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CONSTRUCTIVE APPROXIMATION

Maximal Convergence and Interpolation on Unconnected Sets

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Abstract
A theorem of Grothmann states that interpolating polynomials to a holomorphic function on a compact set E is maximally convergent to f only if a subsequence of the interpolation points converges to the equilibrium distribution of E in the weak* sense. Grothmann's proof applies only for connected sets E. The objective of this paper is to provide a new necessary condition for maximal convergence which is the crucial tool to prove Grothmann's theorem for unconnected sets E.

Keywords Complex approximation · Interpolation · Maximal convergence · Equilibrium measure

Mathematics Subject Classification 30E10 · 41A05 · 41A10

1 Introduction

For $B \subset \mathbb{C}$, we denote by $\overline{B}$ its closure and by ∂B the boundary of B and we use $\|\cdot\|_B$ for the supremum norm over B. Let $\mathcal{A}(B)$ be the class of functions that are holomorphic (i.e., analytic and single-valued) in a neighborhood of B, and we denote by $\mathcal{P}_n$ the set of algebraic polynomials of degree at most n.

Let E be a compact subset of the complex plane $\mathbb{C}$, and let $\mathcal{M}(E)$ be the collection of all probability measures supported on E, then the logarithmic potential of $\mu \in \mathcal{M}(E)$ is defined by

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$$U^{\mu}(z) = \int \log \frac{1}{|z-t|} \, \mathrm{d}\mu(t)$$

and the logarithmic energy $I(\mu)$ by

$$I(\mu) := \int \int \log \frac{1}{|z-t|} \, \mathrm{d}\mu(t) \, \mathrm{d}\mu(z) = \int U^{\mu}(z) \, \mathrm{d}\mu(z).$$

Let

$$V(E) := \inf\{I(\mu) : \mu \in \mathcal{M}(E)\},$$

then $V(E)$ is either finite or $V(E) = +\infty$. The quantity

$$\text{cap } E = e^{-V(E)}$$

is called the logarithmic capacity or capacity of E.

Let E be compact in the complex plane $\mathbb{C}$ with connected complement $\Omega = \overline{\mathbb{C}} \setminus E$ in the extended plane $\overline{\mathbb{C}}$. The domain Ω is called *regular* if the Green function $G(z) = G(z, \infty)$ on Ω with pole at ∞ tends to 0 as $z \in \Omega$ tends to the boundary $\partial\Omega$ of Ω. If Ω is regular, then cap $E > 0$ and there exists a unique measure $\mu_E \in \mathcal{M}(E)$ such that

$$I(\mu_E) = -\log \text{cap } E = V(E)$$

and we have

$$U^{\mu_E}(z) = -G(z) - \log \text{cap } E, \quad z \in \Omega.$$

μ_E is called *equilibrium measure* of E.

In the following, let E be compact in $\mathbb{C}$ with regular connected complement $\Omega = \overline{\mathbb{C}} \setminus E$. Then, we define for $\sigma > 1$ the *Green domains* E_σ by

$$E_\sigma := \{z \in \Omega : G(z) < \log \sigma\} \cup E$$

with boundary $\Gamma_\sigma := \partial E_\sigma$. Since Ω is regular, the Green domain E_σ consists of a finite number of Jordan regions which are mutually exterior (cf. Walsh ([4], Chapter 4, section 4.1). Only in the case that E is connected, each E_σ is a single Jordan region for any $\sigma > 1$.

If $f \in \mathcal{A}(E)$, then there exists $\rho > 1$ and polynomials $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} \leq \frac{1}{\rho},$$

due to a result of Walsh ([4]).

Let $\rho(f)$ denote the maximal parameter $\rho > 1$, $1 < \rho \leq \infty$, such that f is holomorphic on E_ρ. Then, there exist polynomials $p_n \in \mathcal{P}_n$ such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} = \frac{1}{\rho(f)}.$$

Such a sequence $p_n \in \mathcal{P}_n$ is called *maximally convergent*. Moreover, Walsh ([4], Sect 4.7, Theorem 7, Theorem 8 and its Corollary, pp. 79–81) proved that for such maximally convergent polynomials

$$\limsup_{n\to\infty} \|f - p_n\|_{\Gamma_\sigma}^{1/n} = \frac{\sigma}{\rho(f)}, \quad 1 < \sigma < \rho(f). \tag{1.1}$$

Consider Lagrange–Hermite interpolation to f on point sets

$$Z_n : \quad z_{n,0}, z_{n,1}, \ldots, z_{n,n} \in E$$

by polynomials $p_n \in \mathcal{P}_n$. Then, it is known, due to Bernstein–Walsh, that the interpolation on such schemes Z_n yields (1.1) if the normalized counting measures ν_n, i.e.,

$$\nu_n(B) := \frac{\#\{z_{n,j} : z_{n,j} \in B\}}{n+1} \quad (B \subset \mathbb{C}),$$

satisfy

$$\widehat{\nu}_n \xrightarrow[n\to\infty]{*} \mu_E$$

in the weak* sense, where $\widehat{\nu}_n$ denotes the balayage measure of ν_n on the boundary of E, i.e., $\widehat{\nu}_n$ is the measure supported on the boundary of E such that

$$U^{\widehat{\nu}_n}(z) = U^{\nu}(z) \quad \text{for all } z \in \overline{\mathbb{C}} \setminus E.$$

Conversely, Grothmann stated the following theorem.

Theorem *(Grothmann [1]): Let p_n be the polynomial of interpolation on $Z_n \subset E$. If $f \in \mathcal{A}(E)$, $1 < \rho(f) < \infty$, and if*

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} = \frac{1}{\rho(f)},$$

then μ_E is a weak limit point of $\{\widehat{\nu}_n\}_{n\in\mathbb{N}}$.*

The proof given in ([1]) applies only if E is connected or, at least, if $E_{\rho(f)}$ is connected. Hence, one objective of this paper is to provide a proof of Grothmann's theorem even for unconnected sets E. The crucial tool will be a new necessary condition for maximally convergent polynomials which seems to be interesting itself.

2 Maximal Convergence and Interpolation

Let $E \subset \mathbb{C}$ be compact with regular connected complement $\Omega = \overline{\mathbb{C}} \setminus E$ and let $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, denote polynomials such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} = \frac{1}{\rho(f)},$$

where $\rho(f)$ is the maximal parameter of holomorphy of f and $1 < \rho(f) < \infty$.

The Green domains E_r, $1 < r < \infty$, consist of a finite number of disjoint regions ${E_r}^i$,

$$E_r \;=\; E_r^1 \cup E_r^2 \cup \ldots \cup E_r^{l_r}, \quad l_r \in \mathbb{N}. \tag{2.1}$$

Each E_r^i is a Jordan region, and we write $\Gamma_r^i = \partial E_r^i$. Then, the boundary $\Gamma_r = \partial E_r$ is

$$\Gamma_r = \bigcup_{i=1}^{l_r} \Gamma_r^i$$

and we note that Γ_r^i and Γ_r^j may have points in common if $i \neq j$, but only a finite number of points (cf. Walsh ([4], chapter 4, section 4.1).

Our first result is a necessary condition for maximal convergence that is new if E is not connected.

Proposition 1 *Let $f \in \mathcal{A}(E_\rho)$, $1 < \rho < \infty$, and let $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, be polynomials such that*

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} \leq \frac{1}{\rho}. \tag{2.2}$$

If $1 < \sigma < \rho$ *and if*

$$\limsup_{n\to\infty} \min_{1\leq i\leq l_\sigma} \|f - p_n\|_{\Gamma_\sigma^i}^{1/n} < \frac{\sigma}{\rho}, \tag{2.3}$$

then the maximal parameter $\rho(f)$ of holomorphy of f satisfies $\rho(f) > \rho$.

As a consequence of Proposition 1, we get immediately

Theorem 1 *Let $f \in \mathcal{A}(E)$, and let $1 < \sigma < \rho(f) < \infty$. Then, the sequence $\{p_n\}_{n\in\mathbb{N}}$ with $p_n \in \mathcal{P}_n$ is maximally convergent to f if and only if*

$$\frac{\sigma}{\rho(f)} \;=\; \limsup_{n\to\infty} \|f - p_n\|_{\Gamma_\sigma}^{1/n} \;=\; \limsup_{n\to\infty} \min_{1\leq i\leq l_\sigma} \|f - p_n\|_{\Gamma_\sigma^i}^{1/n}. \tag{2.4}$$

If E is connected, then $l_\sigma = 1$ for any σ and Theorem 1 coincides with the well-known characterization of Bernstein–Walsh.

Next, we want to use the above results to characterize interpolating polynomials converging maximally to f by the distribution of the interpolation points.

Proposition 2 *Let $f \in \mathcal{A}(E_\rho)$, $1 < \rho < \infty$, and let $p_n \in \mathcal{P}_n$ be the interpolating polynomial to f on the point set $Z_n \subset E$, $n \in \mathbb{N}$, with*

$$\limsup_{n\to\infty} \|f - p_n\|_{\Gamma_{\sigma^*}}^{1/n} \le \frac{\sigma^*}{\rho} \tag{2.5}$$

for all σ^*, $1 < \sigma^* < \rho$.
Let $\Lambda \subset \mathbb{N}$ and let μ_E be not a weak limit point of $\widehat{\nu}_n$, $n \in \Lambda$. If σ is fixed with $1 < \sigma < \rho$, then for all $n \in \Lambda$ there exists an index $s(n)$, $1 \le s(n) \le l_\sigma$, such that the strict inequality*

$$\limsup_{n\in\Lambda, n\to\infty} \|f - p_n\|_{\Gamma_\sigma^{s(n)}}^{1/n} < \frac{\sigma}{\rho} \tag{2.6}$$

holds.

