (3.1)$$

where $\alpha(v) := \operatorname{Re}(\frac{1}{v})$.

Remark 3.5 As a direct application of Lemma 3.4 we obtain, for every $\mu \in \mathbb{C} \setminus \Lambda_0$, that there exist $\delta > 0$ and $d_\delta, D_\delta > 0$ such that $\overline{B(\mu, \delta)} \cap \Lambda_0 = \emptyset$ and, for every $v \in B(\mu, \delta)$ and $n \in \mathbb{N}_0$, we have that

$$d_\delta D_\delta^{-1} \left(\frac{n-h}{n+1} \right)^{\alpha(v)} \leq \prod_{j=n-h+1}^{n+1} \left| 1 - \frac{1}{jv} \right| \leq D_\delta d_\delta^{-1} \left(\frac{n-h}{n+1} \right)^{\alpha(v)}, \quad (3.2)$$

for all $h \in \{1, \dots, n-1\}$, where $\alpha(v) = \operatorname{Re}(\frac{1}{v})$.

For each $k \in \mathbb{N}$ with $k \geq 2$ define $r_k := (1 - \frac{1}{k})$. Define the norms $\|\cdot\|_k$ and $|||\cdot|||_k$ on $H(\mathbb{D})$ by

$$\|f\|_k := \sum_{n=0}^{\infty} |\hat{f}(n)| r_k^n, \quad f = \sum_{n=0}^{\infty} \hat{f}(n) z^n,$$

and

$$|||f|||_k := \sup_{n \in \mathbb{N}_0} |\hat{f}(n)| r_k^n, \quad f = \sum_{n=0}^{\infty} \hat{f}(n) z^n.$$

Lemma 3.6 *Each of the sequences $\{\|\cdot\|_k\}_{k \geq 2}$ and $\{|||\cdot|||_k\}_{k \geq 2}$ is a fundamental system of norms for $(H(\mathbb{D}), \tau_c)$.*

Proof Given $r \in (0, 1)$ choose any $k \geq 2$ such that $0 < r < (1 - \frac{1}{k})$. Then, for every $f \in H(\mathbb{D})$, we have

$$q_r(f) = \sup_{|z|=r} \left| \sum_{n=0}^{\infty} \hat{f}(n) z^n \right| \leq \sum_{n=0}^{\infty} |\hat{f}(n)| r^n \leq \sum_{n=0}^{\infty} |\hat{f}(n)| \left(1 - \frac{1}{k} \right)^n = \|f\|_k.$$

On the other hand, given $k \geq 2$, let $r_k := (1 - \frac{1}{k}) < (1 - \frac{1}{k+1}) := r_{k+1}$. By the Cauchy inequalities, for $n \in \mathbb{N}_0$, we have

$$|\hat{f}(n)| \leq \frac{1}{r_{k+1}^n} \max_{|z|=r_{k+1}} |f(z)| = \frac{1}{r_{k+1}^n} q_{r_{k+1}}(f), \quad f \in H(\mathbb{D}),$$

and hence,

$$\|f\|_{r_k} = \sum_{n=0}^{\infty} |\hat{f}(n)| r_k^n \leq q_{r_{k+1}}(f) \sum_{n=0}^{\infty} \left(\frac{r_k}{r_{k+1}} \right)^n = c q_{r_{k+1}}(f), \quad f \in H(\mathbb{D}),$$

with $c = \frac{1}{1 - \frac{r_k}{r_{k+1}}} = k^2 > 0$ as $\frac{r_k}{r_{k+1}} < 1$, which is independent of f .

So, the systems $\{q_r\}_{r \in (0,1)}$ and $\{\|\cdot\|_k\}_{k \geq 2}$ are equivalent on $H(\mathbb{D})$.

