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Marcinkiewicz–Zygmund inequalities for scattered and random data on the q-sphere

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ABSTRACT

The recovery of multivariate functions and estimating their integrals from finitely many samples is one of the central tasks in modern approximation theory. Marcinkiewicz–Zygmund inequalities provide answers to both the recovery and the quadrature aspect. In this paper, we put ourselves on the q-dimensional sphere $\mathbb{S}^q$, and investigate how well continuous L_p-norms of polynomials f of maximum degree n on the sphere $\mathbb{S}^q$ can be discretized by positively weighted L_p-sum of finitely many samples, and discuss the distortion between the continuous and discrete quantities, the number and distribution of the (deterministic or randomly chosen) sample points $\xi_1,\dots,\xi_N$ on $\mathbb{S}^q$, the dimension q, and the degree n of the polynomials.

1. Introduction

A typical problem in science is to develop a model for a hidden process from observational data. More precisely, we are given a set of measurements $\{(x_1,f_1),\dots,(x_N,f_N)\}$, where we assume that the set of sampling nodes $\Xi=\left\{x_1,\dots,x_N\right\}$ is a finite subset of a compact metric measure space $\mathbb{X}$ with measure μ and metric ρ. The vector of sampling values $S(f)=(f_1,\dots,f_N)$ has real or complex components. It is usually assumed that the data generating process can be described by a complex-valued function f defined on $\mathbb{X}$, viz. $f(x_j)=f_j$, or at least $f(x_j)\approx f_j$. In order to develop a mathematical method to approximate the function f from its samples it is necessary to make suitable assumptions regarding the nature of f. That is, we assume that f belongs to a (smoothness) function space at least embedded into $C(\mathbb{X})$ in order to make function evaluation available. The question of which function space is suitable is not primarily a mathematical problem but depends more on the specific application. The mathematical problem is to determine an approximation $P\in\Pi_M$ to f from the given data with a certain accuracy, and to give error bounds for

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this approximation. A common strategy is to project f onto this finite-dimensional subspace Π_M spanned by the first M elements of an orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ of $L_2(\mathbb{X},\mu)$ by only using the above mentioned discrete information. This is usually done by interpreting the discrete information as "noisy" samples of a model function $\tilde{f} \in \Pi_M$ and using a least squares approach for the recovery of the coefficients as soon as the sampling operator $\mathcal{S} : \Pi_M \to \mathbb{C}^N$, $\mathcal{S}(P) = (P(x_1),\dots,P(x_N))$ is bounded and boundedly invertible on its range, i.e.,

$$c_1 \|P\|_p \le \|\mathcal{S}(P)\|_{w,p,\mathbb{C}^N} \le c_2 \|P\|_p \tag{1.1}$$

for all $P \in \Pi_M$, where $0 < c_1 \le c_2$, $1 \le p \le \infty$, $\|\cdot\|_{w,p,\mathbb{C}^N}$ is a certain weighted discrete L_p-norm on $\mathbb{C}^N$, and $\|\cdot\|_p$ is the L_p-norm on $\mathbb{X}$.

The present paper is concerned with such inequalities on the q-dimensional unit sphere $\mathbb{X} = \mathbb{S}^q$ in $\mathbb{R}^{q+1}$. Broadly speaking, inequality (1.1) is about the discretization of L_p-norms on Π_n^q, i.e., polynomials on $\mathbb{S}^q$ with maximum degree n, using point samples.

Inequalities of this type have been considered by Marcinkiewicz and Zygmund in their seminal paper [25] in relation to interpolation problems for functions defined on the torus $\mathbb{X} = \mathbb{S}^1$ (resp. 2π-periodic functions) on equidistant nodes $x_j = j/N$. More precisely, the authors of [25] proved that for every trigonometric polynomial P of maximal degree n and every $1 \le p \le \infty$ the following chain of inequalities holds

$$(1-\varepsilon)\left(\frac{1}{N}\sum_{j=1}^{N}\left|P(x_j)\right|^p\right)^{\frac{1}{p}} \le \|P\|_p \le (1+\varepsilon)\left(\frac{1}{N}\sum_{j=1}^{N}\left|P(x_j)\right|^p\right)^{\frac{1}{p}}, \tag{1.2}$$

provided that the number N of sampling points is strictly greater than $(1+\frac{1}{\varepsilon})2n$ for some $\varepsilon > 0$.

Our results cover variants of this *Marcinkiewicz–Zygmund inequality* for both scattered (i.e., deterministically given) sampling points and randomly chosen ones. For scattered data on the q-sphere $\mathbb{S}^q$, we establish the following L_p-result by applying Riesz–Thorin interpolation to the boundary cases $p = 1$ and $p = \infty$. Here L_p-norms are computed with respect to the surface area measure μ_q of $\mathbb{S}^q$, and the weights in the discretized norm are given by the surface areas of the patches Z_ξ of a partition of $\mathbb{S}^q$, to each of which exactly one sampling point $\xi \in \Xi$ belongs. (This one-to-one correspondence is roughly what we mean by saying that $(\Xi,\mathcal{Z})$ is a compatible pair.) The geometry of the partition enters through the partition norm $\|\mathcal{Z}\|$, i.e., the maximum geodesic diameter of its patches. In a sense, Theorem 4.1 can be considered a generalization of (1.2) on the q-sphere, which provides exact constants.

Theorem 4.1. *Let $\eta \in (0,1)$, and let $(\Xi,\mathcal{Z})$ be a compatible pair consisting of a finite set $\Xi \subseteq \mathbb{S}^q$ and a partition $\mathcal{Z}$ of $\mathbb{S}^q$. Assume that*

$$6C_q(n+q^2)\,\|\mathcal{Z}\| \le \eta$$

with $C_q := 3^{q/2}\pi + 2q + 3$. Then, for all $p \in [1,\infty]$ and every $P \in \Pi_n^q$, we have

$$(1-\eta)\,\|P\|_{\mu_q,p} \le \left(\sum_{\xi\in\Xi} \mu_q(Z_\xi)\,|P(\xi)|^p\right)^{\frac{1}{p}} \le (1+\eta)\,\|P\|_{\mu_q,p}.$$

This theorem has an interesting consequence which we state in Corollary 4.7 below. Namely, if $N \gtrsim \dim(\Pi_n^q)\eta^{-q}$, then we find a set $\Xi = \{\xi_1,\dots,\xi_N\}$ of sampling points on $\mathbb{S}^q$ such that the equally weighted Marcinkiewicz–Zygmund inequality

$$(1-\eta)\,\|P\|_{\mu_q,p} \le \left(\frac{\omega_q}{N}\sum_{j=1}^{N}\left|P(\xi_j)\right|^p\right)^{\frac{1}{p}} \le (1+\eta)\,\|P\|_{\mu_q,p}$$

holds true simultaneously for all $p \in [1,\infty]$ and $P \in \Pi_n^q$. The number N of sampling points therefore scales at most like η^{-q} with respect to the parameter $\eta \in (0,1)$.

