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ORIGINAL PAPER

Energy asymptotics for the strongly damped Klein–Gordon equation

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Abstract
We consider the strongly damped Klein–Gordon equation for defocusing nonlinearity and we study the asymptotic behaviour of the energy for periodic solutions. We prove first the exponential decay to zero for zero mean solutions. Then, we characterize the limit of the energy, when the time tends to infinity, for solutions with small enough initial data and we finally prove that such limit is not necessary zero.

Keywords Klein–Gordon equation · Strong and weak damping · Energy decay

Mathematics Subject Classification 35Qxx

1 Introduction

We consider the damped nonlinear Klein–Gordon equation

$$\partial_t^2 \psi + L\partial_t \psi - \Delta\psi + \psi + |\psi|^p \psi = 0, \;\; \psi(0) = \psi_0, \;\; \partial_t \psi(0) = \psi_1, \tag{1}$$

where $\psi : \mathcal{C}(\mathbb{R}^+, X)$, X is a Banach subspace in $L^2(\Omega \subset \mathbb{R}^d)$ and $L : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ is some non-negative operator. Damped semi-linear wave equations particularly Eq. (1) have gained a lot of attention numerically and analytically. The related literature is extensive. See for example [4, 6, 9, 10, 15, 16]. The associated energy $E \in \mathcal{C}(X, \mathbb{R}^+)$ is given by

$$E(\psi) = \frac{1}{2}\int_\Omega |\partial_t \psi|^2 + |\psi|^2 + |\nabla\psi|^2 + \frac{1}{p+2}\int_\Omega |\psi|^{p+2}. \tag{2}$$

The linear part of the system is dissipative in the sense that the linear semigroup action loses energy. Indeed, we have atleast formally

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\psi(t)) = -\int_\Omega |\sqrt{L}\,\partial_t \psi|^2 \le 0. \tag{3}$$

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In general, damping can be *weak* when the semigroup generated by the linear part of the equation is merely continuous (example $L = \gamma(x)I$ with $\gamma(x) \geq 0$), and *strong* when the semigroup is compact (example $L = -\Delta + \gamma(x)I$ with $\gamma(x) \geq 0$). It has been proven that when $L = \gamma(x)$ is a "positive multiplicator", namely,

$$\forall x \in \omega \subset \Omega, \gamma(x) \geq \alpha > 0 \tag{4}$$

for some open set ω and a positive α, the energy E decays exponentially to zero. We refer to [1, 3, 7] for related results. Another related results of polynomially and exponentially decays a for slightly different types of damping are proved by Royer [14] for the linear Klein–Gordon equation and by [11] for the nonlinear equation.

The energy asymptotics for the strongly damped equation hasn't gained enough attention. Xu and Lian [8] have recently considered the following equation

$$\partial_t^2 \psi - \omega \Delta \partial_t \psi + \mu \psi - \Delta \psi + \psi + f(\psi) = 0, \omega \geq 0$$

with logarithmic nonlinearity given by $f(\psi) = -\ln(|\psi|)\psi$ and they proved, under the strong assumption $\mu > \omega\lambda_1$, that the related energy decays exponentially to zero for three different initial energy levels determined by the minima of the potential energy on the so-called Nehari manifold [12, 13]. λ_1 is the first eigenvalue of the operator$-\Delta$ under the homogeneous Dirichlet boundary conditions. Considering a functional setting similar to that used in [8], Cordeiro et al. [2] proved the exponential decay to zero of the energy related to the strongly damped ($L = -\Delta$) Klein–Gordon equation of Kirchhoff–Carrier type

$$\partial_t^2 \psi - \Delta \partial_t \psi - M(\|\nabla \psi\|)\Delta \psi + M_1(\|\psi\|)\ \psi - \ln(|\psi|^2)\psi = 0,$$

where M, M_1 are two continuous non-negative functions defined on $[0, +\infty)$. However, depending on the initial data, the energy in the strongly damped case ($L = -\Delta$) might decay to some conserved quantity different from zero. For example, the initially non-zero-average periodic solutions defined on the $d-$torus $\Omega = \mathbb{T}^d$ have such quantity. This observation has not been studied so far and will be the focus of this paper. To clarify our purpose, let's consider first the linear equation defined by the linear part of (1) and denote

$$\theta(t) := \fint \psi(t) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \psi(t, x)\mathrm{d}x$$

which corresponds to the time-dependent zero Fourier coefficient of $\psi(t, \cdot)$. Thus, the zero average function $\phi := \psi - \theta$ satisfies the linear system