Observe, for every $k \geq 2$, that

$$\|f\|_k = \sup_{n \in \mathbb{N}_0} |\hat{f}(n)| r_k^n \leq \sum_{n=0}^{\infty} |\hat{f}(n)| r_k^n = \|f\|_k, \quad f \in H(\mathbb{D}),$$

and that

$$\begin{aligned} \|f\|_k &= \sum_{n=0}^{\infty} |\hat{f}(n)| r_k^n = \sum_{n=0}^{\infty} |\hat{f}(n)| \left(\frac{r_k}{r_{k+1}}\right)^n r_{k+1}^n \\ &\leq \sup_{n \in \mathbb{N}_0} |\hat{f}(n)| r_{k+1}^n \sum_{n=0}^{\infty} \left(\frac{r_k}{r_{k+1}}\right)^n = k^2 \|f\|_{k+1}, \end{aligned}$$

for $f \in H(\mathbb{D})$, where $\sum_{n=0}^{\infty} \left(\frac{r_k}{r_{k+1}}\right)^n = k^2$. Therefore, the systems $\{\|\cdot\|_k\}_{k \geq 2}$ and $\{\|f\|_k\}_{k \geq 2}$ are equivalent. \square

Proposition 3.7 For each $t \in [0, 1)$ the spectra of the operator $C_t \in \mathcal{L}(H(\mathbb{D}))$ are given by

$$\sigma_{pt}(C_t; H(\mathbb{D})) = \sigma(C_t; H(\mathbb{D})) = \Lambda \quad (3.3)$$

and

$$\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0. \quad (3.4)$$

Proof Let $t \in [0, 1)$ be fixed. For any weight function v on $[0, 1)$ satisfying $\lim_{r \rightarrow 1^-} v(r) = 0$, we have $H_v^\infty \subseteq H(\mathbb{D})$ continuously and $\Phi: H(\mathbb{D}) \rightarrow \omega$ is a continuous imbedding. Accordingly, $\sigma_{pt}(C_t; H_v^\infty) \subseteq \sigma_{pt}(C_t; H(\mathbb{D})) \subseteq \Lambda$; see the proof of Proposition 2.8. Since $\sigma_{pt}(C_t; H_v^\infty) = \Lambda$ (cf. Proposition 2.8) and $\sigma_{pt}(C_t^\omega; \omega) = \Lambda$ [5, Theorem 3.7], it follows that $\sigma_{pt}(C_t; H(\mathbb{D})) = \Lambda$. Moreover, in view of Proposition 2.8 above and Theorem 3.7 in [5], the eigenspace corresponding to each eigenvalue $\frac{1}{n+1} \in \Lambda$ is 1-dimensional. By Proposition 3.3, the operator $C_t: H(\mathbb{D}) \rightarrow H(\mathbb{D})$ is a bi-continuous isomorphism and so $0 \notin \sigma(C_t; H(\mathbb{D}))$.

The claim is that $\mathbb{C} \setminus \Lambda_0 \subseteq \rho(C_t; H(\mathbb{D}))$. To establish this claim, fix $v \in \mathbb{C} \setminus \Lambda_0$. Given $g(z) = \sum_{n=0}^{\infty} c_n z^n \in H(\mathbb{D})$, consider the identity

$$(C_t - vI)f(z) = g(z), \quad z \in \mathbb{D}, \quad (3.5)$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ is to be determined. It follows from (1.6) that $C_t f(z) = \sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \dots + a_n}{n+1} \right) z^n$ from which the identity $(C_t - vI)f(z) = \sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \dots + a_n}{n+1} - v a_n \right) z^n$ is clear. So, (3.5) is satisfied if and only if

$$\sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \dots + a_n}{n+1} - v a_n \right) z^n = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{D},$$

that is, if and only if

$$\frac{t^n a_0 + t^{n-1} a_1 + \dots + a_n}{n+1} - v a_n = c_n, \quad n \in \mathbb{N}_0.$$

In view of this we can argue, as in the proof of [5, Lemma 3.6], to show that if a function $f \in H(\mathbb{D})$ exists which satisfies the identity (3.5), then the Taylor coefficients $(a_n)_{n \in \mathbb{N}_0}$ of

f must verify the following equalities

$$\begin{aligned} a_0 &= \frac{c_0}{1-v} \\ a_n &= \frac{c_n}{(\frac{1}{n+1}-v)} + \sum_{h=1}^n (-1)^h \frac{v^{h-1} t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (\frac{1}{j}-v)} \\ &=: A_n + B_n, \quad n \geq 1. \end{aligned} \quad (3.6)$$

Observe, for each $n \geq 1$ and $h \in \{1, \dots, n\}$, that

$$(-1)^h \prod_{j=n-h+1}^{n+1} \left(\frac{1}{j} - v \right) = - \prod_{j=n-h+1}^{n+1} \left(v - \frac{1}{j} \right) = -v^{h+1} \prod_{j=n-h+1}^{n+1} \left(1 - \frac{1}{jv} \right)$$

and so

$$B_n = - \sum_{h=1}^n \frac{v^{h-1} t^h c_{n-h}}{v^{h+1} (n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{jv})} = - \frac{1}{v^2} \sum_{h=1}^n \frac{t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{jv})}.$$