Finally, we combine Proposition 2 with Proposition 1 to obtain a characterization of maximally converging interpolation polynomials.

Theorem 2 *Let $f \in \mathcal{A}(E)$ with $1 < \rho(f) < \infty$, and let $\{p_n\}_{n\in\mathbb{N}}$ be maximally convergent to f on E. If p_n interpolates f at the interpolation point set $Z_n \subset E$ with counting measure ν_n and balayage measure $\widehat{\nu}_n$ on ∂E, then the following holds:*

(a) μ_E is a weak limit point of $\{\widehat{\nu}_n\}_{n\in\mathbb{N}}$.*
(b) For every fixed σ, $1 < \sigma < \rho(f)$, there exists a subset $\Lambda \subset \mathbb{N}$ such that

$$\frac{\sigma}{\rho(f)} = \limsup_{n\to\infty} \|f - p_n\|_{\Gamma_\sigma}^{1/n} = \lim_{n\in\Lambda, n\to\infty} \min_{1\le i\le l_\sigma} \|f - p_n\|_{\Gamma_\sigma^i}^{1/n} \tag{2.7}$$

and

$$\widehat{\nu}_n \xrightarrow{*} \mu_E \quad \text{as } n \to \infty, n \in \Lambda.$$

The first statement (a) is Grothmann's theorem, even proved here for unconnected E. The second statement (b) describes the subsequences $\Lambda \in \mathbb{N}$ which lead to μ_E as a weak* limit point of $\widehat{\nu}_n$, $n \in \mathbb{N}$.

3 Proof of Proposition 1

The proof is based on constructing a telescoping series of f,

$$f = p_{n_1} + \sum_{j=1}^{\infty} (p_{n_{j+1}} - p_{n_j}),$$

which is holomorphic in a neighborhood of $\overline{E}_\rho$.

Because of (2.2), the Lemma of Bernstein–Walsh induces that there exists for $\varepsilon > 0$ and $1 \leq r < \rho$ a number $n_\varepsilon(r) \in \mathbb{N}$ such that

$$\frac{1}{n} \log \|f - p_n\|_{\Gamma_r} \leq \log \frac{r}{\rho} + \varepsilon \tag{3.1}$$

for $n \geq n_\varepsilon(r)$. Because of (2.3), there exist a map

$$s : \mathbb{N} \to \{1, 2, \ldots, l_\sigma\}$$

and $\delta > 0$ and $n_1(\delta) \in \mathbb{N}$ such that

$$\frac{1}{n} \log \|f - p_n\|_{\Gamma_\sigma^{s(n)}} \leq \log \frac{\sigma}{\rho} - \delta. \tag{3.2}$$

for all $n \geq n_1(\delta)$.

3.1 The Starting Telescoping Series

Let us fix a parameter $\kappa > 1$, then the telescoping series

$$f = p_{m_1} + \sum_{i=1}^{\infty} (p_{m_{i+1}} - p_{m_i}) \tag{3.3}$$

and the sequence $\Lambda_1(\kappa) := \{m_i\}_{i=1}^{\infty}$ is defined recursively:

Set $m_1 := 1$. If m_i is defined and if there exists $m > m_i$ with

$$s(m) = s(m_i) \quad \text{and} \quad m/m_i \leq \kappa,$$

then we define

$$m_{i+1} := m, \quad \textit{otherwise} \quad m_{i+1} := m_i + 1.$$

Hence, $\Lambda_1(\kappa) = \{m_i\}_{i=1}^{\infty}$ has the following properties:

$$m_{i+1}/m_i \leq \kappa \quad \text{and} \quad s(m_{i+1}) = s(m_i) \tag{3.4}$$

or

$$m_{i+1} = m_i + 1 \text{ and } s(m) \neq s(m_i) \text{ for } m_i + 1 \leq m \leq \kappa\, m_i. \tag{3.5}$$

Next, we decompose $\Lambda_1(\kappa)$ into

$$\Lambda_1(\kappa) := \Lambda_{1,1}(\kappa) \cup \Lambda_{1,2}(\kappa) \tag{3.6}$$

with

$$\Lambda_{1,1}(\kappa) \;=\; \{m_i \in \Lambda_1(\kappa) : \; m_i \text{ satisfies } (3.4)\} \tag{3.7}$$

and

$$\Lambda_{1,2}(\kappa) \;=\; \{m_i \in \Lambda_1(\kappa) : \; m_i \text{ satisfies } (3.5)\} \;=\; \Lambda_1(\kappa) \setminus \Lambda_{1,1}(\kappa). \tag{3.8}$$

The next lemma shows how to estimate the norm of the difference

$$p_{m_{i+1}} - p_{m_i} \quad \text{for} \quad m_i \in \Lambda_{1,1}(\kappa)$$

outside of E_r. We use as auxiliary tool the *harmonic measure*

$$h_r^i(z) = \omega(z, \Gamma_r^i, \overline{\mathbb{C}} \setminus \overline{E_r}), \quad 1 \le i \le l_r,$$

where $\Gamma_r^i = \partial E_r^i$ is the boundary of the Jordan region E_r^i in the decomposition of E_r in (2.1), i.e., h_r^i is the harmonic function in $\overline{\mathbb{C}} \setminus \overline{E_r}$ that satisfies the boundary conditions $h_r^i = 1$ on Γ_r^i and $h_r^i = 0$ on $\Gamma_r \setminus \Gamma_r^i$, possibly except of a finite number of points (cf. [3], p. 111, section III, 17 or [2]). Then,

$$h_r^i > 0 \quad \text{for all } z \in \overline{\mathbb{C}} \setminus \overline{E_r}$$

and if we define for $r^* > r$

$$\alpha_r(r^*) := \min_{1 \le i \le l_r} \min_{z \in \Gamma_{r^*}} h_r^i(z),$$

we obtain

$$\alpha_r(r^*) > 0.$$

Lemma 1 *Let $n, m \in \mathbb{N}$ with $m < n$ and $n/m \le \kappa$, and let $1 < r < \rho$ such that*

$$\frac{1}{m} \log \|f - p_m\|_{\Gamma_r^{s(m)}} \;\le\; \log \frac{r}{\rho} - \delta_r \tag{3.9}$$

and

$$\frac{1}{n} \log \|f - p_n\|_{\Gamma_r^{s(n)}} \;\le\; \log \frac{r}{\rho} - \delta_r. \tag{3.10}$$

where $\delta_r > 0$. If $r^* > r$, $s(m) = s(n)$ and if

$$1 < \kappa < 1 + \frac{\alpha_r(r^*)}{\log(\rho/r)}\, \delta_r, \tag{3.11}$$

then there exists $\delta^*_{r^*} > 0$ *and* $n^* = n^*(\kappa) \in \mathbb{N}$ *such that*

$$\frac{1}{n} \log \|p_n - p_m\|_{\Gamma_{r^*}} \leq \log \frac{r^*}{\rho} - \delta^*_{r^*} \tag{3.12}$$

for $m \geq n^*$, *where*

$$\delta^*_{r^*} \geq \frac{1}{2}\left(\frac{\delta_r}{\kappa}\alpha_r(r^*) - \left(1 - \frac{1}{\kappa}\right)\log\frac{\rho}{r}\right). \tag{3.13}$$

Proof (3.9) and (3.10) imply

$$\|p_n - p_m\|_{\Gamma_r^{s(n)}} \leq 2\left(\frac{r}{\rho}\, e^{-\delta_r}\right)^m \leq 2\left(\frac{r}{\rho}\, e^{-\delta_r}\right)^{n/\kappa}$$

or

$$\begin{aligned}\frac{1}{n} \log \|p_n - p_m\|_{\Gamma_r^{s(n)}} &\leq \frac{\log 2}{n} + \left(\log\frac{r}{\rho} - \delta_r\right)\frac{1}{\kappa}\\ &= \log\frac{r}{\rho} - \frac{\delta_r}{\kappa} + \left(\frac{1}{\kappa} - 1\right)\log\frac{r}{\rho} + \frac{\log 2}{n}.\end{aligned} \tag{3.14}$$

Fix $0 < \varepsilon < \log \rho/\sigma$. Because of (3.1), there exists $n_\varepsilon(r) \in \mathbb{N}$ such that

$$\frac{1}{n}\log\|f - p_n\|_{\Gamma_r} \leq \log\frac{r}{\rho} + \varepsilon$$

for $n \geq n_\varepsilon(r)$. Then,

$$\|p_n - p_m\|_{\Gamma_r^{s(n)}} \leq 2\left(\log\frac{r}{\rho}\, e^{\varepsilon}\right)^m \leq 2\left(\log\frac{r}{\rho}\, e^{\varepsilon}\right)^{n/\kappa}$$

or

$$\begin{aligned}\frac{1}{n} \log \|p_n - p_m\|_{\Gamma_r^{s(n)}} &\leq \frac{\log 2}{n} + \frac{1}{\kappa}\log\frac{r}{\rho} + \frac{\varepsilon}{\kappa}\\ &\leq \log\frac{r}{\rho} + \left(\frac{1}{\kappa} - 1\right)\log\frac{r}{\rho} + \frac{\kappa\ \log 2}{n} + \frac{\varepsilon}{\kappa}.\end{aligned} \tag{3.15}$$