$$\partial_t^2 \phi - \Delta \partial_t \phi - \Delta \phi + \phi = 0, \tag{5}$$

which retrieves the exponential decay of $E(\phi)$ to zero. Moreover, $E(\psi)$ decays to a conserved positive quantity; more specifically

$$E(\psi) = E(\phi) + \frac{(2\pi)^d}{2}(|\theta|^2 + |\theta'|^2)$$

and

$$|\theta(t)|^2 + |\theta'(t)|^2 = |\theta(0)|^2 + |\theta'(0)|^2.$$

Indeed, θ satisfies the differential equation

$$\theta'' + \theta = 0.$$

In the spectral level, the damping effect of $L = -\Delta$ is caused by its non-zero eigenvalues, whereas θ oscillates independently, so that quantity $|\theta(t)|^2 + |\theta'(t)|^2$ remains conserved.

This separation is however not clear in the case of the nonlinear equation. The rest of the paper is organised as follows: We prove first that the Cauchy problem of (1) ($L = -\Delta$) is globally well posed on $H^2(\mathbb{T}^d)$. Then, we prove that the energy of zero-average solutions decays exponentially to zero. Finally, we study the energy asymptotics for small initial data solutions.

Useful notations

- The norm $\|\cdot\|$ refers to the L^2 norm.
- The constant C changes in the estimates from line to line unless otherwise noted.

2 The Cauchy problem

In this section we study the Cauchy problem for (1) on $\mathcal{C}(\mathbb{R}^+, H^2(\mathbb{T}^d))$.

We have

Theorem 2.1 *Assume that $d \le 3$, $p \ge 0$ and $\psi_0, \psi_1 \in H^2(\mathbb{T}^d)$. Equation (1) has a unique global solution $\psi \in \mathcal{C}^1(\mathbb{R}^+, H^2(\mathbb{T}^d))$ such that $\psi(0) = \psi_0$ and $\partial_t\psi(0) = \psi_1$. Moreover, there exists a constant $C > 0$ such that*

$$\|\psi(t)\|_{H^2} \le C\left(\|\psi_0\|_{H^2} + \|\psi_0\|_{p+2}^{\frac{p}{2}+1} + \|\psi_1\|\right) \ \forall t \in \mathbb{R}^+. \tag{6}$$

Proof Written as a first order system, Eq. (1) takes the abstract form

$$\partial_t\Psi + \mathcal{A}\Psi + F(\Psi) = 0 \tag{7}$$

with

$$\Psi := \begin{pmatrix} \psi \\ \partial_t\psi \end{pmatrix}, \ \mathcal{A} := \begin{pmatrix} 0 & -1 \\ -\Delta + 1 & -\Delta \end{pmatrix}, \ \text{and } F(\Psi) := \begin{pmatrix} 0 \\ |\psi|^p\psi \end{pmatrix}.$$

We cast (7) in its mild formulation

$$\Psi(t) = \mathrm{e}^{-t\mathcal{A}}\Psi(0) - \int_0^t \mathrm{e}^{(\tau-t)\mathcal{A}} F(\Psi(\tau))\mathrm{d}\tau, \tag{8}$$

where

$$\mathrm{e}^{-t\mathcal{A}} = \mathrm{e}^{\frac{1}{2}t\Delta}\begin{pmatrix} \cosh(tA) - \frac{1}{2}\Delta A^{-1}\sinh(tA) & A^{-1}\sinh(tA) \\ (\Delta - 1)A^{-1}\sinh(tA) & \cosh(tA) + \frac{1}{2}\Delta A^{-1}\sinh(tA) \end{pmatrix}$$

with

$$A := \frac{1}{2}\sqrt{\Delta^2 + 4\Delta - 4}.$$

Taking into account that $d \le 3$, it is classical that the map

$$\Psi \mapsto F(\Psi)$$

leaves $X_2 := H^2 \times H^2$ invariant, and is Lipschitz continuous on the bounded subsets of X_2. Consequently, (8) has a maximal solution $\Psi \in \mathcal{C}([0, T^*), X_2)$, which blows up as t approaches T^* if $T^* < \infty$. We prove now that $T^* = \infty$. Denote

$$J(\psi) := \frac{1}{2}\|\Delta\psi - \partial_t\psi\|^2 + \frac{1}{2}\|\psi\|_{H^1}^2 + \frac{1}{p+2}\|\psi\|_{p+2}^{p+2}.$$

It is clear that

$$J(\psi) = E(\psi) + \frac{1}{2}\|\Delta\psi\|^2 - \mathrm{Re}\int_{\mathbb{T}^d} \partial_t\psi\Delta\bar{\psi}.$$