Accordingly, to verify the claim we need to prove that the power series $\sum_{n=0}^{\infty} a_n z^n$ is convergent in \mathbb{D} , with $(a_n)_{n \in \mathbb{N}_0}$ defined according to (3.6). First, observe that the series $g(z) = \sum_{n=0}^{\infty} c_n z^n$ is convergent in \mathbb{D} and satisfies

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \limsup_{n \rightarrow \infty} \sqrt[n]{\frac{|c_n|}{|\frac{1}{n+1}-v|}} = \limsup_{n \rightarrow \infty} \sqrt[n]{|A_n|}.$$

Therefore, the series $\sum_{n=1}^{\infty} A_n z^n$ has the same radius of convergence as the series $\sum_{n=0}^{\infty} c_n z^n$ and hence, it converges in $H(\mathbb{D})$. Accordingly, $f_1(z) := \sum_{n=1}^{\infty} A_n z^n$, for $z \in \mathbb{D}$, belongs to $H(\mathbb{D})$. On the other hand, the series

$$\begin{aligned} \sum_{n=1}^{\infty} B_n z^n &= - \frac{1}{v^2} \sum_{n=1}^{\infty} \sum_{h=1}^n \frac{t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{jv})} \\ &= - \frac{1}{v^2} \sum_{h=1}^{\infty} t^h z^h \sum_{n=h}^{\infty} \frac{c_{n-h} z^{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{jv})}, \quad z \in \mathbb{D}. \end{aligned}$$

To establish the convergence of the series $\sum_{n=1}^{\infty} B_n z^n$ in $H(\mathbb{D})$, fix $z \in \mathbb{D} \setminus \{0\}$ and $r \in (|z|, 1)$. Recall, for every $n \in \mathbb{N}_0$, that the Taylor coefficients of g satisfy (as $\frac{1}{r} > 1$)

$$|c_n| = \left| \frac{g^{(n)}(0)}{n!} \right| = \left| \frac{1}{2\pi i} \int_{|\xi|=r} \frac{g(\xi)}{\xi^{n+1}} d\xi \right| \leq \frac{1}{r^n} \max_{|\xi|=r} |g(\xi)| \leq \frac{C}{r^{n+1}}$$

where $C := \max_{|\xi|=r} |g(\xi)|$. Therefore, setting $\alpha := \alpha(v) = \operatorname{Re}(\frac{1}{v})$ and $d := d_\delta$ and $D := D_\delta$ for a suitable $\delta > 0$ (cf. Remark 3.5), we obtain via (3.1) and (3.2) that

$$\begin{aligned} &\sum_{h=1}^{\infty} t^h |z|^h \sum_{n=h}^{\infty} \frac{|c_{n-h}| |z|^{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{jv}|} \\ &\leq C \sum_{h=1}^{\infty} t^h |z|^{h-1} \left(\frac{|z|}{r} d^{-1} (h+1)^{-\alpha-1} + \sum_{n=h+1}^{\infty} \left(\frac{|z|}{r} \right)^{n-h+1} D d^{-1} \left(\frac{n+1}{n-h} \right)^\alpha \right) \\ &= C d^{-1} \sum_{h=1}^{\infty} t^h (h+1)^{-\alpha-1} |z|^h + C D d^{-1} \sum_{h=1}^{\infty} t^h |z|^{h-1} \sum_{n=h+1}^{\infty} \left(\frac{|z|}{r} \right)^{n-h+1} \left(\frac{n+1}{n-h} \right)^\alpha \end{aligned}$$

$$\leq C d^{-1} \left(\sum_{h=1}^{\infty} t^h (h+1)^{-\alpha-1} |z|^h + D \sum_{h=1}^{\infty} t^h |z|^{h-1} \max\{1, (2+h)^\alpha\} \sum_{n=h+1}^{\infty} \left(\frac{|z|}{r} \right)^{n-h+1} \right),$$

which is finite after observing that if $\alpha \leq 0$, then $\left(\frac{n+1}{n-h}\right)^\alpha = \left(\frac{n-h}{n+1}\right)^{-\alpha} \leq 1$ for every $h \in \mathbb{N}$ and every $n \geq h+1$, whereas if $\alpha > 0$, then $\left(\frac{n+1}{n-h}\right)^\alpha = \left(1 + \frac{h+1}{n-h}\right)^\alpha \leq (2+h)^\alpha$. This implies that the series $\sum_{n=1}^{\infty} B^n z^n$ converges in $H(\mathbb{D})$. Accordingly, $f_2(z) := \sum_{n=1}^{\infty} B_n z^n$, for $z \in \mathbb{D}$, belongs to $H(\mathbb{D})$.