For $p = 2$, this dependence can be improved to η^{-2} by using randomly drawn points at the expense of an additional logarithmic factor in the dimension of the vector space Π_n^q. More precisely, we show the following L_2-version of the Marcinkiewicz–Zygmund inequality for sampling points drawn independently and identically distributed according to the normalized surface area measure σ_q. As in [4] the problem boils down to the analysis of the extreme singular values of random matrices created from the orthonormal basis of the subspace evaluated at random points. It has been already observed in the 1990s that a logarithmic oversampling allows to control the singular values of random matrices, see for instance [30]. Here we use the more recent result by Tropp [33] together with the (3.4) which allows to consider complex random matrices and control the involved constants.

Theorem 3.2. *Let $\eta, \varepsilon \in (0,1)$. Suppose $\xi_1,\dots,\xi_N \in \mathbb{S}^q$ are drawn i.i.d. according to σ_q. If*

$$N > \frac{3\dim(\Pi_n^q)}{\eta^2}\log\left(\frac{2\dim(\Pi_n^q)}{\varepsilon}\right),$$

then with probability exceeding $1-\varepsilon$ with respect to the product measure $\mathbb{P} = \sigma_q^{\otimes N}$, we have

$$(1-\eta)\,\|P\|_{\sigma_q,2}^2 \le \frac{1}{N}\sum_{j=1}^{N}\left|P(\xi_j)\right|^2 \le (1+\eta)\,\|P\|_{\sigma_q,2}^2$$

for all $P \in \Pi_n^q$.

Also note that the addition formula (2.1) together with (3.4) implies a $(2,\infty)$-Nikolskii inequality which allows to utilize the results in [18, D.15] and [32, Section 6].

The last main result of the paper at hand is a combination of our deterministic Marcinkiewicz–Zygmund inequality in Theorem 4.1, the well-known coupon collector problem from probability theory [11, p. 36], and partitioning results for $\mathbb{S}^q$ [22, Theorem 3.1.3]. This leads us to the following L_p-Marcinkiewicz–Zygmund inequality for sets of random sampling points and $1 \le p \le \infty$.

Theorem 5.2. *Let* $n, q \in \mathbb{N}$, $\eta, \varepsilon \in (0,1)$, $\omega_q := \mu_q(\mathbb{S}^q)$, $\alpha_q := 8\left(\frac{\omega_q q}{\omega_{q-1}}\right)^{\frac{1}{q}}$, *and* $C_q = 3^{q/2}\pi + 2q + 3$. *Choose* $N \in \mathbb{N}$ *large enough such that*

$$6C_q\alpha_q\left(\frac{N}{4\log(N)}\right)^{-\frac{1}{q}}(n+q^2) < \eta.$$

Draw points $\xi_1,\dots,\xi_N \in \mathbb{S}^q$ *i.i.d. according to* σ_q. *Then with probability* $\ge 1-\frac{1}{N}$ *with respect to the product measure* $\mathbb{P} = \sigma_q^{\otimes N}$, *there exists weights* $w_1,\dots,w_N > 0$ *such that* $\sum_{j=1}^N w_j = 1$ *and*

$$(1-\eta)\,\|P\|_{\sigma_q,p} \le \left(\sum_{j=1}^{N} w_j\left|P(\xi_j)\right|^p\right)^{\frac{1}{p}} \le (1+\eta)\,\|P\|_{\sigma_q,p}$$

for all $p \in [1,\infty]$ *and all* $P \in \Pi_n^q$.

The original Marcinkiewicz–Zygmund inequality (1.2) has been generalized in many directions as to univariate and multivariate algebraic polynomials, to non-equidistant, scattered, or random samplings point sets, and to general manifolds. These generalizations have many applications in various fields in applied mathematics such as interpolation and approximation, quadrature and optimal design, sampling theory, and phase retrieval. The number of papers dealing with approximation problems on the sphere related to Marcinkiewicz–Zygmund inequalities is too large to present an exhaustive list here, exemplary we mention the papers [2,3,6–9,12–16,18,20,21,23,24,26,27,31,32]. An elaborate discussion on the various relationships in the literature is given by Gröchenig [17] and by Kashin et al. [18].

The reason for revisiting the problem of the Marcinkiewicz–Zygmund inequalities on the unit sphere $\mathbb{S}^q$ in this paper is at least twofold. First, classical proofs of the Marcinkiewicz–Zygmund inequalities for cases $p=1$ and $p=\infty$ are based on the Bernstein inequality. To get the intermediate cases $1<p<\infty$, commonly a Riesz–Thorin interpolation argument has been employed, see, e.g., [19, Theorem 1] and [27, Theorem 3.1]. However, there is a pitfall in this argument. The space Π_n^q does not contain the simple functions and therefore the use of the Riesz–Thorin interpolation theorem is not justified. The authors of [13] found a workaround to this problem in a rather abstract way. In the paper at hand, we present a more direct solution to this problem by constructing an operator related to the Marcinkiewicz–Zygmund inequalities which is defined on the entire space L_p and for which the Riesz–Thorin argument is justified, see Theorem 4.1. Second, we derive probabilistic Marcinkiewicz–Zygmund inequalities in Theorems 3.2 and 5.2. These are to some extent easier to set up because unlike in the deterministic version, no partition of the sphere is required.

We have organized the paper as follows. We start by collecting some basic material regarding the analysis on the q-dimensional unit sphere in Section 2. Section 3 is devoted to a first look to Marcinkiewicz–Zygmund inequalities for sets of random sampling points in the special case of L_2. The entire Section 4 is concerned with the proof of the Marcinkiewicz–Zygmund inequalities for deterministic sets of scattered sampling points for L_p, $1 \le p \le \infty$. In Section 5 we consider the case of random point sets again and we show how to derive L_p-versions of the desired inequalities for those sampling sets and $1 \le p \le \infty$.