Differentiating J with respect to time for $t < T^*$ and using (3), we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J(\psi) &= \frac{\mathrm{d}}{\mathrm{d}t}E(\psi) + \mathrm{Re}\int_{\mathbb{T}^d} \partial_t\Delta\psi\Delta\bar{\psi} - \partial_t^2\psi\Delta\bar{\psi} - \partial_t\psi\Delta\partial_t\bar{\psi} \\ &= \mathrm{Re}\int_{\mathbb{T}^d}(-\Delta\psi + \psi + |\psi|^p\psi)\Delta\bar{\psi} \\ &= -\|\Delta\psi\|^2 - \|\nabla\psi\|^2 - \int_{\mathbb{T}^d}|\psi|^p|\nabla\psi|^2 + p|\psi|^{p-2}|\mathrm{Re}(\psi\nabla\bar{\psi})|^2\end{aligned}$$

which means that $t \mapsto J(\psi(t))$ is decreasing and we have

$$J(\psi(t)) \le J(\psi(0)) \ \forall t \in [0, T^*).$$

Consequently, there exists a constant $C > 0$ depending on ψ_0 and ψ_1 which can be expressed in the form of the r.h. side of (6) such that

$$\|\psi(t)\|_{H^2} \le C \ \forall t \in [0, T^*). \tag{9}$$

Using (8) together with (9), we find that there exists a constant $C_1 > 0$ depending on ψ_0 and ψ_1 such that for any $0 < T < T^*$, we have

$$\|\Psi(t)\|_{L^\infty([0,T],X_2)} \le \|\Psi(0)\|_{X_2} + C_1 T,$$

which implies that Ψ can be extended as a global solution in $\mathcal{C}(\mathbb{R}^+, X_2)$. □

Remark 2.2 Giving more restrictions on the nonlinearity, a global well posedness result for the Cauchy problem of (1) could be established on a larger spacial domain. If, for example, $\psi_0, \psi_1 \in H^1(\mathbb{T}^d)$ and

$$0 < p \begin{cases} \le \frac{4}{d-2} & \text{if } d \ge 3, \\ < \infty & \text{if } d = 1, 2, \end{cases}$$

then (1) has a unique global solution

$$\psi \in \mathcal{C}(\mathbb{R}^+, H^1(\mathbb{T}^d)) \cap \mathcal{C}^1(\mathbb{R}^+, L^2(\mathbb{T}^d)) \cap \mathcal{C}^2(\mathbb{R}^+, H^{-1}(\mathbb{T}^d))$$

with $\partial_t\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^+, H^1(\mathbb{T}^d))$. The proof for such result follows the same steps of Theorem 3.1 in [5] for the the existence of a unique maximal solution. The fact that the energy is decreasing implies the boundness of $t \mapsto \|\psi(t)\|_{H^1}$ which in terns imply that the solution is global. However, the spacial regularity given by Theorem 2.1 is needed in the main result presented in Theorem 3.2.

3 Asymptotic behaviour of energy

We consider now the energy decay for zero mean solutions.

Theorem 3.1 *Let ψ denote a solution of* (1) *in the same functional settings mentioned in Remark 2.2. Assume further that*

$$⨍ \psi(t) = 0, \ \ \forall t \in \mathbb{R}^+. \tag{10}$$

Then, there exists $C, \alpha > 0$ such that

$$E(\psi(t)) \leq C\mathrm{e}^{-\alpha t}, \;\; \forall t \in \mathbb{R}^+. \tag{11}$$

Proof The proof uses the same technique used to prove Proposition 2.6 in [7] together with the Poincaré inequality

$$\|\partial_t \psi\| = \left\| \partial_t \left(\psi - \fint \psi \right) \right\| \leq C \|\nabla \partial_t \psi\|$$

for some constant $C > 0$. We introduce the modified energy

$$E_\varepsilon(\psi) \;:=\; E(\psi) + \varepsilon \int_{\mathbb{T}^d} \mathrm{Re}(\bar{\psi}\partial_t \psi)$$

with $\varepsilon > 0$. For ε small enough, there exist two constants $C_1, C_2 > 0$ depending on ε such that

$$C_1(\|\psi\|^2 + \|\partial_t\psi\|^2) \leq \frac{1}{2}(\|\psi\|^2 + \|\partial_t\psi\|^2) + \varepsilon \int_{\mathbb{T}^d} \mathrm{Re}(\bar{\psi}\partial_t\psi) \leq C_2(\|\psi\|^2 + \|\partial_t\psi\|^2),$$