Set $f(z) := \frac{c_0}{1-v} + f_1(z) + f_2(z)$, for $z \in \mathbb{D}$. Then $f \in H(\mathbb{D})$. Moreover, the arguments above imply that f satisfies (3.5). The identities (3.6) imply that f is the unique solution of (3.5). Accordingly, the inverse operator $(C_t - vI)^{-1}: H(\mathbb{D}) \rightarrow H(\mathbb{D})$ exists. In particular, $(C_t - vI)^{-1} \in \mathcal{L}(H(\mathbb{D}))$ as it is the inverse of a continuous linear operator on a Fréchet space.

Since $v \in \mathbb{C} \setminus \Lambda_0$ is arbitrary and $0 \in \rho(C_t; H(\mathbb{D}))$, we can conclude that $\sigma(C_t; H(\mathbb{D})) = \Lambda$.

It remains to show that $\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0$. To establish this, fix $\mu \in \mathbb{C} \setminus \Lambda_0$ and observe, by Lemma 3.4, that there exist $\delta > 0$ and constants $d_\delta, D_\delta > 0$ such that $\overline{B}(\mu, \delta) \cap \Lambda_0 = \emptyset$ and the inequalities (3.1) and (3.2) are satisfied. We will show that $B(\mu, \delta) \subset \rho(C_t; H(\mathbb{D}))$ and that the set $\{(C_t - vI)^{-1} : v \in B(\mu, \delta)\}$ is equicontinuous in $\mathcal{L}(H(\mathbb{D}))$. To see this, first observe that the function $v \in \overline{B}(\mu, \delta) \mapsto \operatorname{Re}(\frac{1}{v}) \in \mathbb{R}$ is continuous and hence, $\alpha_0 := \max_{v \in \overline{B}(\mu, \delta)} \{\operatorname{Re}(\frac{1}{v})\}$ exists. For the sake of simplicity of notation set $d := d_\delta$ and $D := D_\delta$.

Let $v \in B(\mu, r)$, where $r := \frac{1}{2}d(\Lambda_0, \overline{B}(\mu, \delta)) > 0$ has the property that $|v - \frac{1}{j}| > r$ for all $j \in \mathbb{N}$. It was proved above, for any fixed $g(z) = \sum_{n=0}^{\infty} c_n z^n \in H(\mathbb{D})$, that

$$(C_t - vI)^{-1} g(z) = \frac{c_0}{1-v} + \sum_{n=1}^{\infty} \left(\frac{c_n}{\frac{1}{n+1} - v} - \frac{1}{v^2} \sum_{h=1}^n \frac{(-1)^h t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{jv})} \right) z^n,$$

for each $z \in \mathbb{D}$. So, for $k \geq 2$ fixed, consider the norm $\|\cdot\|_k$ in $H(\mathbb{D})$. Then we have, via (3.6), that

$$\begin{aligned} \|(C_t - vI)^{-1} g\|_k &\leq \frac{|c_0|}{|1-v|} + \sum_{n=1}^{\infty} \left| \frac{c_n}{\frac{1}{n+1} - v} - \frac{1}{v^2} \sum_{h=1}^n \frac{(-1)^h t^h c_{n-h}}{(n+1) \prod_{j=n-h+1}^{n+1} (1 - \frac{1}{jv})} \right| \left(1 - \frac{1}{k}\right)^n \\ &\leq \left(\frac{1}{r} \sum_{n=0}^{\infty} |c_n| \left(1 - \frac{1}{k}\right)^n \right) + \frac{1}{|v|^2} \sum_{n=1}^{\infty} \sum_{h=1}^n \frac{t^h |c_{n-h}|}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^n \\ &= \frac{1}{r} \|g\|_k + \frac{1}{|v|^2} \sum_{h=1}^{\infty} t^h \left(1 - \frac{1}{k}\right)^h \sum_{n=h}^{\infty} \frac{|c_{n-h}|}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^{n-h}. \end{aligned}$$