Define

$$A(\kappa) := \left(1 - \frac{1}{\kappa}\right)\log\frac{\rho}{r}$$

and

$$H(z) := \frac{1}{n}\log|(p_n(z) - p_m(z))| - G(z) + \log\rho.$$

Then, H(z) is subharmonic in $\overline{\mathbb{C}} \setminus \overline{E}_r$ and the harmonic function

$$-\frac{\delta_r}{\kappa}\, h_r^{s(n)}(z) + A(\kappa) + \frac{\log 2}{n} + \frac{\varepsilon}{\kappa}$$

is an upper bound of $H(z)$ in $\mathbb{C} \setminus \overline{E}_r$, where we have taken into account (3.14) and (3.15) and the definition of h_r^i for $i = s(n) = s(m)$. Inserting $z \in \Gamma_{r^*}$, we obtain

$$H(z) < -\frac{\delta_r}{\kappa}\, \alpha_r(r^*) + A(\kappa) + \frac{\log 2}{n} + \frac{\varepsilon}{\kappa}$$

and

$$\frac{1}{n}\, \log \|p_n - p_m\|_{\Gamma_{r^*}} \leq \log \frac{r^*}{\rho} - \frac{\delta_r}{\kappa}\alpha_r(r^*) + A(\kappa) + \frac{\log 2}{n} + \frac{\varepsilon}{\kappa}. \tag{3.16}$$

If we choose κ as in (3.11), we get with

$$\begin{aligned} -\frac{\delta_r}{\kappa}\, \alpha_r(r^*) + A(\kappa) &= -\frac{\delta_r}{\kappa}\, \alpha_r(r^*) + \log \frac{\rho}{r} - \frac{1}{\kappa} \log \frac{\rho}{r} \\ &= -\frac{1}{\kappa}\left(\delta_r\, \alpha_r(r^*) + \log \frac{\rho}{r}\right) + \log \frac{\rho}{r} \\ &< 0. \end{aligned}$$

Now, let us define

$$\delta_{r^*}^* := \frac{1}{2}\left(\frac{\delta_r}{\kappa}\, \alpha_r(r^*) - A(\kappa)\right),$$

then $\delta_{r^*}^* > 0$, and we can choose n large enough and ε sufficiently small such that

$$\frac{\log 2}{n} + \frac{\varepsilon}{\kappa} < \delta_{r^*}^*.$$

Hence, we can find $n^* = n^*(\kappa) \in \mathbb{N}$ such that by (3.16) we obtain

$$\frac{1}{n}\, \log \|p_n - p_m\|_{\Gamma_{r^*}} \leq \log \frac{r^*}{\rho} - \delta_{r^*}^*$$

for $m \geq n^*$, and (3.12) and (3.13) are proven. □

We know from (3.2) that

$$\frac{1}{n} \log \|f - p_n\|_{\Gamma_\sigma^{s(n)}} \;\leq\; \log \frac{\sigma}{\rho} - \delta, \quad n \geq n_1(\delta). \tag{3.17}$$

Let

$$\kappa_1^* := 1 + \frac{\alpha_\sigma(\rho)}{\log(\rho/\sigma)}\,\delta \tag{3.18}$$

and let $1 < \kappa < \kappa_1^*$ in the definition of $\Lambda_1(\kappa)$ in (3.6)–(3.8), then we obtain by Lemma 1 and choosing $r = \sigma$ and $r^* = \rho$:

There exists $\delta_1^* > 0$ and $n_1^* = n_1^*(\kappa) \in \mathbb{N}$ such that

$$\frac{1}{m_{i+1}} \log \|p_{m_{i+1}} - p_{m_i}\|_{\Gamma_\rho} \le -\delta_1^* \tag{3.19}$$

for all $m_i \in \Lambda_{1,1}(\kappa)$, $m_i \ge n_1^*$, and

$$\delta_1^* \ge \frac{1}{2}\left(\frac{\delta}{\kappa}\alpha_\sigma(\rho) - \left(1 - \frac{1}{\kappa}\right)\log\frac{\rho}{\sigma}\right).$$

Corollary 1 *Let $1 < \kappa < \kappa_1^*$ and assume that $\Lambda_{1,2}(\kappa)$ is a finite set in the decomposition of $\Lambda_1(\kappa)$ in (3.6). Then, the telescoping series (3.3) converges uniformly on compact sets of a neighborhood of $\overline{E_\rho}$, i.e., $\rho(f) > \rho$.*

Proof We apply the Bernstein–Walsh Lemma to the differences

$$p_{m_{i+1}} - p_{m_i}$$

and use the inequality (3.19). □

Hence, Proposition 1 is proved for this special situation.

3.2 The Auxiliary Parameter $1 < \sigma_0 < \sigma$

To restrict κ in the definition of $\Lambda_1(\kappa)$ completely, we start with the decomposition of E_σ in (2.1),

$$E_\sigma = E_\sigma^1 \cup E_\sigma^2 \cup \ldots \cup E_\sigma^{l_\sigma}, \quad l_\sigma \in \mathbb{N}.$$

Then, we can define a parameter σ_0, $1 < \sigma_0 < \sigma$, such that the decomposition of E_{σ_0} into disjoint Jordan regions $E_{\sigma_0}^i$,

$$E_{\sigma_0} = E_{\sigma_0}^1 \cup E_{\sigma_0}^2 \cup \ldots \cup E_{\sigma_0}^{l_{\sigma_0}}, \quad l_{\sigma_0} \in \mathbb{N},$$

satisfies

$$l_{\sigma_0} = l_\sigma \quad \text{and} \quad \overline{E_{\sigma_0}^i} \subset E_\sigma^i \quad \text{for } 1 \le i \le l_{\sigma_0}.$$

This can be achieved by the strict monotonicity of E_r with respect to r and the fact that the Green function $G(z)$ of Ω has only a finite number of critical points in $\mathbb{C} \setminus E$ (cf. Walsh [4], chapter 4, section 4.1).

Lemma 2 *Let $\delta > 0$ and let $1 < \sigma_0 < \sigma$ such that according to (3.2)*

$$\frac{1}{n} \log \|f - p_n\|_{\Gamma_\sigma^{s(n)}} \leq \log \frac{\sigma}{\rho} - \delta, \quad n \geq n_1(\delta).$$

Then, there exist $\delta_0 > 0$ and $n_0(\delta)$ such that

$$\frac{1}{n} \log \|f - p_n\|_{\Gamma_{\sigma_0}^{s(n)}} \leq \log \frac{\sigma_0}{\rho} - \delta_0 \ \ for \quad n \geq n_0(\delta). \tag{3.20}$$

Proof Because of (3.1), there exists $n_\varepsilon(1)$ such that

$$\frac{1}{n} \log \|f - p_n\|_E \leq \log \frac{1}{\rho} + \varepsilon, \quad n \geq n_\varepsilon(1).$$

Let us consider the Dirichlet problem for the harmonic function $g_i(z)$ in the region

$$E_\sigma^i \setminus E, \quad 1 \leq i \leq l_\sigma,$$

with the boundary conditions

$$g_i(z) = -\delta \text{ for } z \in \Gamma_\sigma^i \quad \text{and} \quad g_i(z) = 0 \text{ for } z \in E_\sigma^i \cap \Gamma,$$

where $\Gamma = \partial E$. Then, $g_i(z) < 0$ for $z \in E_\sigma^i \setminus E$. Define

$$\beta_i := \max_{z \in \Gamma_{\sigma_o}^i} g_i(z),$$

then $\beta_i < 0$ and also

$$\beta := \max_{1 \leq i \leq l_\sigma} \beta_i < 0.$$

Moreover, the function

$$g_{s(n)}(z) + \varepsilon$$

is a harmonic majorant of

$$\frac{1}{n} \log |f(z) - p_n(z)| - G(z) + \log \rho \quad \text{in} \quad E_\sigma^{s(n)} \setminus E.$$

That leads to

$$\frac{1}{n}\,\log\|f-p_n\|_{\Gamma_{\sigma_0}^{s(n)}}-\log\frac{\sigma_o}{\rho}\;\le\max_{z\in\Gamma_{\sigma_o}^{s(n)}}\;g_{s(n)}(z)+\varepsilon$$

or

$$\frac{1}{n}\,\log\|f-p_n\|_{\Gamma_{\sigma_0}^{s(n)}}\le\log\frac{\sigma_o}{\rho}+\beta+\varepsilon.$$

If we define $\varepsilon:=-\beta/2$, then we get

$$\frac{1}{n}\,\log\|f-p_n\|_{\Gamma_{\sigma_0}^{s(n)}}\le\log\frac{\sigma_0}{\rho}+\frac{\beta}{2}$$

for all $n\ge n_\varepsilon(1)$. Therefore,

$$n_0(\delta):=n_\varepsilon(1)\quad\text{and}\quad\delta_0:=-\beta/2$$

satisfy the statement of Lemma 2. □

Lemma 3 *Let $n,m\in\mathbb{N}$ with $m<n$ and $n/m\le\kappa$. Let $\delta_0>0$ and $n_0(\delta)\in\mathbb{N}$ such that (3.20) holds according to Lemma 2. Moreover, let $s(m)=s(n)$ and let*

$$1<\kappa<\kappa_2^*:=1+\frac{\alpha_{\sigma_0}(\sigma)}{\log(\rho/\sigma_0)}\,\delta_0,\tag{3.21}$$

then there exists $\delta_0^*>0$ and $n_2^*=n_2^*(\kappa)\in\mathbb{N}$ such that

$$\frac{1}{n}\log\|p_n-p_m\|_{\Gamma_\sigma}\le\log\frac{\sigma}{\rho}-\delta_0^*\tag{3.22}$$

for $m\ge n_2^*$ and

$$\delta_0^*\ge\frac{1}{2}\left(\frac{\delta_0}{\kappa}\alpha_{\sigma_0}(\sigma)-\left(1-\frac{1}{\kappa}\right)\log\frac{\rho}{\sigma_0}\right).\tag{3.23}$$