2. Preliminaries

We start with some notation and basic results on harmonic analysis on the sphere, which can be found, e.g., in [1]. Let $q \ge 2$ be an arbitrary but fixed integer. The q-dimensional unit sphere $\mathbb{S}^q$ embedded in $\mathbb{R}^{q+1}$ is the set

$$\mathbb{S}^q = \left\{x \in \mathbb{R}^{q+1} \;:\; |x|_2 = 1\right\},$$

where $|x|_2$ denotes the Euclidean norm of $x \in \mathbb{R}^{q+1}$. For the inner product of two vectors $x, y \in \mathbb{R}^{q+1}$ we write $x \cdot y$. The geodesic distance on $\mathbb{S}^q$ is given by $d(x,y) = \arccos(x \cdot y)$. It defines a metric on $\mathbb{S}^q$. The surface measure on $\mathbb{S}^q$ will be denoted by μ_q and we assume that

$$\mu_q(\mathbb{S}^q) = \frac{2\pi^{\frac{q+1}{2}}}{\Gamma(\frac{q+1}{2})} =: \omega_q.$$

The spaces $L_p(\mathbb{S}^q) := L_p(\mathbb{S}^q, \mu_q)$ are defined as usual. The inner product on the Hilbert space $L_2(\mathbb{S}^q)$ is given by

$$\langle f, g\rangle = \int_{\mathbb{S}^q} f(x)\overline{g(x)}\mathrm{d}\mu_q(x).$$

Recall that using polar coordinates the kth component of the vector $x \in \mathbb{S}^q$ satisfies

$$x_k = \begin{cases} \prod_{j=1}^q \sin(\theta_j) & \text{if } k = 1, \\ \cos(\theta_{k-1}) \prod_{j=k}^q \sin(\theta_j) & \text{if } 2 \le k \le q, \\ \cos(\theta_q) & \text{if } k = q+1, \end{cases}$$

where $\theta_1 \in [-\pi, \pi]$ and $\theta_2, \dots, \theta_q \in [0, \pi]$. In polar coordinates the surface measure reads as

$$\mathrm{d}\mu_q = \prod_{k=1}^q \sin(\theta_k)^{k-1}\, \mathrm{d}\theta_k = \sin(\theta_q)^{q-1}\, \mathrm{d}\theta_q\, \mathrm{d}\mu_{q-1},$$

or equivalently

$$\mathrm{d}\mu_q = w_q(t)\, \mathrm{d}t\, \mathrm{d}\mu_{q-1},$$

with the Jacobi weight function $w_q(t) = (1-t^2)^{\frac{q}{2}-1}$ and $t = \cos(\theta_q)$.

According to the weight w_q the spaces $L_{w_q,p}([-1,1]) := L_p([-1,1], w_q(t)\,\mathrm{d}t)$ are defined in the usual manner. Using the above decomposition of $\mathrm{d}\mu_q$ it can be easily seen that for any $\phi \in L_{w_q,1}([-1,1])$ and any $y \in \mathbb{S}^q$

$$\int_{\mathbb{S}^q} \phi(x\cdot y)\mathrm{d}\mu_q(x) = \omega_{q-1} \int_{-1}^{1} \phi(t) w_q(t)\mathrm{d}t.$$

Let $n \ge 0$ be a fixed integer. The restriction of a harmonic homogeneous polynomial of degree n to $\mathbb{S}^q$ is called a *spherical harmonic* of degree n. Spherical harmonics of degree at most n form a vector space Π_n^q. The vector space of spherical harmonics of degree equal to n shall be denoted by $\mathcal{H}_n^q$. The spaces $\mathcal{H}_n^q$ are mutually orthogonal with respect to the inner product on $L_2(\mathbb{S}^q)$ and, moreover, we have the following decomposition $\Pi_n^q = \bigoplus_{\ell=0}^n \mathcal{H}_\ell^q$. Clearly, the spaces $\mathcal{H}_\ell^q$ are finite-dimensional and the dimension of Π_n^q is given by the sum of the dimensions of the spaces $\mathcal{H}_\ell^q$, $\ell = 0, \dots, n$. More precisely,

$$\dim(\mathcal{H}_\ell^q) = \frac{(2\ell+q-1)(\ell+q-2)}{\ell!\,(q-1)!} =: h_q(\ell), \qquad \dim(\Pi_n^q) = \sum_{\ell=0}^n h_q(\ell) =: d_q(n).$$

The dimension $\dim(\Pi_n^q)$ of the space Π_n^q is therefore asymptotically equivalent to n^q. Let $\{Y_{n,k} : k = 1, \dots, h_q(n)\}$ be an orthonormal basis for $\mathcal{H}_n^q$. The following relation of the basis elements $Y_{n,k}$ to the ultraspherical polynomials, known as the addition formula, is of fundamental importance to our analysis

$$\sum_{k=1}^{h_q(n)} Y_{n,k}(x)\overline{Y_{n,k}(y)} = \frac{h_q(n)}{\omega_q}\, R_n^{(\frac{q}{2}-1,\frac{q}{2}-1)}(x\cdot y), \tag{2.1}$$

where $R_n^{(\frac{q}{2}-1,\frac{q}{2}-1)}$ is the ultraspherical polynomial corresponding to the weight w_q and normalized such that $R_n^{(\frac{q}{2}-1,\frac{q}{2}-1)}(1) = 1$. The addition formula, the orthogonality relation

$$\int_{-1}^{1} R_n^{(\frac{q}{2}-1,\frac{q}{2}-1)}(t)\, R_m^{(\frac{q}{2}-1,\frac{q}{2}-1)}(t)\, w_q(t)\,\mathrm{d}t = \frac{\omega_q}{\omega_{q-1}\, h_q(n)}\, \delta_{n,m}$$

and the subsequent properties of these polynomials can be found in [10, Section 1.2]. In order to simplify the notation we will write R_n instead of $R_n^{(\frac{q}{2}-1,\frac{q}{2}-1)}$.

The space $L_2(\mathbb{S}^q)$ can be decomposed in terms of the spaces $\mathcal{H}_n^q$ as

$$L_2(\mathbb{S}^q) = c\ell \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}_n^q = c\ell\ \mathrm{span}\left\{Y_{n,k} : n \in \mathbb{N}_0,\ k = 1, \dots, h_q(n)\right\}.$$

Consequently, the orthogonal projection of $f \in L_2(\mathbb{S}^q)$ onto $\mathcal{H}_n^q$ reads as

$$\mathcal{P}_n f(x) = \sum_{k=1}^{h_q(n)} \left\langle f, Y_{n,k}\right\rangle Y_{n,k}(x) = \frac{h_q(n)}{\omega_q} \int_{\mathbb{S}^q} f(y)\, R_n(x\cdot y)\, \mathrm{d}\mu_q(y),$$

where the second identity is an implication of the addition formula (2.1). The orthogonal projection onto the space Π_n^q is therefore given as

$$\mathcal{S}_n f(x) = \sum_{k=0}^{n} \mathcal{P}_k f(x) = \frac{1}{\omega_{q-1}} \int_{\mathbb{S}^q} f(y)\, K_n(x\cdot y, 1)\, \mathrm{d}\mu_q(y),$$

where

$$K_n(t,t') = \sum_{k=0}^{n} \left\|R_k\right\|_2^{-2} R_k(t) R_k(t') \tag{2.2}$$

is the Christoffel–Darboux kernel for the ultraspherical polynomials and $\left\|R_k\right\|_2 = (\int_{-1}^{1} \left|R_k(t)\right|^2 \mathrm{d}t)^{\frac{1}{2}}$. In order to simplify our notation we will write $K_n(t)$ for $K_n(t,1)$.