which means that E_ε is equivalent to E for small enough ε and it is sufficient to prove the exponential decay for E_ε. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_\varepsilon(\psi(t)) &= \frac{\mathrm{d}}{\mathrm{d}t} E(\psi(t)) + \varepsilon \|\partial_t \psi\|^2 + \varepsilon \int_{\mathbb{T}^d} \mathrm{Re}(\bar{\psi}\partial_t^2 \psi) \\
&= -\|\nabla \partial_t \psi\|^2 + \varepsilon \|\partial_t \psi\|^2 - \varepsilon \|\psi\|^2 - \varepsilon \|\nabla \psi\|^2 \\
&\quad -\varepsilon \|\psi\|_{p+2}^{p+2} - \varepsilon \int_{\mathbb{T}^d} \mathrm{Re}(\nabla \bar{\psi} \nabla \partial_t \psi) \\
&\leq -(C^{-1} - \varepsilon - \frac{\varepsilon}{2} C^{-1}) \|\partial_t \psi\|^2 - \varepsilon \|\psi\|^2 - \frac{\varepsilon}{2} \|\nabla \psi\|^2 - \varepsilon \|\psi\|_{p+2}^{p+2} \\
&\leq -\beta E_\varepsilon(\psi(t))
\end{aligned}$$

with $\beta = \beta(\varepsilon) > 0$ which implies the exponential decay of E_ε and thus of E. □

We study now the energy decay for solutions with small initial data. Namely,

Theorem 3.2 *Let $\psi_0, \psi_1 \in H^2(\mathbb{T}^d)$. Denote ψ the solution of (1) given by Theorem 2.1 such that*

$$\psi(0) = \psi_0, \;\; \partial_t \psi(0) = \psi_1.$$

Denote further

$$\theta := \fint \psi, \;\; \phi := \psi - \theta \;\; \textit{and}$$

$$Q(\theta) := (2\pi)^d \left(\frac{1}{2}|\theta|^2 + \frac{1}{2}|\theta'|^2 + \frac{1}{p+2}|\theta|^{p+2} \right).$$

Then, if $\|\psi_0\|_{H^2}, \|\psi_1\|_{H^2}$ are small enough, there exist $\tilde{C}, \beta, \alpha > 0$ depending on ψ_0 and ψ_1 and there exists $C > 0$ such that

$$\|\phi\|_{H^2} + \|\partial_t \phi\|_{H^2} \leq C(\|\phi(0)\|_{H^2} + \|\partial_t \phi(0)\|_{H^2})\mathrm{e}^{-\beta t}. \tag{12}$$

Moreover, the limit $\lim_{t\to\infty} Q(\theta(t))$ exists and we have

$$|E(\psi(t)) - Q(\theta(t))| \leq \tilde{C}\,\mathrm{e}^{-\alpha\, t}. \tag{13}$$

Proof Denote $\phi = \psi - \theta$ and $f(z) = |z|^p z$. Thus, (ϕ, θ) satisfies the system

$$\partial_t^2 \phi - \Delta \partial_t \phi - \Delta \phi + \phi + f(\psi) - \fint f(\psi) = 0, \tag{14a}$$

$$\theta'' + \theta + \fint f(\psi) = 0. \tag{14b}$$

Denote

$$\Phi := \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix} \text{ and } G(\psi) := \begin{pmatrix} 0 \\ f(\psi) - \fint f(\psi) \end{pmatrix}.$$

Thus, as in (8), Φ satisfies the mild equation

$$\Phi(t) = \mathrm{e}^{-t\mathcal{A}} \Phi(0) - \int_0^t \mathrm{e}^{(\tau - t)\mathcal{A}} G(\psi(\tau)) \mathrm{d}\tau. \tag{15}$$

Using the Pioncaré inequality, there is a constant $C > 0$ such that

$$\|G(\psi)\|_{H^2} \le C \|\nabla f(\psi)\|_{H^1}. \tag{16}$$

Moreover, we have

$$|\nabla f(\psi)| \le (p+1)|\psi|^p |\nabla \psi| = (p+1)|\psi|^p |\nabla \phi|,$$

and

$$|\Delta f(\psi)| \le (p+1)(|\psi|^p |\Delta \psi| + p|\psi|^{p-1} |\nabla \psi|^2) = (p+1)(|\psi|^p |\Delta \phi| + p|\psi|^{p-1} |\nabla \phi|^2).$$