Moreover, (3.1) and (3.2) with $\alpha(v) = \operatorname{Re}(\frac{1}{v}) \leq \alpha_0$ imply, for each $h \in \mathbb{N}$, that

$$\begin{aligned} \sum_{n=h}^{\infty} \frac{|c_{n-h}|}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^{n-h} &= \sum_{l=0}^{\infty} \frac{|c_l|}{(l+h+1) \prod_{j=l+1}^{l+h+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^l \\ &= \frac{|c_0|}{(h+1) \prod_{j=1}^{h+1} |1 - \frac{1}{jv}|} + \sum_{l=1}^{\infty} \frac{|c_l|}{(l+h+1) \prod_{j=l+1}^{l+h+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k}\right)^l \\ &\leq d^{-1} |c_0| (h+1)^{\alpha(v)-1} + d^{-1} D \sum_{l=1}^{\infty} \frac{|c_l|}{l+h+1} \left(\frac{l+h+1}{l} \right)^{\alpha(v)} \left(1 - \frac{1}{k}\right)^l \end{aligned}$$

$$\begin{aligned} &\leq d^{-1}|c_0|(h+1)^{\alpha_0-1} + d^{-1}D \sum_{l=1}^{\infty} \frac{|c_l|}{l+h+1} \left(\frac{l+h+1}{l} \right)^{\alpha_0} \left(1 - \frac{1}{k} \right)^l \\ &\leq \max\{d^{-1}, d^{-1}D\}(2+h)^{\alpha_0} \sum_{l=0}^{\infty} |c_l| \left(1 - \frac{1}{k} \right)^l = K(2+h)^{\alpha_0} \|g\|_k, \end{aligned}$$

with $K := \max\{d^{-1}, d^{-1}D\}$, and hence, since $|v| > r$ for all $v \in B(\mu, \delta)$, that

$$\begin{aligned} &\frac{1}{|v|^2} \sum_{h=1}^{\infty} t^h \left(1 - \frac{1}{k} \right)^h \sum_{n=h}^{\infty} \frac{|c_{n-h}|}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{jv}|} \left(1 - \frac{1}{k} \right)^{n-h} \\ &\leq \frac{K}{r^2} \|g\|_k \sum_{h=1}^{\infty} t^h \left(1 - \frac{1}{k} \right)^h (2+h)^{\alpha_0} = K' \|g\|_k, \end{aligned}$$

with $K' = \frac{K}{r^2} \sum_{h=1}^{\infty} t^h \left(1 - \frac{1}{k} \right)^h (2+h)^{\alpha_0} < \infty$, by the ratio test, for instance.

We have established, for every $v \in B(\mu, \delta)$, that

$$\|(C_t - vI)^{-1}g\|_k \leq \left(\frac{1}{r} + K' \right) \|g\|_k.$$

Since $g \in H(\mathbb{D})$ and $k \geq 2$ are arbitrary, this shows that the set $\{(C_t - vI)^{-1} : v \in B(\mu, \delta)\}$ is equicontinuous. Hence, $\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0$. \square

Proposition 3.8 *For each $t \in [0, 1)$ the operator $C_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ is power bounded, uniformly mean ergodic but, it fails to be supercyclic. Moreover, $(I - C_t)(H(\mathbb{D}))$ is the closed subspace of $H(\mathbb{D})$ given by*

$$(I - C_t)(H(\mathbb{D})) = \{g \in H(\mathbb{D}) : g(0) = 0\} \quad (3.7)$$

and we have the decomposition

$$H(\mathbb{D}) = \text{Ker}(I - C_t) \oplus (I - C_t)(H(\mathbb{D})). \quad (3.8)$$

Proof Fix $t \in [0, 1)$. We first prove that C_t is power bounded. Once this is established, C_t is necessarily uniform mean ergodic because $H(\mathbb{D})$ is a Fréchet–Montel space (see [1, Proposition 2.8]).