In the one-dimensional case, i.e., on $\mathbb{S}^1 = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, the $(2n+1)$-dimensional polynomial spaces $\Pi_n^1 = T(n)$ consists of the trigonometric polynomials $f : \mathbb{T} \to \mathbb{C}$, $f(x) = \sum_{k=-n}^{n} c_k \exp(\mathrm{i}kx)$, where $c_{-n},\dots,c_n \in \mathbb{C}$. The well-known Bernstein inequality for trigonometric polynomials reads as follows, see [35, Theorem III.3.16].

Lemma 2.1. *For $p \in [1,\infty]$ and $P \in T(n)$, we have*

$$\left\|P'\right\|_{\mathbb{T},p} \le n \left\|P\right\|_{\mathbb{T},p},$$

where the L_p-norm of a function defined on the torus is given as

$$\|f\|_{\mathbb{T},p} := \begin{cases} \left((2\pi)^{-1} \int_{-\pi}^{\pi} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \sup_{x\in[-\pi,\pi]} |f(x)| & \text{if } p = \infty. \end{cases} \tag{2.3}$$

Analogously to Equation (2.3), we may define

$$\|f\|_{\mu,p} := \begin{cases} \left(\int_{\mathbb{S}^q} |f(x)|^p \,\mathrm{d}\mu(x)\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \mu\text{-}\operatorname{ess\,sup}\{|f(x)| \;:\; x \in \mathbb{S}^q\} & \text{if } p = \infty, \end{cases}$$

and

$$\|x\|_{\mathbb{C}^D,p} := \begin{cases} \left(\sum_{j=1}^{D} \left|x_j\right|^p\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \sup\left\{\left|x_1\right|,\dots,\left|x_D\right|\right\} & \text{if } p = \infty, \end{cases}$$

for any measure μ on $\mathbb{S}^q$, $f : \mathbb{S}^q \to \mathbb{C}$, and $x = (x_1,\dots,x_D) \in \mathbb{C}^D$.

For our analysis it will be necessary to consider partitions of $\mathbb{S}^q$ and related sets of points. A family $\mathcal{Z} = \left\{Z_1,\dots,Z_N\right\}$ of measurable subsets $Z_k \subseteq \mathbb{S}^q$ is called a *partition* of $\mathbb{S}^q$ if their interiors are pairwise disjoint, i.e., $\operatorname{int}(Z_k) \cap \operatorname{int}(Z_{k'}) = \emptyset$ for all $k,k' \in \{1,\dots,N\}$, and $\mathbb{S}^q = \bigcup_{k=1}^{N} Z_k$. An element $Z \in \mathcal{Z}$ is called a *patch*. A finite subset Ξ of $\mathbb{S}^q$ is called *compatible* with the partition if there is precisely one element of Ξ in the interior of every patch of $\mathcal{Z}$, viz. $\Xi \cap Z_k = \{\xi\}$ for every $k \in \{1,\dots,N\}$. We will call the pair $(\Xi,\mathcal{Z})$ compatible if the set Ξ is compatible with the partition $\mathcal{Z}$ and we will write Z_ξ to indicate the patch from $\mathcal{Z}$ which contains the element $\xi \in \Xi$, see Fig. 1. There are two parameters related to Ξ resp. $\mathcal{Z}$ which will be relevant for our analysis. These are the *mesh norm* of Ξ defined as

$$\delta_\Xi := \max_{x\in\mathbb{S}^q} \min_{y\in\Xi} d(x,y),$$

and the *partition norm* related to $\mathcal{Z}$ given by

$$\|\mathcal{Z}\| := \max_{Z\in\mathcal{Z}} \sup_{x,y\in Z} d(x,y).$$

In view of the Marcinkiewicz–Zygmund inequality, we discretize a measure μ on $\mathbb{S}^q$ using the data of a compatible pair $(\Xi,\mathcal{Z})$ by $\mu(\Xi,\mathcal{Z}) := \sum_{\xi\in\Xi} \mu(Z_\xi)\delta_\xi$, where $\delta_\xi(A) := \mathbf{1}_A(\xi)$ denotes the Dirac measure at $\xi \in \Xi$.

Before we get to L_p-Marcinkiewicz–Zygmund inequalities for general $1 \le p \le \infty$ later, we have a look at the special case $p = 2$ through the lens of random matrix theory in the next section. These proof techniques are tailored to the $p = 2$ case, and yield a first version of the Marcinkiewicz–Zygmund inequality for randomly chosen sampling points.

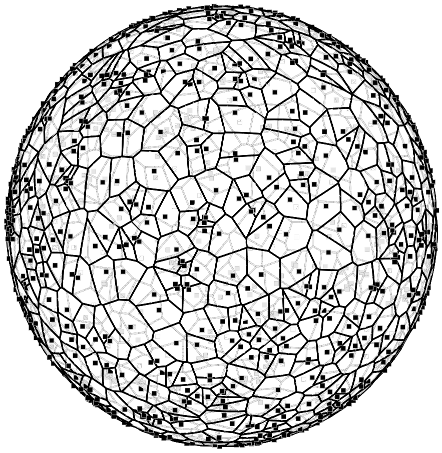

Fig. 1. Example of a compatible pair $(\Xi, \mathcal{Z})$.

3. A first look at random points

In this section, we consider the following randomized setting. Denote by $\Xi = \{\xi_1, \dots, \xi_N\}$ a set of points on $\mathbb{S}^q$ drawn i.i.d. according to the normalized surface area measure $\sigma_q = \frac{1}{\omega_q}\mu_q$ on $\mathbb{S}^q$. The aim is to provide a relationship between the number N of samples, the dimension q, the degree n of the polynomials, and the parameter $\eta \in (0,1)$ such that

$$(1-\eta)\,\|P\|^2_{\sigma_q,2} \le \frac{1}{N}\sum_{j=1}^{N}\left|P(\xi_j)\right|^2 \le (1+\eta)\,\|P\|^2_{\sigma_q,2}, \tag{3.1}$$

holds with high probability for every $P \in \Pi_n^q$.