Proof Because of Lemma 2, there exists $n_0(\delta)$ such that

$$\frac{1}{m}\log\|f-p_m\|_{\Gamma_{\sigma_0}^{s(m)}}\le\log\frac{\sigma_0}{\rho}-\delta_0$$

and

$$\frac{1}{n}\log\|f-p_n\|_{\Gamma_{\sigma_0}^{s(n)}}\le\log\frac{\sigma_0}{\rho}-\delta_0$$

for $m < n$ and $m \geq n_0(\delta)$. Since $s(m) = s(n)$ and κ satisfies (3.21), then Lemma 1 yields that there exists $n_2^* = n_2^*(\kappa) \in \mathbb{N}$ and $\delta_0^* > 0$ such that

$$\frac{1}{n} \log \|p_m - p_n\|_{\Gamma_\sigma} \leq \log \frac{\sigma}{\rho} - \delta_0^*$$

for $m \geq n_2^*$ and δ_0^* satisfies (3.23). □

3.3 The Final Telescoping Series

We start with the telescoping series associated with

$$\Lambda_1(\kappa) = \{m_i\}_{i=1}^{\infty}$$

satisfying (3.4) and (3.5) and choosing the parameter κ such that

$$1 < \kappa < \min\left(\kappa_1^*, \kappa_2^*, \kappa_3^*\right).$$

κ_1^* is defined by (3.18), i.e.,

$$\kappa_1^* \;=\; 1 + \frac{\alpha_\sigma(\rho)}{\log(\rho/\sigma)}\,\delta$$

and δ satisfies (3.17). κ_2^* is defined by (3.21), i.e.,

$$\kappa_2^* \;=\; 1 + \frac{\alpha_{\sigma_0}(\sigma)}{\log(\rho/\sigma_0)}\,\delta_0,$$

and δ_0 satisfies (3.20). κ_3^* will be defined by

$$\kappa_3^* \;:=\; 1 + \frac{\alpha_\sigma(\rho)}{\log(\rho/\sigma)}\,\frac{\delta_0^*}{2} \tag{3.24}$$

and δ_0^* satisfies (3.23). The role of κ_3^* will be seen in the proof of Lemma 5. As above, we use the decomposition

$$\Lambda_1(\kappa) := \Lambda_{1,1}(\kappa) \;\cup\; \Lambda_{1,2}(\kappa)$$

Hence, by (3.19) there exist $\delta_1^* > 0$ and $n_1^* = n_1^*(\kappa) \in \mathbb{N}$ such that

$$\frac{1}{m_{i+1}} \log\, \|p_{m_{i+1}} - p_{m_i}\|_{\Gamma_\rho} \leq -\delta_1^* \tag{3.25}$$

for all $m_i \in \Lambda_{1,1}(\kappa),\;\; m_i \geq n_1^*$. So, as critical differences in the telescoping series with respect to $\Lambda_1(\kappa)$ remain $p_{m_{i+1}} - p_{m_i}$, where

$$m_{i+1} = m_i + 1 \text{ and } s(m_{i+1}) \neq s(m_i).$$

In Corollary 1, we have given already the proof of Proposition 1 for the case that $\Lambda_{1,2}(\kappa)$ is a finite sequence. Therefore, we assume henceforth that

$$\Lambda_{1,2}(\kappa) \ = \ \{\lambda_1 < \lambda_2 < \lambda_3 < \ \ \dots\} \tag{3.26}$$

is an infinite sequence.

In the following, we use a real parameter $c \in \mathbb{R}$, $0 < c < 1$.

Lemma 4 *Let $\lambda_k \in \Lambda_{1,2}(\kappa)$ be fixed. Then, there exist at most l_σ elements of $\Lambda_{1,2}(\kappa)$ in the interval*

$$(\lambda_k \ , \ \kappa\lambda_k] \, .$$

Moreover, let the parameter $c \in \mathbb{R}$, $0 < c < 1$, be fixed and let the semi-open intervals $I(\lambda_k, j)$ be defined by

$$I(\lambda_k, j) := \left(\lambda_k \left(1 + \left(\frac{c}{1+c}\right)^{j+1} (\kappa - 1)\right), \ \lambda_k \left(1 + \left(\frac{c}{1+c}\right)^{j} (\kappa - 1)\right)\right]$$

for $0 \le j \le l_\sigma - 1$. Then, there exists $\widetilde{l_k}$, $0 \le \widetilde{l_k} \le l_\sigma - 1$, such that

$$I(\lambda_k, \widetilde{l_k}) \ \cap \ \Lambda_{1,2}(\kappa) = \emptyset .$$

Proof Let us assume that there exist at least l_σ elements of $\Lambda_{1,2}(\kappa)$ in the interval $(\lambda_k, \kappa\lambda_k]$. Then, the definition of $\Lambda_1(\kappa)$, resp. $\Lambda_{1,2}(\kappa)$, implies that the values of the function s at the points

$$\lambda_k, \lambda_{k+1}, \dots, \lambda_{k+l_\sigma}$$

are all different, which contradicts the definition of l_σ.

Let us assume that the second part of the Lemma is false. Then, in each

$$I(\lambda_k, j), \ \ 0 \le j \le l_\sigma - 1,$$

there exists at least one element of $\Lambda_{1,2}(\kappa)$. Hence, the interval

$$\left(\lambda_k \left(1 + \left(\frac{c}{1+c}\right)^{l_\sigma} (\kappa - 1)\right) \ , \ \lambda_\kappa\right]$$

contains at least l_σ elements of $\Lambda_{1,2}(\kappa)$, contradicting the first part of the lemma. □

3.3.1 The Telescoping Defining Sequence $\Lambda(\kappa, c)$

Let

$$\Lambda_1(\kappa) = \{m_i\}_{i=1}^{\infty}$$

satisfies (3.4) and (3.5) with a parameter κ where

$$1 < \kappa < \min\left(\kappa_1^*, \kappa_2^*, \kappa_3^*\right).$$

κ_1^* is defined by (3.18), κ_2^* by (3.21), κ_3^* by (3.24). As in (3.6) - (3.8), we decompose

$$\Lambda_1(\kappa) := \Lambda_{1,1}(\kappa) \cup \Lambda_{1,2}(\kappa).$$

Then, we define the sequence

$$\Lambda(\kappa, c) = \left\{n_j\right\}_{j=1}^{\infty}$$

as follows: If $\Lambda_{1,2}(\kappa)$ is a finite sequence, then $\Lambda(\kappa, c) := \Lambda_1(\kappa)$. If $\Lambda_{1,2}(\kappa)$ is an infinite sequence, we define

$$\gamma := \left\lfloor \frac{1}{\left(\frac{c}{1+c}\right)^{l_\sigma}(\kappa - 1)} \right\rfloor + 1 \tag{3.27}$$

and we set

$$M := \min\{m_i \in \Lambda_1(\kappa) : \ m_i > \gamma\}.$$

Then, we define $n_j := m_j$ for $1 \leq m_j \leq M$. The remaining elements $n_j \in \Lambda(\kappa, c)$, $n_j > M$, will be defined recursively:
If $n_j \geq M$ is already constructed, we note that we obtain by (3.27) for $0 \leq \widetilde{l_j} \leq l_\sigma - 1$

$$n_j \left(\frac{c}{1+c}\right)^{\widetilde{l_j}+1}(\kappa - 1) \geq n_j \left(\frac{c}{1+c}\right)^{l_\sigma}(\kappa - 1) > 1. \tag{3.28}$$

Then, we fix

$$m := \min\left\{m_i \in \Lambda_1(\kappa) : \ m_i > n_j\right\}$$

and distinguish 2 cases:

(i) If $s(m) = s(n_j)$, then $n_{j+1} := m$.