Thus, using the interpolation inequality

$$\| |\nabla \psi|^2 \| \le \sqrt{2} \|\nabla \psi\| \|\Delta \psi\|$$

together with the Sobolev inequality

$$\|\psi\|_{L^\infty} \le C \|\psi\|_{H^2},$$

there exists a constant $C > 0$ such that

$$\|\nabla f(\psi)\|_{H^1} \le C \|\psi\|_{H^2}^p \|\phi\|_{H^2}. \tag{17}$$

For any $\varphi \in H^2(\mathbb{T}^d)$ with $\fint \varphi = 0$, we have

$$\|\mathrm{e}^{\frac{1}{2} t \Delta} \sinh(tA) \varphi\|_{H^2} \le \mathrm{e}^{-\frac{1}{2} t} \|\varphi\|_{H^2}, \quad \|\mathrm{e}^{\frac{1}{2} t \Delta} \cosh(tA) \varphi\|_{H^2} \le \mathrm{e}^{-\frac{1}{2} t} \|\varphi\|_{H^2},$$

which implies that, for any $\Upsilon \in X_2$ with $\fint \Upsilon = 0$, we have

$$\left\| \mathrm{e}^{-t\mathcal{A}} \Upsilon \right\|_{X_2} \le C \mathrm{e}^{-\frac{1}{2} t} \|\Upsilon\|_{X_2} \tag{18}$$

for some $C > 0$. Then, combining (15)–(18) together with estimate (6), we get

$$\begin{aligned} \|\Phi(t)\|_{X_2} &\le C_1 \mathrm{e}^{-\frac{1}{2} t} \|\Phi(0)\|_{X^2} + C_2 \int_0^t \mathrm{e}^{-\frac{1}{2}(t-\tau)} \|\psi(\tau)\|_{H^2}^p \|\Phi(\tau)\|_{X_2} \mathrm{d}\tau \\ &\le C_1 \mathrm{e}^{-\frac{1}{2} t} \|\Phi(0)\|_{X_2} + C_2 C(\psi_0, \psi_1) \int_0^t \mathrm{e}^{-\frac{1}{2}(t-\tau)} \|\Phi(\tau)\|_{X_2} \mathrm{d}\tau, \end{aligned}$$

for some constants $C_1, C_2 > 0$. Since $H^2(\mathbb{T}^d)$ is embedded continuously in $L^{p+2}(\mathbb{T}^d)$ and using (6), we find that for small enough $\|\psi_0\|_{H^2}$, $\|\psi_1\|$, we have

$$\beta := \frac{1}{2} - C_2 C(\psi_0, \psi_1) > 0,$$

which implies, using Gronwall inequality,

$$\|\Phi(t)\|_{X^2} \leq C_1 \|\Phi(0)\|_{X^2} \mathrm{e}^{-\beta t}. \tag{19}$$

Since

$$\left||\phi + \theta|^{p+2} - |\theta|^{p+2}\right| \leq (p+2)2^p |\phi| (|\phi|^{p+1} + |\theta|^{p+1})$$

and using the continuous embedding $H^2(\mathbb{T}^d) \subset L^{p+2}(\mathbb{T}^d)$ together with (19), we get

$$\begin{aligned} |E(\psi(t)) - Q(t)| &\leq E(\phi(t)) + \frac{1}{p+2} \left| \|\psi(t)\|_{p+2}^{p+2} - (2\pi)^d |\theta(t)|^{p+2} \right| \\ &\leq C \mathrm{e}^{-\alpha t} \end{aligned} \tag{20}$$

for some constants $C, \alpha > 0$ depending on $\|\psi_0\|_{H^2}$ and $\|\psi_1\|$. Since $t \mapsto E(\psi(t))$ is decreasing [see (3)], the limit $\lim_{t\to\infty} Q(t)$ exists and (13) holds. □

We prove in the following theorem that $E(\psi)$ doesn't decay necessarily to zero.

Theorem 3.3 *Assume that $\frac{3}{2} \leq p \leq 4$. Keeping the other settings of Theorem 3.2, there exist $\psi_0, \psi_1 \in H^2(\mathbb{T}^d)$ for which we have*

$$\lim_{t\to\infty} Q(t) > 0.$$

Proof Writing $\psi = \phi + \theta$ and using the Taylor series with integral remainder, we find that

$$\begin{aligned} f(\psi) &= f(\theta) + \frac{\mathrm{d}}{\mathrm{d}x} f(x\phi + \theta)|_{x=0} + \int_0^1 \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x\phi + \theta)|(1-x)\mathrm{d}x \\ &= |\theta|^p \theta + \frac{\varepsilon}{2} |\theta|^{p-2} ((p+2)|\theta|^2 \phi + p\,\theta^2 \bar{\phi}) \\ &\quad + \frac{p(p+2)}{4} \int_0^1 |x\phi + \theta|^{p-2} (2(x\phi + \theta)|\phi|^2 + \overline{(x\phi + \theta)} \phi^2)(1-x)\mathrm{d}\sigma \\ &\quad + \frac{p(p-2)}{4} \int_0^1 |x\phi + \theta|^{p-4} (x\phi + \theta)^3 \bar{\phi}^2 (1-x)\mathrm{d}\sigma. \end{aligned} \tag{21}$$