To keep the notation simple we will write d instead of $d_q(n)$ for the dimension of Π_n^q. Let $(e_k)_{k=1}^d$ be an orthonormal basis of Π_n^q with respect to the inner product

$$\langle f, g\rangle := \int_{\mathbb{S}^q} f(x)\overline{g(x)}\mathrm{d}\sigma_q(x).$$

Parseval's identity yields

$$\|P\|_{\sigma_q,2} = \|u\|_{\mathbb{C}^d,2}$$

where $P \in \Pi_n^q$ and $u = (\langle P, e_k\rangle)_{k=1}^d$. Now consider

$$L = \begin{pmatrix} e_1(\xi_1) & e_2(\xi_1) & \cdots & e_d(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ e_1(\xi_N) & e_2(\xi_N) & \cdots & e_d(\xi_N) \end{pmatrix}, \tag{3.2}$$

and note that

$$(Lu)_j = \sum_{k=1}^{d} \langle P, e_k\rangle\, e_k(\xi_j) = P(\xi_j)$$

for $j = 1, \dots, N$. Thus the inequality (3.1) can be rewritten as

$$(1-\eta)\,\|u\|^2_{\mathbb{C}^d,2} \le \left\|\frac{1}{\sqrt{N}}Lu\right\|^2_{\mathbb{C}^N,2} \le (1+\eta)\,\|u\|^2_{\mathbb{C}^d,2}. \tag{3.3}$$

Obviously, the best possible constants $1 \pm \eta$ in (3.3) are given by the minimal resp. maximal eigenvalue of $\frac{1}{N}L^*L$. In [28, Theorem 2.1], Moeller and Ullrich proved the following concentration inequality for the smallest and largest eigenvalue of such random Gram matrices. The result is based on Tropp [33].

Theorem 3.1. *Let $s, N, M \in \mathbb{N}$, $t \in (0,1)$, $\Omega \subseteq \mathbb{R}^s$ a set, ϱ a probability measure on Ω and $(e_k)_{k=1}^D$ be an orthonormal system in $L_2(\Omega, \varrho)$. Let $\xi_1, \dots, \xi_N \in \Omega$ be drawn i.i.d. according to ϱ, $L = (e_k(\xi_j))_{j,k=1}^{N,D}$, and $\mathbb{P} = \varrho^{\otimes N}$ the product measure. Then the following concentration inequalities for the extremal eigenvalues of $\frac{1}{N}L^*L$ hold*

$$\mathbb{P}\left(\lambda_{\min}\left(\frac{1}{N}L^*L\right) < 1-t\right) < (D+1)\exp\left(-\frac{N\log((1-t)^{1-t}\mathrm{e}^{t})}{\sup_{x\in\Omega}\sum_{k=1}^{D}\left|e_k(x)\right|^2}\right),$$
$$\mathbb{P}\left(\lambda_{\max}\left(\frac{1}{N}L^*L\right) > 1+t\right) < (D+1)\exp\left(-\frac{N\log((1+t)^{1+t}\mathrm{e}^{-t})}{\sup_{x\in\Omega}\sum_{k=1}^{D}\left|e_k(x)\right|^2}\right).$$

To apply this result to our case let $s = q+1$, $\Omega = \mathbb{S}^q$, $\varrho = \sigma_q$, and let $(e_k)_{k=1}^d$ be an orthonormal basis of the Hilbert space $(\Pi_n^q, \|\cdot\|_{\sigma_q,2})$.

To compute the expression $\sup_{x\in\mathbb{S}^q}\sum_{k=1}^d \left|e_k(x)\right|^2$, note that orthonormal bases of $(\Pi_n^q, \|\cdot\|_{\sigma_q,2})$ are obtained from orthonormal bases of $(\Pi_n^q, \|\cdot\|_{\mu_q,2})$ by multiplying each element by the constant scalar $\omega_q^{\frac{1}{2}}$. Using the addition formula (2.1) an easy computation shows that

$$\sum_{k=1}^{d}\left|e_k(x)\right|^2 = d = \sum_{\ell=0}^{n}\frac{(2\ell+q-1)(\ell+q-2)!}{\ell!(q-1)!} \tag{3.4}$$

for every $x \in \mathbb{S}^q$.

Theorem 3.2. *Let $\eta, \varepsilon \in (0,1)$. Suppose $\xi_1, \ldots, \xi_N \in \mathbb{S}^q$ are drawn i.i.d. according to σ_q. If*

$$N > \frac{3d_q(n)}{\eta^2}\log\left(\frac{2d_q(n)}{\varepsilon}\right),$$

then with probability exceeding $1-\varepsilon$ with respect to the product measure $\mathbb{P} = \sigma_q^{\otimes N}$, we have

$$(1-\eta)\,\|P\|_{\sigma_q,2}^2 \le \frac{1}{N}\sum_{j=1}^{N}\left|P(\xi_j)\right|^2 \le (1+\eta)\,\|P\|_{\sigma_q,2}^2$$

for all $P \in \Pi_n^q$.

Proof. We will again use d for $d_q(n)$. Let $(e_k)_{k=1}^d$ be an orthonormal basis for $(\Pi_n^q, \|\cdot\|_{\sigma_q})$ and $L := (e_k(\xi_j))_{j,k=1}^{N,d}$. By Theorem 3.1 and (3.4), we have

$$\mathbb{P}\left(\lambda_{\min}\left(\frac{1}{N}L^*L\right) < 1-\eta\right) < d\exp\left(-\frac{N\log((1-\eta)^{1-\eta}\mathrm{e}^{t})}{d}\right) \tag{3.5}$$

and

$$\mathbb{P}\left(\lambda_{\max}\left(\frac{1}{N}L^*L\right) > 1+\eta\right) < d\exp\left(-\frac{N\log((1+\eta)^{1+\eta}\mathrm{e}^{-\eta})}{d}\right). \tag{3.6}$$

For $\eta \in (0,1)$, we have

$$\max\left\{\log((1-\eta)^{1-\eta}\mathrm{e}^{\eta}), \log((1+\eta)^{1+\eta}\mathrm{e}^{-\eta})\right\} \ge \frac{\eta^2}{3},$$

so the right-hand sides of (3.5) and (3.6) are each $\le \exp\left(-\frac{N\eta^2}{3d}\right)$. Note that $N > \frac{3d}{\eta^2}\log\left(\frac{2d}{\varepsilon}\right)$ is equivalent to $d\exp\left(-\frac{N\eta^2}{3d}\right) < \frac{\varepsilon}{2}$. Thus

$$\begin{aligned}&\mathbb{P}\left(\lambda_{\min}\left(\frac{1}{N}L^*L\right) < 1-\eta\right) + \mathbb{P}\left(\lambda_{\max}\left(\frac{1}{N}L^*L\right) > 1+\eta\right)\\ &< 2d\exp\left(-\frac{N\eta^2}{3d}\right)\\ &< \varepsilon.\end{aligned}$$

This concludes the proof. □

Note that the statement of Theorem 3.2 holds verbatim for any direct sum $\bigoplus_{\ell\in J}\mathcal{H}_\ell^q$ for some index set $J \subseteq \mathbb{N}$ in place of Π_n^q, with $d_q(n) = \dim(\Pi_n^q)$ replaced by $\dim(\bigoplus_{\ell\in J}\mathcal{H}_\ell^q)$. Like in (3.4), this is due to fact that the addition formula for orthonormal bases holds true for the summands $\mathcal{H}_\ell^q$, see again [15, equation (2.8)].