(ii) If $s(m) \neq s(n_j)$, then we apply Lemma 4. Hence, there exists $k_0 \in \mathbb{N}$ and $0 \le \widetilde{l_j} \le l_\sigma - 1$, such that

$$\lambda_{k_0} \le n_j \left(1 + \left(\frac{c}{1+c}\right)^{\widetilde{l_j}+1} (\kappa - 1)\right)$$

and

$$\lambda_{k_{0+1}} > n_j \left(1 + \left(\frac{c}{1+c}\right)^{\widetilde{l_j}} (\kappa - 1)\right),$$

where we have used (3.28) and the enumeration of $\Lambda_{1,2}(\kappa)$ as in (3.26). Then, we define

$$n_{j+1} := \lambda_{k_0}.$$

Properties of $\Lambda(\kappa, c)$ We have always $n_{j+1} \le \kappa\, n_j$. If $s(n_{j+1}) \neq s(n_j)$, then

$$n_{j+1} - n_j \;\le\; n_j \left(\frac{c}{1+c}\right)^{\widetilde{l_j}+1} (\kappa - 1) \tag{3.29}$$

and

$$\min_{\lambda \in \Lambda_{1,2}(\kappa)} \left\{\lambda : \lambda > n_{j+1}\right\} > n_j \left(1 + \left(\frac{c}{1+c}\right)^{\widetilde{l_j}} (\kappa - 1)\right), \tag{3.30}$$

where $0 \le \widetilde{l_j} \le l_\sigma - 1$. Moreover, $s(m) = s(n_{j+1})$ for

$$m \in \Lambda_1(\kappa),\; \text{where } n_{j+1} \le m < n_j \left(1 + \left(\frac{c}{1+c}\right)^{\widetilde{l_j}} (\kappa - 1)\right). \tag{3.31}$$

In the following, we use the decomposition

$$\Lambda(\kappa, c) := \Lambda_1(\kappa, c) \;\cup\; \Lambda_2(\kappa, c),$$

where

$$\Lambda_1(\kappa, c) = \left\{n_j \in \Lambda(\kappa, c) : n_{j+1}/n_j \le \kappa \text{ and } s(n_{j+1}) = s(n_j)\right\}$$

and

$$\Lambda_2(\kappa, c) = \left\{n_j \in \Lambda(\kappa, c) : n_{j+1}/n_j \le \kappa \text{ and } s(n_{j+1}) \neq s(n_j).\right\}$$

Lemma 5 *Let $n_j \in \Lambda_2(\kappa, c)$, then there exists $\delta_{2,j} > 0$ and $n_3^* = n_3^*(\kappa) \in \mathbb{N}$ such that*

$$\frac{1}{n_{j+1}} \log \|f - p_{n_{j+1}}\|_{\Gamma_\sigma} \;\le\; \log \frac{\sigma}{\rho} - \delta_{2,j} \quad \textit{for } n_j \ge n_3^*. \tag{3.32}$$

Moreover, $\delta_{2,j}$ can be chosen in such a way that

$$\delta_{2,j} \;\ge\; \frac{1}{2} \min \left(\delta_0^*, \; \frac{1}{2\kappa} \left(\frac{c}{1+c} \right)^{\widetilde{l}_j} \frac{\kappa - 1}{1+c} \log \frac{\rho}{\sigma} \right) \tag{3.33}$$

with δ_0^* satisfying (3.22) and (3.23) of Lemma 3.

Proof We consider the telescoping series

$$f = p_{n_{j+1}} + \sum_{k=1}^{\infty} (p_{n_{j+k+1}} - p_{n_{j+k}})$$

and define

$$k_j := \sup \left\{ k : s(n_{j+1}) = s(n_{j+2}) = \ldots = s(n_{j+k}) \right\}. \tag{3.34}$$

Because of (3.30), we have

$$n_{j+k_j} \ge n_j \left(1 + \left(\frac{c}{1+c} \right)^{\widetilde{l}_j} (\kappa - 1) \right) - 1, \tag{3.35}$$

keeping in mind that $n_{j+k_j} \in \mathbb{N}$. Now, we write

$$\sum_{k=1}^{\infty} (p_{n_{j+k+1}} - p_{n_{j+k}}) = \; A_j + B_j,$$

where

$$A_j = \sum_{k=1}^{k_j - 1} (p_{n_{j+k+1}} - p_{n_{j+k}})$$

and

$$B_j = \sum_{k=k_j}^{\infty} (p_{n_{j+k+1}} - p_{n_{j+k}}).$$

Estimation of A_j on Γ_σ Because of the definition of k_j in (3.34), we use (3.31) and apply Lemma 3 for all differences

$$p_{n_{j+k+1}} - p_{n_{j+k}}$$

occurring in A_j. We obtain with $\delta_0^* > 0$ and $n_2^* = n^*(\kappa) \in \mathbb{N}$ that

$$\frac{1}{n_{j+k+1}} \log \|p_{n_{j+k+1}} - p_{n_{j+k}}\|_{\Gamma_\sigma} \le \log \frac{\sigma}{\rho} - \delta_0^*, \quad n_{j+k} \ge n_2^*,$$

where δ_0^* satisfies the inequality (3.23), since

$$1 < \kappa < \kappa_2^* = 1 + \frac{\alpha_{\sigma_0}(\sigma)}{\log(\rho/\sigma_0)} \delta_0$$

and δ_0 is defined by Lemma 2 in (3.20). Then,

$$\begin{aligned} \|A_j\|_{\Gamma_\sigma} &\le \sum_{k=1}^{k_j-1} \|p_{n_{j+k+1}} - p_{n_{j+k}}\|_{\Gamma_\sigma} \\ &\le \sum_{k=1}^{\infty} \left(\frac{\sigma}{\rho} e^{-\delta_0^*}\right)^{n_{j+k+1}} \\ &= \beta_1 \left(\frac{\sigma}{\rho} e^{-\delta_0^*}\right)^{n_{j+1}} \end{aligned} \tag{3.36}$$

for all j with $n_j \ge n_2^*$ and β_1 is a constant independent of j.

Estimation of B_j on Γ_σ Let us define

$$\gamma_j := 1 + \left(\frac{c}{1+c}\right)^{\widetilde{l}_j} (\kappa - 1). \tag{3.37}$$

Because of (3.1), there exists $n_\varepsilon(\sigma)$ such that for $n \ge n_\varepsilon(\sigma)$

$$\frac{1}{n} \log \|f - p_n\|_{\Gamma_\sigma} \le \log \frac{\sigma}{\rho} + \varepsilon,$$

where $0 < \varepsilon < \log(\rho/\sigma)$ is fixed. Then, for $n_j \ge n_\varepsilon(\sigma)$

$$\|p_{n_{j+k_j+1}} - p_{n_{j+k_j}}\|_{\Gamma_\sigma} \le 2 \left(\frac{\sigma}{\rho} e^{\varepsilon}\right)^{n_{j+k_j}}$$

and with (3.35) and (3.37) we obtain

$$\begin{aligned}\|p_{n_j+k_j+1}-p_{n_j+k_j}\|_{\Gamma_\sigma} &\le 2\left(\frac{\sigma}{\rho}\,e^{\varepsilon}\right)^{\gamma_j n_j}\left(\frac{\sigma}{\rho}\,e^{\varepsilon}\right)^{-1}\\ &\le \beta_2\left(\frac{\sigma}{\rho}\,e^{\varepsilon}\right)^{\gamma_j n_j}\end{aligned}$$

with

$$\beta_2 = 2\left(\frac{\sigma}{\rho}\,e^{\varepsilon}\right)^{-1}.$$

Analogously,

$$\|p_{n_j+k+1}-p_{n_j+k}\|_{\Gamma_\sigma} \;\le\; \beta_2\left(\frac{\sigma}{\rho}\,e^{\varepsilon}\right)^{\gamma_j n_j+k-k_j}$$

for all $k \ge k_j$ and $n_j \ge n_\varepsilon(\sigma)$. Hence, for such n_j

$$\begin{aligned}\|B_j\|_{\Gamma_\sigma} &\le \sum_{k=k_j}^{\infty}\|p_{n_{j+k+1}}-p_{n_{j+k}}\|_{\Gamma_\sigma}\\ &\le \beta_2\left(\frac{\sigma}{\rho}e^{\,\varepsilon}\right)^{\gamma_j n_j}\sum_{\nu=0}^{\infty}\left(\frac{\sigma}{\rho}\,e^{\varepsilon}\right)^{\nu}\\ &= \beta_2\,\frac{\rho}{\rho-\sigma\,e^{\varepsilon}}\left(\frac{\sigma}{\rho}\,e^{\varepsilon}\right)^{\gamma_j n_j}\\ &= \beta_3\left(\frac{\sigma}{\rho}\,e^{\varepsilon}\right)^{\gamma_j n_j},\end{aligned}\tag{3.38}$$

where

$$\beta_3 \;=\; \beta_2\,\frac{\rho}{\rho-\sigma\,e^{\varepsilon}}.$$

Because of (3.29),

$$1 \;\le\; \frac{n_j}{n_{j+1}}\left(1+\left(\frac{c}{1+c}\right)^{\tilde{l}_j+1}(\kappa-1)\right)$$

and therefore

$$\begin{aligned} \frac{\gamma_j n_j}{n_{j+1}} &\geq \frac{1+\left(\frac{c}{1+c}\right)^{\widetilde{l}_j}(\kappa-1)}{1+\left(\frac{c}{1+c}\right)^{\widetilde{l}_j+1}(\kappa-1)} \\ &= 1+\frac{\left(\frac{c}{1+c}\right)^{\widetilde{l}_j}\frac{\kappa-1}{1+c}}{1+\left(\frac{c}{1+c}\right)^{\widetilde{l}_j+1}(\kappa-1)}. \end{aligned} \tag{3.39}$$

For abbreviation, we define

$$\widetilde{\delta}_j := \frac{\left(\frac{c}{1+c}\right)^{\widetilde{l}_j}\frac{\kappa-1}{1+c}}{1+\left(\frac{c}{1+c}\right)^{\widetilde{l}_j+1}(\kappa-1)}$$

and note that

$$\frac{1}{\kappa}\left(\frac{c}{1+c}\right)^{\widetilde{l}_j}\frac{\kappa-1}{1+c} < \widetilde{\delta}_j < \kappa-1. \tag{3.40}$$