Since $\frac{3}{2} \leq p \leq 4$, for any $x, y, z \in \mathbb{C}$, we have

$$|z||y|^{p-1} \leq C(|y| + |z|^2 + |y|^{p+2})$$

and then

$$\begin{aligned} |x|^2 |x+y|^{p-1} |z| &\leq C(|x|^{p+1} + |x|^2)(|z||y|^{p-1} + |z|) \\ &\leq C(|x|^{p+1} + |x|^2)(|z| + |y| + |z|^2 + |y|^{p+2}) \end{aligned}$$

for some constant $C = C(p) > 0$. Thus, we can write

$$|Q'| = (2\pi)^d \left| \mathrm{Re} \left(\left(⨍ f(\psi) - |\theta|^p \theta \right) \bar{\theta'} \right) \right|$$

$$
\begin{aligned}
&\leq C \fint \int_0^1 |\phi|^2 |x\phi + \theta|^{p-1} |\theta'| (1-x) \mathrm{d}x \\
&\leq C(|\theta'| + |\theta| + |\theta'|^2 + |\theta|^{p+2}) \fint (|\phi|^{p+1} + |\phi|^2) \\
&\leq C(\|\phi\|_{p+1}^{p+1} + \|\phi\|^2)(Q + \sqrt{Q}),
\end{aligned}
$$

where $C = C(p) > 0$. Denote now $S := \frac{1}{2} Q$ and $\eta(t) := \frac{1}{2} C \int_0^t (\|\phi(\tau)\|_{p+1}^{p+1} + \|\phi(\tau)\|^2) \mathrm{d}\tau$. Thus, we have

$$
|S'| \leq \frac{1}{2} C(\|\phi\|_{p+1}^{p+1} + \|\phi\|^2)(S+1).
$$

A direct application of Gronwall lemma implies

$$
S(t) \geq \mathrm{e}^{-\eta(t)} S(0) - \frac{1}{2} C \int_0^t (\|\phi(\tau)\|_{p+1}^{p+1} + \|\phi(\tau)\|^2) \mathrm{e}^{(\eta(\tau)-\eta(t))} \mathrm{d}\tau.
$$

Using (19) together with the Sobolev embedding $H^2(\mathbb{T}^d) \subset L^{p+1}(\mathbb{T}^d)$, we find that there exist $\Gamma = \Gamma(\|\phi(0)\|_{H^2}, \|\partial_t \phi(0)\|_{H^2}) > 0$ and $\gamma = \gamma(\|\psi_0\|_{H^2}, \|\psi_1\|) > 0$ such that

$$
\eta(t) \leq \frac{\Gamma}{\gamma}, \quad \forall t \geq 0.
$$

Moreover, following the development of the constant β in (19) and for fixed $(\theta(0), \theta'(0)) \neq (0,0)$, we have

$$
\lim_{(\|\phi(0)\|_{H^2}, \|\partial_t \phi(0)\|_{H^2}) \to (0,0)} \frac{\Gamma}{\gamma} = 0.
$$

Thus, there exist $\psi_0, \psi_1 \in H^2$ and $\delta > 0$ such that

$$
S(0) > \left(\frac{\Gamma}{\gamma} + \delta \right) \mathrm{e}^{\frac{\Gamma}{\gamma}} \geq (\eta(t) + \delta) \mathrm{e}^{\eta(t)},
$$

which implies that

$$
\lim_{t \to \infty} S(t) \geq \delta > 0
$$

and completes the proof. □

4 Numerical tests

In this section we study numerically the energy asymptotics for (1) with $L = -\Delta$, $p = 2$ and $d = 1, 2$. The undamped equation ($L = 0$) has a conserved quantity $E(\psi)$. Then, for long time simulations, one wants to construct numerical methods that approximately conserve this energy. When using Fourier spectral methods, we primarily need to ensure that the time discretization preserves these property, since the spectral spatial discretization will typically automatically satisfy it. We suggest therefore to use the time stepping discretization

$$
UD_n := \frac{\psi_{n+1} - 2\psi_n + \psi_{n-1}}{(\delta t)^2} + (I - \Delta) \frac{\psi_{n+1} + 2\psi_n + \psi_{n-1}}{4} + |\psi_n|^2 \psi_n,
$$