In order to illustrate Theorem 3.2 we fix $q = 2$, $\eta = 0.9$ and $\varepsilon = 0.01$, randomly draw $N = \frac{3d_q(n)}{\eta^2}\log\left(\frac{2d_q(n)}{\varepsilon}\right)$ points $\xi_1, \ldots, \xi_N \in \mathbb{S}^q$ and compute the minimum and maximum eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$ of the matrix $\frac{1}{N}L^*L$. We repeated this 1000 times for different degrees n of the polynomials and depicted in Fig. 2 the average minimum and maximum eigenvalues as well as the 1 and 99 percent quantiles. According to our experiment those are safely within the range $[1-\eta, 1+\eta]$ as stated by Theorem 3.2.

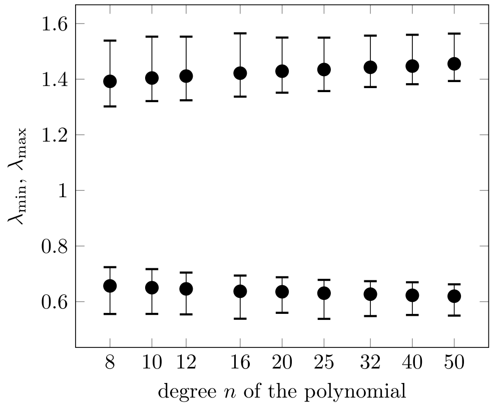

Fig. 2. Concentration of the minimum and maximum eigenvalues of the matrix $\frac{1}{N}L^*L$ for random sample sets $\xi_1, \dots, \xi_N \in \mathbb{S}^2$ and different degrees n of the polynomials. The number of sampling points N is chosen according to the lower bound in Theorem 3.2, where we have used the constants $\eta = 0.9$ and $\varepsilon = 0.01$. Displayed are the mean minimum and maximum eigenvalues as well as the 1 and 99 percent quantiles.

As a first step towards general p, we obtain the following statement when $q = 2$ and p is even.

Corollary 3.3. *Let $\eta, \varepsilon \in (0, 1)$, $q = 2$, and let $p \in \mathbb{N}$ be an even number. Assume $\xi_1, \dots, \xi_N \in \mathbb{S}^q$ are drawn i.i.d. according to σ_q. If*

$$N > \frac{3d_q(np/2)}{\eta^2} \log\left(\frac{2d_q(np/2)}{\varepsilon}\right),$$

then with probability exceeding $1 - \varepsilon$ with respect to the product measure $\mathbb{P} = \sigma_q^{\otimes N}$, we have

$$(1 - \eta) \|P\|^p_{\sigma_q, p} \le \frac{1}{N} \sum_{j=1}^N \left|P(\xi_j)\right|^p \le (1 + \eta) \|P\|^p_{\sigma_q, p},$$

for all $P \in \Pi^q_n$.

Proof. For abbreviation we put $\tilde{d} = d_q(np/2)$. Let $(e_k)_{k=1}^{\tilde{d}}$ be the orthonormal basis (spherical harmonics) of the Hilbert space $(\Pi^q_{\frac{np}{2}}, \|\cdot\|_{\sigma_q, 2})$. Since the product of two spherical harmonics of degree n_1 and n_2 belongs to the span of the spherical harmonics up to degree $n_1 + n_2$, cf. [34, Section 5.6.2], we have $P^{\frac{p}{2}} \in \Pi^q_{\frac{np}{2}}$ when $P \in \Pi^q_n$.

Together with Theorem 3.2 this implies for $N > \frac{3\tilde{d}}{\eta^2} \log\left(\frac{2\tilde{d}}{\varepsilon}\right)$ that

$$(1 - \eta) \left\|P^{\frac{p}{2}}\right\|^2_{\sigma_q, 2} \le \frac{1}{N} \sum_{j=1}^N \left|P(\xi_j)^{\frac{p}{2}}\right|^2 \le (1 + \eta) \left\|P^{\frac{p}{2}}\right\|^2_{\sigma_q, 2}$$

with probability $\ge 1 - \varepsilon$ with respect to the product measure $\mathbb{P} = \sigma_q^{\otimes N}$. This is equivalent to the assertion. $\square$

In order to obtain Marcinkiewicz–Zygmund inequalities for random sampling points and general $p \in [1, \infty]$ we first reconsider the case were the sampling points are deterministic scattered points on $\mathbb{S}^q$.

4. Marcinkiewicz–Zygmund inequalities for scattered data

In this section, we give a proof for a deterministic Marcinkiewicz–Zygmund inequality on $\mathbb{S}^q$ which holds for all p simultaneously. Reasoning from the Riesz–Thorin interpolation theorem has been attempted in the literature several times, however (to our best knowledge) always fraught with problems. The authors of [13] are aware of this issue and prove deterministic Marcinkiewicz–Zygmund inequality in a manifold setting by different means. The aim of this section is to provide a self-contained and rather elementary proof for the sphere by proper use of Riesz–Thorin interpolation, which, in addition, simplifies some of the technical calculations in [15,27]. The main theorem in this section reads as follows.

Theorem 4.1. *Let $\eta \in (0, 1)$, and let $(\Xi, \mathcal{Z})$ be a compatible pair consisting of a finite set $\Xi \subseteq \mathbb{S}^q$ and a partition $\mathcal{Z}$ of $\mathbb{S}^q$. Assume that*

$$6C_q(n + q^2) \|\mathcal{Z}\| \le \eta$$

with $C_q := 3^{q/2}\pi + 2q + 3$.