Since $\varepsilon < \log(\rho/\sigma)$, multiplication of (3.39) by $\log(\sigma/\rho)+\varepsilon$ yields

$$\frac{\gamma_j n_j}{n_{j+1}}\left(\log\frac{\sigma}{\rho}+\varepsilon\right) \leq \log\frac{\sigma}{\rho}+\widetilde{\delta}_j\log\frac{\sigma}{\rho}+\varepsilon\left(1+\widetilde{\delta}_j\right).$$

Hence, the upper bound in (3.40) leads to

$$\frac{\gamma_j n_j}{n_{j+1}}\left(\log\frac{\sigma}{\rho}+\varepsilon\right) < \log\frac{\sigma}{\rho}-\widetilde{\delta}_j\log\frac{\rho}{\sigma}+\varepsilon\,\kappa.$$

Next, we define

$$\varepsilon := \frac{1}{2\kappa^2}\left(\left(\frac{c}{1+c}\right)^{\widetilde{l}_j}\frac{\kappa-1}{1+c}\log\frac{\rho}{\sigma}\right). \tag{3.41}$$

Then the general condition $\varepsilon < \log(\rho/\sigma)$ is satisfied and the lower bound of (3.40) yields

$$\begin{aligned} -\widetilde{\delta}_j\log\frac{\rho}{\sigma}+\varepsilon\,\kappa &< -\widetilde{\delta}_j\log\frac{\rho}{\sigma}+\frac{1}{2}\,\widetilde{\delta}_j\log\frac{\rho}{\sigma} \\ &= -\frac{1}{2}\,\widetilde{\delta}_j\log\frac{\rho}{\sigma}. \end{aligned}$$

Therefore, for such ε we obtain by (3.38) for $n_j \in \Lambda_2(\kappa, c)$ and $n_j \geq n_\varepsilon(\sigma)$

$$\|B_j\|_{\Gamma_\sigma} \;<\; \beta_3 \left(\frac{\sigma}{\rho} e^{-\delta^*_{2.j}}\right)^{n_{j+1}}, \tag{3.42}$$

where $\delta^*_{2.j}$ is defined by

$$\delta^*_{2.j} \;:=\; \frac{\widetilde{\delta_j}}{2} \log \frac{\rho}{\sigma}$$

and ε is defined by (3.41).

Summarizing, by (3.36) and (3.42) we have got for $n_j \in \Lambda_2(\kappa, c)$ and $n_j \geq \max(n_2^*, n_\varepsilon(\sigma))$

$$\begin{aligned}\|f - n_{j+1}\|_{\Gamma_\sigma} \leq \;& \|A_j\|_{\Gamma_\sigma} + \|B_j\|_{\Gamma_\sigma} \\ \leq \;& \beta_1 \left(\frac{\sigma}{\rho} e^{-\delta^*_0}\right)^{n_{j+1}} + \beta_3 \left(\frac{\sigma}{\rho} e^{-\delta^*_{2,j}}\right)^{n_{j+1}}.\end{aligned}$$

β_1 and β_3 are constants, independent of n_j. Hence, if we define

$$\delta_{2,j} \;:=\; \frac{1}{2} \min\left(\delta^*_0,\; \frac{1}{2\kappa}\left(\frac{c}{1+c}\right)^{\widetilde{l_j}} \frac{\kappa - 1}{1+c} \log \frac{\rho}{\sigma}\right)$$

and if we use the lower bound in (3.40), then there exists $n_3^* = n_3^*(\kappa, c)$ such that

$$\frac{1}{n_{j+1}} \log \|f - p_{n_{j+1}}\|_{\Gamma\sigma} \;\leq\; \log \frac{\sigma}{\rho} - \delta_{2,j}, \quad n_j \geq n_3^*,$$

and (3.32) and (3.33) of Lemma 5 are proven. □

3.3.2 Fixing the Parameter c in $\Lambda(\kappa, c)$

In the case that $n_j \in \Lambda_2(\kappa, c)$, we have by Lemma 5: There exists $\delta_{2,j} > 0$ and $n_3^* = n_3^*(\kappa, c)$ such that

$$\frac{1}{n_{j+1}} \log \|f - p_{n_{j+1}}\|_{\Gamma_\sigma} \;\leq\; \log \frac{\sigma}{\rho} - \delta_{2,j} \tag{3.43}$$

for all $n_j \geq n_3^*(\kappa, c)$. Moreover,

$$\delta_{2,j} \geq \frac{1}{2} \min\left(\delta^*_0,\; \frac{1}{2\kappa}\left(\frac{c}{1+c}\right)^{\widetilde{l_j}} \frac{\kappa - 1}{1+c} \log \frac{\rho}{\sigma}\right) \tag{3.44}$$

with $\delta_0^* > 0$ as in Lemma 3. Because of (3.43), we have a fortiori

$$\frac{1}{n_{j+1}} \log \|f - p_{n_{j+1}}\|_{\Gamma_\sigma^{s(n_j)}} \;\leq\; \log \frac{\sigma}{\rho} - \delta_{2,j}. \tag{3.45}$$

On the other hand, we have by (3.2)

$$\frac{1}{n_j} \log \|f - p_{n_j}\|_{\Gamma_\sigma^{s(n_j)}} \;\leq\; \log \frac{\sigma}{\rho} - \delta \tag{3.46}$$

for all $n_j \geq n_1(\delta)$ with $\delta > 0$. Now, we can apply Lemma 1 by taking into account (3.45) and (3.46): There exists $n_4^* = n_4^*(\kappa)$ such that

$$\frac{1}{n_{j+1}} \log \|p_{n_{j+1}} - p_{n_j}\|_{\Gamma_\rho} \;\leq\; -\delta_3^*, \quad n_j \geq n_4^*, \tag{3.47}$$

where

$$\delta_3^* \;\geq\; \frac{1}{2}\left(\frac{\alpha_\sigma(\rho)}{\kappa}\min(\delta,\ \delta_{2,j}) - \left(1 - \frac{1}{\kappa}\right)\log\frac{\rho}{\sigma}\right) > 0,$$

if we can achieve, i.e., if we can arrange c with 0 < c <1 such that

$$\kappa_j := \frac{n_{j+1}}{n_j} \;\leq\; \kappa \;<\; 1 + \frac{\alpha_\sigma(\rho)}{\log(\rho/\sigma)}\min(\delta, \delta_{2,j}). \tag{3.48}$$

Since

$$\kappa_j = \frac{n_{j+1}}{n_j} \;\leq\; \kappa \;<\; \kappa_1^* \;=\; 1 + \frac{\alpha_\sigma(\rho)}{\log(\rho/\sigma)}\,\delta,$$

the inequalities (3.48) are fulfilled if

$$\kappa_j = \frac{n_{j+1}}{n_j} \;<\; 1 + \frac{\alpha_\sigma(\rho)}{\log(\rho/\sigma)}\,\delta_{2,j}.$$

Taking into account (3.44) and

$$\kappa \;<\; \kappa_3^* \;=\; 1 + \frac{\alpha_\sigma(\rho)}{\log(\rho/\sigma)}\,\frac{\delta_0^*}{2},$$

the inequality (3.48) is satisfied if

$$\kappa_j \;\leq\; \kappa \;<\; 1 \;+\; \frac{\alpha_\sigma(\rho)}{\log(\rho/\sigma)}\,\frac{1}{4\kappa}\,\left(\frac{c}{1+c}\right)^{\tilde{l}_j}\,\frac{\kappa - 1}{1+c}\,\log\frac{\rho}{\sigma}.$$

Because of (3.29), we know that

$$\kappa_j = \frac{n_{j+1}}{n_j} \;\le\; 1 + \left(\frac{c}{1+c}\right)^{\widetilde{l}_j+1} (\kappa - 1)\,.$$

Therefore, (3.48) is guaranteed if

$$c \;<\; \frac{1}{4\,\kappa}\,\alpha_\sigma(\rho). \tag{3.49}$$

3.4 Conclusions

We consider the telescoping series

$$f \;=\; p_{n_1} + \sum_{j=1}^{\infty} \left(p_{n_{j+1}} - p_{n_j}\right),$$

associated with the sequence

$$\Lambda(\kappa, c) = \left\{n_j\right\}_{j=1}^{\infty} = \Lambda_1(\kappa, c) \cup \Lambda_2(\kappa, c).$$

The parameter κ satisfies

$$1 < \kappa \;< \min\left(\kappa_1^*, \kappa_2^*, \kappa_3^*\right),$$

and we fix a parameter c such that

$$0 < c < \frac{1}{4\,\kappa}\,\alpha_\sigma(\rho),$$

where $\kappa_1^*, \kappa_2^*, \kappa_3^*$ are defined by (3.18), (3.21), (3.24). If $n_j \in \Lambda_1(\kappa, c)$, then according to (3.25)

$$\frac{1}{n_{j+1}} \log\, \|p_{n_{j+1}} - p_{n_j}\|_{\Gamma_\rho} \le -\delta_1^*$$

for all $n_j \in \Lambda_1(\kappa, c)$, $n_j \ge n_1^*(\kappa)$. If $n_j \in \Lambda_2(\kappa, c)$, then according to (3.47)

$$\frac{1}{n_{j+1}} \log\, \|p_{n_{j+1}} - p_{n_j}\|_{\Gamma_\rho} \le -\delta_3^*.$$

for all $n_j \in \Lambda_2(\kappa, c)$, $n_j \ge n_4^*(\kappa)$, since c satisfies (3.49). Therefore,

$$\frac{1}{n_{j+1}} \log\, \|p_{n_{j+1}} - p_{n_j}\|_{\Gamma_\rho} \le -\min\left(\delta_1^*, \delta_3^*\right) < 0$$

for all $n_j \in \Lambda(\kappa, c)$ with $n_j \geq \max\left(n_1^*(\kappa), n_4^*(\kappa)\right)$. Finally, the Lemma of Bernstein–Walsh implies that f is holomorphic in a neighborhood of $\overline{E_\rho}$, i.e.,

$$\rho(f) > \rho,$$

and Proposition 1 is proven.