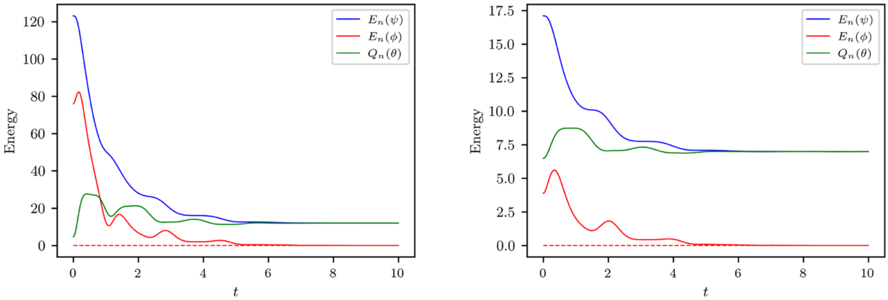

Fig. 1 Approximate representation for long time asymptotics of $E(\psi)$, $E(\phi)$ and Q by $E_n(\psi)$, $E_n(\phi)$ and Q_n respectively with $d = 1$, $\psi_1(x) = 0$ and left: $\psi_0(x) = 1 + 3\cos(x)$; right: $\psi_0(x) = (1 + 0.5\cos(x))^2$

where ψ_n is the approximation of $\psi(n\,\delta t)$ and δt is the time step. Thus, the scheme is given by

$$UD_n - \Delta\frac{\psi_{n+1} - \psi_n}{\delta t} = 0. \tag{22}$$

For the approximation of $E(\psi)$, we consider the quantity

$$E_n(\psi) := \frac{1}{2}\int_{\mathbb{T}^d}\left|\frac{\psi_n - \psi_{n-1}}{\delta t}\right|^2 + \left|\frac{\psi_n + \psi_{n-1}}{2}\right|^2 + \left|\nabla\frac{\psi_n + \psi_{n-1}}{2}\right|^2 + \frac{1}{4}\int_{\mathbb{T}^d}\left|\frac{\psi_n + \psi_{n-1}}{2}\right|^4$$

which is approximately conserved for the scheme

$$UD_n = 0$$

discretizing the undamped equation (1) with ($L = 0$). We study the long time asymptotic behaviour of $E_n(\psi)$,

$$E_n(\phi) := \frac{1}{2}\int_{\mathbb{T}^d}\left|\frac{\phi_n - \phi_{n-1}}{\delta t}\right|^2 + \left|\frac{\phi_n + \phi_{n-1}}{2}\right|^2 + \left|\nabla\frac{\phi_n + \phi_{n-1}}{2}\right|^2 + \frac{1}{4}\int_{\mathbb{T}^d}\left|\frac{\phi_n + \phi_{n-1}}{2}\right|^4$$

and

$$Q_n(\theta) := (2\pi)^d\left(\frac{1}{2}|\theta_n|^2 + \frac{1}{2}\left|\frac{\theta_n - \theta_{n-1}}{\delta t}\right|^2 + \frac{1}{4}|\theta_n|^4\right),$$

where θ_n is the zero coefficient of the discrete Fourier transform applied on ψ_n and $\phi_n := \psi_n - \theta_n$.

5 Discussion

Our numerical results show, for the solutions defined on the one and two dimensional spacial domains ($\mathbb{T}^d$, $d = 1, 2$), that $E(\psi)$ and $Q(\theta)$ converge to the same limit (Figs. 1, 2) and that $E(\phi)$ decays exponentially to zero (Figs. 3, 4) regardless of the assumptions of Theorem 3.2 made on the initial data ψ_0, ψ_1. In other words, we believe that the results of Theorem 3.2 are still valid for weaker assumptions on the initial data and we leave the proof of such results as an open problem. In general, the methods used in the literature (for example in [1, 3, 7]) to study the energy decay for the weakly damped Klein–Gordon equation having the damping operator $L = \gamma(x)I$ relay essentially on the fact that L is bounded in the given functional