Then, for all $p \in [1,\infty]$ and every $P \in \Pi_n^q$, we have

$$(1-\eta)\,\|P\|_{\mu_q,p} \le \|P\|_{\mu_q(\Xi,\mathcal{Z}),p} \le (1+\eta)\,\|P\|_{\mu_q,p}\,.$$

The proof of Theorem 4.1 is essentially based on a generalized de la Vallée Poussin kernel $v_n\colon [-1,1]\to\mathbb{R}$, $n\in\mathbb{N}$, for the system of ultraspherical polynomials which was defined in [15] as

$$v_n(t) = \frac{1}{\omega_{q-1}}\,\frac{K_{\lfloor \frac{n}{2}\rfloor}(t)K_{\lfloor \frac{3n}{2}\rfloor}(t)}{K_{\lfloor \frac{n}{2}\rfloor}(1)},$$

where K_n is the Christoffel–Darboux kernel defined in (2.2). The generalized de la Vallée Poussin kernel v_n is a polynomial of degree $2n$ that reproduces polynomials $P\in\Pi_n^q$, up to degree n viz.

$$P(x) = \int_{\mathbb{S}^q} P(y)\, v_n(x\cdot y)\, \mathrm{d}\mu_q(y),$$

as it does the Christoffel–Darboux kernel K_n. Additionally, the kernels v_n, $n\in\mathbb{N}$, have bounded $L_{w_q,1}$-norm

$$\int_{-1}^{1} \left|v_n(t)\right| (1-t^2)^{\frac{q}{2}-1}\mathrm{d}t \le \frac{3^{\frac{q}{2}}}{\omega_{q-1}} \tag{4.1}$$

and satisfy

$$\sup_{t\in[-1,1]} \left|v_n(t)\right| \le \frac{1}{\omega_{q-1}}\,\frac{2^{-q+1}(\left\lfloor \frac{3n}{2}\right\rfloor + q)^q}{\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)} \le \frac{1}{\omega_{q-1}}\,\frac{2\max\{n,2q\}^q}{\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)}. \tag{4.2}$$

For the proof of these statements we refer to [15, Section 3.3]. We prepare the proof of Theorem 4.1 by first showing an integral bound of the derivative of the generalized de la Vallée Poussin kernel.

Lemma 4.2. *For $n,q\in\mathbb{N}$ with $q\ge 2$, the following estimate holds*

$$\int_0^{\pi} \left|v_n'(\cos(\tau))\sin(\tau)^q\right| \mathrm{d}\tau \le C_q\,\frac{n+q^2}{\omega_{q-1}}$$

where $C_q = 3^{q/2}\pi + 2q + 3$ as in Theorem 4.1.

Proof. Let $\theta := 1/\max\{n,2q\}$. We split the integral over $[0,\pi]$ into three parts

$$\int_0^{\pi} \left|v_n'(\cos(\tau))\sin(\tau)^q\right| \mathrm{d}\tau = \Big(\int_0^{\theta} + \int_{\theta}^{\pi-\theta} + \int_{\pi-\theta}^{\pi}\Big) \left|v_n'(\cos(\tau))\sin(\tau)^q\right| \mathrm{d}\tau.$$

By the trigonometric Bernstein inequality (Lemma 2.1) and Equation (4.2), we obtain for the first integral

$$\begin{aligned} A_1 &:= \int_0^{\theta} \left|v_n'(\cos(\tau))\sin(\tau)^q\right| \mathrm{d}\tau = \int_0^{\theta} \left|(v_n\circ\cos)'(\tau)\sin(\tau)^{q-1}\right| \mathrm{d}\tau \\ &\le 2n\left\|v_n\circ\cos\right\|_{\mathbb{T},\infty} \int_0^{\theta} \tau^{q-1}\mathrm{d}\tau \le \frac{4n}{\omega_{q-1}\,q\,\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)} \le \frac{2n}{\omega_{q-1}}. \end{aligned}$$

Thanks to symmetry the same upper bound is valid for the third integral

$$A_3 := \int_{\pi-\theta}^{\pi} \left|v_n'(\cos(\tau))\sin(\tau)^q\right| \mathrm{d}\tau.$$

Using the product rule followed by the triangle inequality we split the middle integral into

$$A_2 := \int_{\theta}^{\pi-\theta} \left|v_n'(\cos(\tau))\sin(\tau)^q\right| \mathrm{d}\tau$$

$$\le \underbrace{\int_{\theta}^{\pi-\theta} \left|((v_n \circ \cos)\sin^{q-1})'(\tau)\right| \mathrm{d}\tau}_{=:I_1} + \underbrace{\int_{\theta}^{\pi-\theta} \left|v_n(\cos(\tau))(q-1)\sin(\tau)^{q-2}\cos(\tau)\right| \mathrm{d}\tau}_{=:I_2}$$

Applying the trigonometric Bernstein inequality to $(v_n \circ \cos)\sin^{q-1}$ we obtain in conjunction with Equation (4.1)

$$I_1 \le (2n+q-1)\int_0^{\pi} \left|v_n(\cos(\tau))\sin(\tau)^{q-1}\right| \mathrm{d}\tau \le (2n+q-1)\frac{3^{\frac{q}{2}}}{\omega_{q-1}}$$

and

$$I_2 = (q-1)\int_{\theta}^{\pi-\theta} \left|v_n(\cos(\tau))\right| \sin(\tau)^{q-1}\frac{|\cos(\tau)|}{\sin(\tau)}\mathrm{d}\tau$$
$$\le (q-1)\frac{\pi}{2\theta}\int_{\theta}^{\pi-\theta} \left|v_n(\cos(\tau))\sin(\tau)^{q-1}\right| \mathrm{d}\tau \le (q-1)\frac{\pi}{2\theta}\frac{3^{\frac{q}{2}}}{\omega_{q-1}},$$

where we made use of $|\cos(\tau)| \le 1$ for all $\tau \in \mathbb{R}$, $\frac{1}{\sin(\tau)} \le \frac{\pi}{2\tau} \le \frac{\pi}{2\theta}$ when $\theta \le \tau \le \frac{\pi}{2}$, and $\frac{1}{\sin(\tau)} \le \frac{\pi}{2(\pi-\tau)} \le \frac{\pi}{2\theta}$ when $\frac{\pi}{2} \le \tau \le \pi - \theta$. Finally we arrive at

$$\int_0^{\pi} \left|v_n'(\cos(\tau))\sin(\tau)^q\right| \mathrm{d}\tau \le A_1 + A_3 + I_1 + I_2$$
$$\le \frac{4n}{\omega_{q-1}} + \frac{3^{\frac{q}{2}}}{\omega_{q-1}}\Big(2n + (q-1)\Big(1 + \pi \max\Big(\frac{n}{2}, q\Big)\Big)\Big)$$
$$\le \frac{4 + 2\cdot 3^{\frac{q}{2}} + \frac{q-1}{2}\pi}{\omega_{q-1}} n + \frac{3^{\frac{q}{2}}(q-1)(1+\pi q)}{\omega_{q-1}}$$
$$\le \frac{3^{\frac{q}{2}}\pi + 2q + 2}{\omega_{q-1}}(n + q^2)$$

which concludes the proof. □

A key step in the proof of Theorem 4.1 is to show that for every compatible pair $(\Xi, \mathcal{Z})$

$$T_{\Xi,\mathcal{Z},n}(f)(x) := \int_{\mathbb{S}^q} \sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x)\big(v_n(x\cdot y) - v_n(\xi\cdot y)\big) f(y)\mathrm{d}\mu_q(y), \tag{4.3}$$

defines a bounded operator $T_{\Xi,\mathcal{Z},n} : L_p(\mathbb{S}^q, \mu_q) \to L_p(\mathbb{S}^q, \mu_q)$ for all $p \in [1, \infty]$. We concentrate on the extreme cases $p = 1$ and $p = \infty$ in the following two lemmas and start with $p = 1$.