4 Proof of Proposition 2

We choose r and R such that

$$1 < r < \sigma < R < \rho$$

under the additional condition that in the decomposition of E_R, resp. E_ρ, analogous to (2.1), the numbers l_R and l_ρ satisfy $l_R = l_\rho$. For abbreviation, we define

$$l := l_R = l_\rho.$$

Now, for all $z \in \Omega = \overline{\mathbb{C}} \setminus E$ we have

$$\left(U^{\nu_n} - U^{\mu_E}\right)(z) = U^{\nu_n}(z) + G(z) + \log \operatorname{cap} E$$

and therefore

$$\begin{aligned} \max_{z\in\Gamma_r} \left(U^{\nu_n} - U^{\mu_E}\right)(z) &= \max_{z\in\Gamma_r} U^{\nu_n}(z) + \log r + \log \operatorname{cap} E \\ &= \max_{z\in\Gamma_r} U^{\nu_n}(z) + \log \operatorname{cap} \overline{E_r}. \end{aligned}$$

Hence, the uniqueness of the equilibrium measure of $\overline{E_r}$ implies

$$\delta_n := \max_{z\in\Gamma_r} \left(U^{\nu_n} - U^{\mu_E}\right)(z) > 0. \tag{4.1}$$

Next, we fix $z_n \in \Gamma_r$ such that

$$\delta_n = \left(U^{\nu_n} - U^{\mu_E}\right)(z_n) \tag{4.2}$$

and we choose $s^*(n) \in \mathbb{N}$ such that

$$1 \leq s^*(n) \leq l \text{ and } z_n \in E_R^{s^*(n)}. \tag{4.3}$$

Consider

$$D_{R,r}^{s^*(n)} := E_R^{s^*(n)} \setminus \overline{E_r},$$

then $D_{R,r}^{s^*(n)}$ is a region with boundary

$$\Gamma_R^{s^*(n)} \,\cup\, \left(\Gamma_r \cap E_R^{s^*(n)}\right),$$

where

$$\Gamma_R^{s^*(n)} \,\cap\, \left(\Gamma_r \cap E_R^{s^*(n)}\right) = \emptyset.$$

The Lagrange–Hermite formula for the error $f - p_n$ at $z \in \Gamma_r$ is

$$f(z) - p_n(z) \;=\; \frac{1}{2\pi i} \int_{\Gamma_R} \frac{w_n(z)}{w_n(t)} \, \frac{f(t)}{t-z} \, d(t)$$

with

$$w_n(t) \;=\; \prod_{i=0}^{n} \left(t - z_{n,i}\right), \quad t \in \mathbb{C},$$

where $z_{n,i},\ 0 \le i \le n$, are the interpolation points of Z_n. Moreover, we can write

$$f(z) - p_n(z) \;=\; \frac{1}{2\pi i} \int_{\Gamma_R} \frac{w_n(z)}{w_n(t)} \, \frac{f(t) - p_n(t)}{t-z} \, d(t)$$

for $z \in \Gamma_r$. If $z \in \Gamma_r \cap E_R^{s^*(n)}$, we may reduce the path of integration to $\Gamma_R^{s^*(n)}$, hence

$$f(z) - p_n(z) \;=\; \frac{1}{2\pi i} \int_{\Gamma_R^{s^*(n)}} \frac{w_n(z)}{w_n(t)} \, \frac{f(t) - p_n(t)}{t-z} \, d(t). \tag{4.4}$$

Let $\varepsilon > 0$, then (2.5) implies that there exists $n_0(\varepsilon)$ such that

$$\frac{1}{n} \log \|f - p_n\|_{\Gamma_R} \;\le\; \log \frac{R}{\rho} + \varepsilon$$

and

$$\frac{1}{n} \log \|f - p_n\|_{\Gamma_r} \;\le\; \log \frac{r}{\rho} + \varepsilon$$

for all $n \ge n_0(\varepsilon)$. Using (4.4), we may choose $n_0(\varepsilon)$ in such a way that for all $z \in \Gamma_r^{s^*(n)}$ and $n \ge n_0(\varepsilon)$

$$\begin{aligned}\frac{1}{n} \log |f(z) - p_n(z)| &\le -U^{\nu_n}(z) + \max_{t \in \Gamma_R} U^{\nu_n}(t) + \frac{1}{n} \log \|f - p_n\|_{\Gamma_R} + \varepsilon \\ &\le -U^{\nu_n}(z) + \max_{t \in \Gamma_R} U^{\nu_n}(t) + \log \frac{R}{\rho} + 2\,\varepsilon.\end{aligned}$$

Since

$$U^{\mu_E}(t) = U^{\mu_E}(z) + \log\frac{r}{R} \quad \text{for } t \in \Gamma_R \quad \text{and } z \in \Gamma_r,$$

we get for $z \in \Gamma_r^{s^*(n)}$

$$\begin{aligned} \frac{1}{n}\log|f(z) - p_n(z)| \;\le\; & \left(U^{\mu_E} - U^{\nu_n}\right)(z) + \max_{t\in\Gamma_R}\left(U^{\nu_n} - U^{\mu_E}\right)(t) \\ & + \log\frac{r}{\rho} + 2\,\varepsilon. \end{aligned} \tag{4.5}$$

Now, let us consider the difference

$$U^{\nu_n}(z) - U^{\mu_E}(z), \quad z \in \Omega,$$

which is a harmonic function in Ω. Then, the maximum of this difference on the level curve Γ_{σ^*} is increasing with decreasing σ^*, $1 < \sigma^* < \infty$. Consequently,

$$\max_{t\in\Gamma_r}\left(U^{\nu_n} - U^{\mu_E}\right)(t) \;>\; \max_{t\in\Gamma_R}\left(U^{\nu_n} - U^{\mu_E}\right)(t). \tag{4.6}$$

We note for further applications that (4.6) holds also if we replace ν_n by any probability measure $\nu \neq \mu_E$ with support in E.

Because of (4.1) and (4.2) and the choice of $s^*(n)$, we have

$$\begin{aligned} \max_{z\in\Gamma_r^{s^*(n)}}\left(U^{\nu_n} - U^{\mu_E}\right)(z) \;=\; & \max_{z\in\Gamma_r}\left(U^{\nu_n} - U^{\mu_E}\right)(z) \\ >\; & \max_{t\in\Gamma_R}\left(U^{\nu_n} - U^{\mu_E}\right)(t). \end{aligned}$$

Next, we define the Dirichlet problem for the harmonic function $\Phi_n(z)$ in the region

$$D_{R,r}^{s^*(n)} = E_R^{s^*(n)} \setminus \overline{E_r}$$

with the boundary conditions

$$\Phi_n(z) \;=\; 0,\; z \in \Gamma_R^{s^*(n)}$$

and

$$\Phi_n(z) = \min\left(0, \left(U^{\mu_E} - U^{\nu_n}\right)(z) + c(\nu_n; \Gamma_R)\right),\; z \in \Gamma_r \cap E_R^{s^*(n)}, \tag{4.7}$$

where

$$c(\nu_n; \Gamma_R) \;:=\; \max_{t\in\Gamma_R}\left(U^{\nu_n} - U^{\mu_E}\right)(t).$$

Because of (4.3) and (4.6), $\Phi_n(z_n) < 0$ and therefore $\Phi_n(z) < 0$ for all $z \in D_{R,r}^{s^*(n)}$. Thus, if we define

$$\Phi_n^\sigma := \max_{t \in \Gamma_\sigma \cap E_R^{s^*(n)}} \Phi_n(t), \quad r < \sigma < R,$$

then

$$\Phi_n^\sigma < 0 \quad \text{for all } n \in \Lambda.$$

Moreover, the maximum principle for harmonic functions, together with (4.5), implies that the harmonic function

$$\Phi_n(z) + G(z) - \log \rho + 2\,\varepsilon,$$

is an upper bound for the subharmonic function

$$\frac{1}{n} \log |f(z) - p_n(z)|,$$

i.e.,

$$\frac{1}{n} \log |f(z) - p_n(z)| \ \leq \ \Phi_n(z) + G(z) - \log \rho + 2\,\varepsilon$$

for all $z \in D_{R,r}^{s^*(n)}$. Hence, we obtain

$$\frac{1}{n} \log |f(z) - p_n(z)| \ \leq \ \log \frac{\sigma}{\rho} + \Phi_n^\sigma + 2\,\varepsilon \tag{4.8}$$

for all $z \in \Gamma_\sigma \cap E_R^{s^*(n)}$ and all $n \geq n_0(\varepsilon)$.