Given $k \geq 2$ we have, for every $f \in H(\mathbb{D})$ and with $r_k := (1 - \frac{1}{k})$, that

$$\begin{aligned} |||C_t f|||_k &= \sup_{n \in \mathbb{N}_0} \left| \frac{1}{n+1} \sum_{j=0}^n t^{n-j} \hat{f}(j) \right| r_k \leq \sup_{n \in \mathbb{N}_0} \frac{1}{n+1} \sum_{j=0}^n |\hat{f}(j)| r_k^n \\ &\leq \sup_{n \in \mathbb{N}_0} \frac{1}{n+1} \sum_{j=0}^n |\hat{f}(j)| r_k^j \leq \sup_{j \in \mathbb{N}_0} |\hat{f}(j)| r_k^j = |||f|||_k, \end{aligned}$$

because $r_k^n \leq r_k^j$ for all $j \in \{0, 1, \dots, n\}$. It follows, for every $n \in \mathbb{N}$, that

$$|||C_t^n f|||_k \leq |||f|||_k, \quad f \in H(\mathbb{D}).$$

Since $k \geq 2$ is arbitrary, the operator $C_t \in \mathcal{L}(H(\mathbb{D}))$ is indeed power bounded.

To establish that $C_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ is not supercyclic, note that the continuous embedding $\Phi : H(\mathbb{D}) \rightarrow \omega$ has dense range. The operator $C_t^\omega \in \mathcal{L}(\omega)$ satisfies $\Phi \circ C_t = C_t^\omega \circ \Phi$ as an identity in $\mathcal{L}(H(\mathbb{D}), \omega)$, which implies if $C_t : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ is supercyclic, then also

$C_t^\omega: \omega \rightarrow \omega$ must be supercyclic as $\Phi \circ C_t^n = \Phi \circ C_t \circ C_t^{n-1} = C_t^\omega \circ \Phi \circ C_t^{n-1} = \dots = (C_t^\omega)^n \circ \Phi$, for all $n \in \mathbb{N}$, and $\Phi(H(\mathbb{D}))$ is dense in ω . A contradiction with [5, Theorem 6.1].

To establish (3.7) note that $(I - C_t)(H(\mathbb{D})) \subseteq \{g \in H(\mathbb{D}) : g(0) = 0\}$ because $C_t f(0) = f(0)$ for every $f \in H(\mathbb{D})$. To show the reverse inclusion, let $g \in H(\mathbb{D})$ satisfy $g(0) = 0$. Then $h(z) := zg'(z) + g(z)$, for $z \in \mathbb{D}$, is holomorphic and $h(0) = 0$. Accordingly, also $z \mapsto \frac{h(z)}{z}$, for $z \in \mathbb{D} \setminus \{0\}$, and taking the value $h'(0)$ at $z = 0$ is holomorphic in \mathbb{D} . Define $f \in H(\mathbb{D})$ by

$$f(z) := \frac{1}{tz - 1} \int_0^z (1 - t\xi) \frac{h(\xi)}{\xi} d\xi, \quad z \in \mathbb{D},$$

and note that $f(0) = 0$. Direct calculation reveals that

$$\frac{f(z)}{1 - tz} - (zf(z))' = h(z) = (zg(z))', \quad z \in \mathbb{D},$$

from which it follows that

$$\int_0^z \frac{f(\xi)}{1 - t\xi} d\xi - zf(z) = zg(z), \quad z \in \mathbb{D}.$$

Since $f(0) = 0$, we can conclude that

$$\frac{1}{z} \int_0^z \frac{f(\xi)}{1 - t\xi} d\xi - f(z) = g(z), \quad z \in \mathbb{D},$$

that is, $(C_t - I)f = g$ and so $g \in (I - C_t)(H(\mathbb{D}))$. Hence, (3.7) is valid.

To show the validity of (3.8) it suffices to repeat the argument given in the proof of Proposition 3.2. \square

Acknowledgements The research of J. Bonet was partially supported by the Project PID2020-119457GB-I00 funded by MCIN/AEI/10.13039/501100011033 and by “ERFD A way of making Europe” and by the project GV AICO/2021/170.

Funding Open access funding provided by Università del Salento within the CRUI-CARE Agreement.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Albanese, A.A., Bonet, J., Ricker, W.J.: Mean ergodic operators in Fréchet spaces. *Ann. Acad. Sci. Fenn. Math.* **34**, 401–436 (2009)
2. Albanese, A.A., Bonet, J., Ricker, W.J.: The Cesàro operator in growth Banach spaces of analytic functions. *Integr. Equ. Oper. Theory* **86**, 97–112 (2016)
3. Albanese, A.A., Bonet, J., Ricker, W.J.: The Cesàro operator on Korenblum type spaces of analytic functions. *Collect. Math.* **69**, 263–281 (2018)
4. Albanese, A.A., Bonet, J., Ricker, W.J.: The Cesàro operator on duals of power series spaces of infinite type. *J. Oper. Theory* **79**, 373–402 (2018)
5. Albanese, A.A., Bonet, J., Ricker, W.J.: Spectral properties of generalized Cesàro operator in sequence spaces. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* 117, Article number 140 (2023). <https://doi.org/10.48550/arXiv.2305.04805>