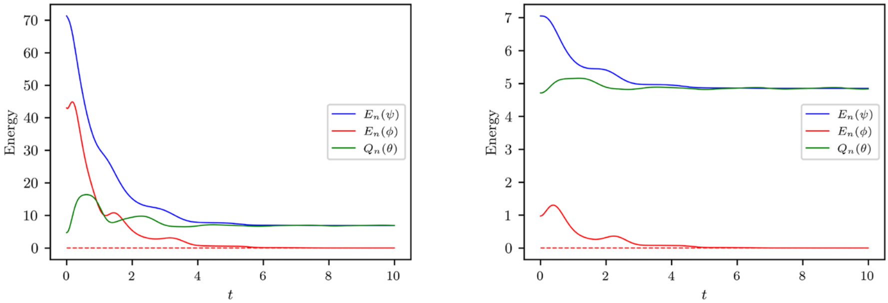

Fig. 2 Approximate representation for long time asymptotics of $E(\psi)$, $E(\phi)$ and Q by $E_n(\psi)$, $E_n(\phi)$ and Q_n respectively with $d = 2$, and left: $\psi_0(x, y) = 1 + \cos(x) + 2\cos(y)$, $\psi_1(x, y) = \sin(x) + 2\sin(y)$; right: $\psi_0(x, y) = 1 + 0.2\cos(x) + 0.5\cos(y)$, $\psi_1(x, y) = 0$

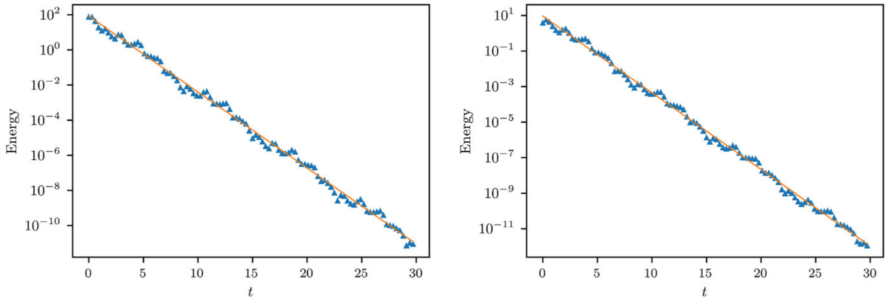

Fig. 3 Time decay of $E_n(\phi)$ when $d = 1$, $\psi_1(x) = 0$ and left: $\psi_0(x) = 1 + 3\cos(x)$; right: $\psi_0(x) = (1 + 0.5\cos(x))^2$

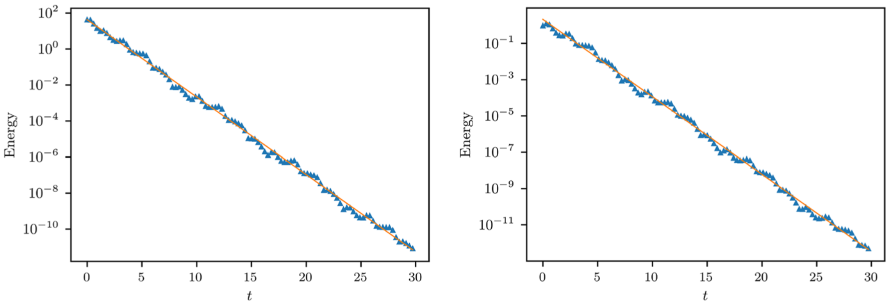

Fig. 4 Time decay of $E_n(\phi)$ when $d = 2$, and left: $\psi_0(x, y) = 1 + \cos(x) + 2\cos(y)$, $\psi_1(x, y) = \sin(x) + 2\sin(y)$; right: $\psi_0(x, y) = 1 + 0.2\cos(x) + 0.5\cos(y)$, $\psi_1(x, y) = 0$

setting which is not the case for the strongly damped equation with $L = -\Delta$. Moreover, in view of Theorem 3.3, the energy for the strongly damped equation doesn't decay necessarily to zero. These two main differences make the methods used for the weakly damped equation unapplicable to study the energy asymptotics for the strongly damped one and lead to think differently to address the above open question.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

Research statement In this paper I consider an important model of semi linear wave equations called the Klein–Gordon equation with damping effect and I study the asymptotic behaviour of the energy for periodic solutions. This phenomena has gained recently a considerable attention in the literature, but for a specific type of damping called in many references weak damping in which the semigroup generated by the linear part of the equation is merely continuous. However, the other type of damping, called the strong damping, considered in this paper hasn't gained enough attention. I believe that this work will be an interesting contribution to the field of semi linear wave equations, since the characterisation of the energy asymptotics, supported by numerical results, is new and leads to further important open question.

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