Now, we claim: There exists $\delta > 0$ such that

$$\Phi_n^\sigma \ \leq \ -\delta \quad \text{for all } n \in \Lambda. \tag{4.9}$$

Let us assume that the claim is false:
Then, there exists a subsequence $\Lambda_1 \subset \Lambda$ such that

$$\lim_{n \in \Lambda_1, n \to \infty} \Phi_n^\sigma \ = \ 0.$$

By Helly's theorem, there exists a subsequence $\Lambda_2 \subset \Lambda_1$ such that

$$\lim_{n \in \Lambda_2, n \to \infty} \widehat{\nu}_n \ = \ \nu$$

with $\operatorname{supp}(\nu) \subset E$ and $\nu \neq \mu_E$. Since there are only l different sets

$$\Gamma_\sigma \cap E_R^i, \quad 1 \leq i \leq l,$$

we can finally choose Λ_2 such that the sets

$$\Gamma_\sigma \cap E_R^{s^*(n)}$$

are fixed for all $n \in \Lambda_2$, i.e., $s^*(n) = j_0$ is fixed for all $n \in \Lambda_2$.

Because of

$$\widehat{\nu}_n \quad \underset{n\in\Lambda_2, n\to\infty}{\overset{*}{\longrightarrow}} \quad \nu,$$

there exists $n_1(\varepsilon) \geq n_0(\varepsilon)$ such that

$$|U^\nu(z) - U^{\nu_n}(z)| \ < \ \varepsilon, \quad z \in \Gamma_r \cup \Gamma_R,$$

for all $n \in \Lambda_2$, $n \geq n_1(\varepsilon)$. Then, for $z \in \Gamma_r$ and $n \geq n_1(\varepsilon)$

$$\begin{aligned}
&\left(U^{\mu_E} - U^{\nu_n}\right)(z) + c(\nu_n; \Gamma_R) \\
&\quad = \ \left(U^{\mu_E} - U^{\nu_n}\right)(z) + \max_{t\in\Gamma_R} \left(U^{\nu_n} - U^{\mu_E}\right)(t) \\
&= \ \left(U^{\mu_E} - U^{\nu}\right)(z) + \left(U^{\nu} - U^{\nu_n}\right)(z) \\
&\quad + \max_{t\in\Gamma_R} \left(\left(U^{\nu_n} - U^{\nu}\right)(t) + \left(U^{\nu} - U^{\mu_E}\right)(t)\right) \\
&\leq \ \left(U^{\mu_E} - U^{\nu}\right)(z) + c(\nu; \Gamma_R) + 2\,\varepsilon,
\end{aligned}$$

where we have defined

$$c(\nu; \Gamma_R) := \max_{t\in\Gamma_R} \left(U^{\nu} - U^{\mu_E}\right)(t). \tag{4.10}$$

Therefore, the boundary condition (4.7) can be estimated by

$$\begin{aligned}
&\min\left(0, \left(U^{\mu_E} - U^{\nu_n}\right)(z) + c(\nu_n; \Gamma_R)\right) \\
&\quad \leq \ \min\left(0, \left(U^{\mu_E} - U^{\nu}\right)(z) + c(\nu; \Gamma_R) + 2\,\varepsilon\right)
\end{aligned} \tag{4.11}$$

for $z \in \Gamma_r \cap E_R^{j_0}$.

Now, we consider the Dirichlet Problem for the function $\Phi(z)$ in the region $D_{R,r}^{j_0} = E_R^{j_0} \setminus \overline{E_r}$ with the boundary conditions

$$\Phi(z) = 0 \text{ for } z \in \Gamma_R^{j_0} \tag{4.12}$$

and

$$\Phi(z) = \min\left(0, \left(U^{\mu_E} - U^{\nu}\right)(z) + c(\nu; \Gamma_R) + 2\,\varepsilon\right), \ z \in \Gamma_r \cap E_R^{j_0} \qquad (4.13)$$

where $c(\nu; \Gamma_R)$ is defined by (4.10).The continuous functions $U^{\mu_E} - U^{\nu_n}$ converge in Ω uniformly on compact sets, especially on $\Gamma_r \cup \Gamma_R$, as $n \in \Lambda_2, n \to \infty$. Hence, by (4.1) and (4.2)

$$\begin{aligned}
\max_{z \in \Gamma_r \cap E_R^{j_0}} \left(U^{\nu} - U^{\mu_E}\right)(z) &= \lim_{n \in \Lambda_2, n \to \infty} \max_{z \in \Gamma_r \cap E_R^{j_0}} (U^{\nu_n} - U^{\mu_E})(z) \\
&= \lim_{n \in \Lambda_2, n \to \infty} \max_{z \in \Gamma_r} (U^{\nu_n} - U^{\mu_E})(z) \\
&= \max_{z \in \Gamma_r} (U^{\nu} - U^{\mu_E})(z) \\
&> \max_{t \in \Gamma_R} (U^{\nu} - U^{\mu_E})(t). \\
&= c(\nu; \Gamma_R).
\end{aligned}$$

The last inequality follows from $\nu \neq \mu_E$, mentioned in the remark following (4.6).

Next, we choose $\varepsilon > 0$ such that

$$\max_{z \in \Gamma_r \cap E_R^{j_0}} \left(U^{\nu} - U^{\mu_E}\right)(z) - 2\,\varepsilon \ > \ c(\nu; \Gamma_R).$$

Hence, the boundary conditions for the harmonic function $\Phi(z)$ in (4.12) and (4.13) read as $\Phi(z) \leq 0$, but $\Phi(z)$ is not identically 0 on $\Gamma_r \cap E_R^{j_0}$. Then, the maximum principle for harmonic functions yields

$$\Phi(t) < 0 \quad \text{for} \quad t \in D_{R,r}^{j_0} = E_{R,r}^{j_0} \setminus \overline{E_r}.$$

If we compare the Dirichlet problems for Φ_n and Φ, then by (4.11)

$$\Phi_n(t) \leq \Phi(t) \quad \text{for } t \in D_{R,r}^{j_0} \quad \text{and for all } n \in \Lambda_2, n \geq n_1(\varepsilon).$$

Therefore,

$$\Phi_n^{\sigma} = \max_{t \in \Gamma_\sigma \cap E_R^{j_0}} \Phi_n(t) \leq \max_{t \in \Gamma_\sigma \cap E_R^{j_0}} \Phi(t) < 0$$

for $n \in \Lambda_2, n \geq n_1(\varepsilon)$, contradicting our assumption that (4.9) is not true.

Hence, (4.8) and (4.9) imply that

$$\frac{1}{n}\,|f(z) - p_n(z)| \ \leq \ \log\frac{\sigma}{\rho} - \delta + 2\,\varepsilon$$

for all $z \in \Gamma_\sigma \cap E_R^{s^*(n)}$ and $n \geq n_0(\varepsilon)$, $n \in \Lambda$. If we choose $\varepsilon = \delta/4$, then we finally get

$$\frac{1}{n}\,|f(z) - p_n(z)| \;\leq\; \log\frac{\sigma}{\rho} - \frac{\delta}{2} \tag{4.14}$$

for $z \in \Gamma_\sigma \cap E_R^{s^*(n)}$ and $n \geq n_0(\varepsilon)$, $n \in \Lambda$.

We note that each $\Gamma_\sigma \cap E_R^{s^*(n)}$, $1 \leq s^*(n) \leq l$, consists of a finite number of connected components of Γ_σ. Therefore, because of (4.14) we can define for each $n \in \Lambda$ a number $s(n)$, $1 \leq s(n) \leq l_\sigma$ such that

$$\limsup_{n\in\Lambda, n\to\infty} \|f - p_n\|_{\Gamma_\sigma^{s(n)}}^{1/n} \;<\; \frac{\sigma}{\rho}.$$

Hence, (2.6) of Proposition 2 is proven.

5 Proof of the Theorems

We have already mentioned that Theorem 1 is a direct consequence of Proposition 1. More precisely, if the condition (2.4) of Theorem 1 is true, then the sequence $p_n \in \mathcal{P}_n$ is maximally convergent to f, due to Bernstein–Walsh. Conversely, if the condition (2.4) is not true for some σ, $1 < \sigma < \rho(f) < \infty$, i.e.,

$$\limsup_{n\to\infty} \min_{1\leq i\leq l_\sigma} \|f - p_n\|_{\Gamma_\sigma^i}^{1/n} \;<\; \frac{\sigma}{\rho(f)} \;=\; \limsup_{n\to\infty} \|f - p_n\|_{\Gamma_\sigma}^{1/n},$$

then Proposition 1 shows that $\rho(f)$ is not the maximal parameter of holomorphy of f, which is a contradiction.

Concerning part (a) of Theorem 2, let us assume that μ_E is not a weak* limit point of $\widehat{\nu}_n$, $n \in \mathbb{N}$. Then, Proposition 2 yields—using $\Lambda = \mathbb{N}$—that there exist parameter $s(n)$, $1 \leq s(n) \leq l_\sigma$, such that

$$\limsup_{n\to\infty} \|f - p_n\|_{\Gamma_\sigma^{s(n)}}^{1/n} \;<\; \frac{\sigma}{\rho(f)}.$$

But according to Theorem 1, then $\rho(f)$ could not be the maximal parameter of holomorphy of f. This is a contradiction to the maximal convergence of $\{p_n\}_{n\in\mathbb{N}}^{\infty}$.

Concerning part (b), we know already that there exists a subsequence $\Lambda \in \mathbb{N}$ such that (2.7) holds. Let us assume that μ_E is not a weak* limit point of $\widehat{\nu}_n$, $n \in \Lambda$. Then, Proposition 2 implies that there exist

$$s(n),\; 1 \leq s(n) \leq l_\sigma,$$

such that

$$\limsup_{n\in\Lambda,n\to\infty} \|f - p_n\|_{\Gamma_\sigma^{s(n)}}^{1/n} < \frac{\sigma}{\rho(f)}.$$

This contradicts (2.7) and Theorem 2 is proven.

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