6. Bayart, F., Matheron, E.: Dynamics of Linear Operators. Cambridge Tracts in Mathematics, vol. 179. Cambridge University Press, Cambridge (2009)
7. Bierstedt, K.D., Bonet, J., Galbis, A.: Weighted spaces of holomorphic functions on balanced domains. *Mich. Math. J.* **40**, 271–297 (1993)
8. Bierstedt, K.D., Summers, W.H.: Biduals of weighted Banach spaces of analytic functions. *J. Aust. Math. Soc.* **54**, 70–79 (1993)
9. Blasco, O.: Cesàro-type operators on Hardy spaces. *J. Math. Anal. Appl.* **529**, Article number 127017 (2024). <https://doi.org/10.1016/j.jmaa.2023.127017>
10. Bonet, J.: Weighted Banach spaces of analytic functions with sup-norms and operators between them: a survey. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **116**(4), Article number 184 (2022)
11. Bonet, J., Ricker, W.J.: Mean ergodicity of multiplication operators in weighted spaces of holomorphic functions. *Arch. Math.* **92**, 428–437 (2009)
12. Curbera, G.P., Ricker, W.J.: The Cesàro operator and unconditional Taylor series in Hardy spaces. *Integr. Equ. Oper. Theory* **83**, 179–195 (2015)
13. Curbera, G.P., Ricker, W.J.: Fine spectra and compactness of generalized Cesàro operators in Banach lattices in C_0^N . *J. Math. Anal. Appl.* **507**, 125824 (2022)
14. Danikas, N., Siskakis, A.: The Cesàro operator on bounded analytic functions. *Analysis* **13**, 295–299 (1993)
15. Edwards, R.E.: Functional Analysis, Theory and Applications. Holt, Rinehart and Winston, New York (1965)
16. Galanopoulos, P., Girela, D., Merchán, N.: Cesàro-like operators acting on spaces of analytic functions. *Anal. Math. Phys.* **12**, Article number 51 (2022)
17. Grosse-Erdmann, K.-G., Peris Manguillot, A.: Linear Chaos, Universitext. Springer, London (2011)
18. Grothendieck, A.: Topological Vector Spaces. Gordon and Breach, London (1973)
19. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
20. Jarchow, H.: Locally Convex Spaces. Teubner, Stuttgart (1981)
21. Köthe, G.: Topological Vector Spaces I, 2nd edn. Springer, Berlin (1983)
22. Köthe, G.: Topological Vector Spaces II. Springer, Berlin (1979)
23. Krengel, U.: Ergodic Theorems. Walter de Gruyter, Berlin (1985)
24. Lusky, W.: On the isomorphism classes of weighted spaces of harmonic and holomorphic functions. *Studia Math.* **175**, 19–40 (2006)
25. Rhaly, H.C., Jr.: Discrete generalized Cesàro operators. *Proc. Am. Math. Soc.* **86**, 405–409 (1982)
26. Rhaly, H.C., Jr.: Generalized Cesàro matrices. *Canad. Math. Bull.* **27**, 417–422 (1984)
27. Rhoades, B.E.: Spectra of some Hausdorff operators. *Acta Sci. Math. (Szeged)* **32**, 91–100 (1971)
28. Sawano, Y., El-Shabrawy, S.R.: Fine spectra of the discrete generalized Cesàro operator on Banach sequence spaces. *Monatsh. Math.* **192**, 185–224 (2020)
29. Waelbrock, L.: Topological Vector Spaces and Algebras, LNM, vol. 230. Springer, Berlin (1971)
30. Yildirim, M., Durna, N.: The spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on ℓ^p ($1 < p < \infty$). *J. Inequal. Appl.* **2017**(1), 1–13 (2017)
31. Yildirim, M., Mursaleen, M., Dogan, C.: The spectrum and fine spectrum of the generalized Rhaly–Cesàro matrices on c_0 and c . *Oper. Matrices* **12**(4), 955–975 (2018)