Lemma 4.3. *The mapping $T_{\Xi,\mathcal{Z},n}$ defines a bounded linear operator from $L_1(\mathbb{S}^q)$ to $L_1(\mathbb{S}^q)$ with norm*

$$\left\|T_{\Xi,\mathcal{Z},n}\right\|_{1\to 1} \le \left(\frac{2^{q+4}}{q\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)} + 4C_q\right)(n+q^2)\,\|\mathcal{Z}\|,$$

provided that $(n+q^2)\,\|\mathcal{Z}\| < 1$.

Proof. Using the triangle inequality, Fubini's theorem, and Hölder's inequality, we obtain

$$\left\|T_{\Xi,\mathcal{Z},n}(f)\right\|_{\mu_q,1}$$
$$= \int_{\mathbb{S}^q}\left|\int_{\mathbb{S}^q}\sum_{\xi\in\Xi}\mathbf{1}_{Z_\xi}(x)\big(v_n(x\cdot y) - v_n(\xi\cdot y)\big)\, f(y)\mathrm{d}\mu_q(y)\right|\mathrm{d}\mu_q(x)$$
$$\le \int_{\mathbb{S}^q}\int_{\mathbb{S}^q}\sum_{\xi\in\Xi}\mathbf{1}_{Z_\xi}(x)\left|v_n(x\cdot y) - v_n(\xi\cdot y)\right|\,|f(y)|\,\mathrm{d}\mu_q(y)\mathrm{d}\mu_q(x)$$

$$\leq \|f\|_{\mu_q,1} \operatorname*{ess\,sup}_{y\in\mathbb{S}^q} \int_{\mathbb{S}^q} \sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x) \left|v_n(x\cdot y) - v_n(\xi\cdot y)\right| \mathrm{d}\mu_q(x).$$

Now fix $y \in \mathbb{S}^q$. The fundamental theorem of calculus and the triangle inequality give

$$\begin{aligned}
&\int_{\mathbb{S}^q} \sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x) \left|v_n(x\cdot y) - v_n(\xi\cdot y)\right| \mathrm{d}\mu_q(x)\\
&= \int_{\mathbb{S}^q} \sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x) \left| \int_{[d(x,y),d(\xi,y)]} (v_n\circ\cos)'(t)\mathrm{d}t \right| \mathrm{d}\mu_q(x)\\
&\leq \int_{\mathbb{S}^q} \sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x) \int_{d(x,y)-\|\mathcal{Z}\|}^{d(x,y)+\|\mathcal{Z}\|} \left|(v_n\circ\cos)'(t)\right| \mathrm{d}t\,\mathrm{d}\mu_q(x).
\end{aligned}$$

Now integration is independent of ξ, and since $\sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x) = 1$ for μ_q-almost all $x \in \mathbb{S}^q$, this factor can be omitted. Parametrizing $\mathbb{S}^q$ with north pole y yields

$$= \omega_{q-1} \int_0^\pi \sin(\tau)^{q-1} \int_{\tau-\|\mathcal{Z}\|}^{\tau+\|\mathcal{Z}\|} \left|(v_n\circ\cos)'(t)\right| \mathrm{d}t\mathrm{d}\tau$$

where we only resolved the outer integral in the last step. Having in mind that $\|\mathcal{Z}\| < \frac{\pi}{2}$, we split the integration over $[0,\pi]$ into pieces:

$$\int_0^\pi \sin(\tau)^{q-1} \int_{\tau-\|\mathcal{Z}\|}^{\tau+\|\mathcal{Z}\|} \left|(v_n\circ\cos)'(t)\right| \mathrm{d}t\mathrm{d}\tau = B_1 + B_2 + B_3.$$

Upper bounds for the summands

$$\begin{aligned}
B_1 :=& \int_0^{2\|\mathcal{Z}\|} \sin(\tau)^{q-1} \int_{\tau-\|\mathcal{Z}\|}^{\tau+\|\mathcal{Z}\|} \left|(v_n\circ\cos)'(t)\right| \mathrm{d}t\mathrm{d}\tau\\
&\leq 2\|\mathcal{Z}\| \left\|(v_n\circ\cos)'\right\|_{\mathbb{T},\infty} \int_0^{2\|\mathcal{Z}\|} \sin(\tau)^{q-1}\mathrm{d}\tau\\
&\leq 4n\|\mathcal{Z}\| \left\|v_n\circ\cos\right\|_{\mathbb{T},\infty} \int_0^{2\|\mathcal{Z}\|} \tau^{q-1}\mathrm{d}\tau\\
&= 4n\|\mathcal{Z}\| \left\|v_n\circ\cos\right\|_{\mathbb{T},\infty} q^{-1}2^q\|\mathcal{Z}\|^q\\
&\leq n\max\{n,2q\}^q \|\mathcal{Z}\|^{q+1} \frac{2^{q+3}}{q\omega_{q-1}\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)}\\
&\leq (n+q^2)^{q+1} \|\mathcal{Z}\|^{q+1} \frac{2^{q+3}}{q\omega_{q-1}\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)}
\end{aligned}$$

and likewise

$$\begin{aligned}
B_2 :=& \int_{\pi-2\|\mathcal{Z}\|}^{\pi} \sin(\tau)^{q-1} \int_{\tau-\|\mathcal{Z}\|}^{\tau+\|\mathcal{Z}\|} \left|\frac{\mathrm{d}}{\mathrm{d}t}(v_n\circ\cos)'(t)\right| \mathrm{d}t\mathrm{d}\tau\\
&\leq 2\|\mathcal{Z}\| \left\|(v_n\circ\cos)'\right\|_{\mathbb{T},\infty} \int_{\pi-2\|\mathcal{Z}\|}^{\pi} \sin(\tau)^{q-1}\mathrm{d}\tau\\
&\leq (n+q^2)^{q+1} \|\mathcal{Z}\|^{q+1} \frac{2^{q+3}}{q\omega_{q-1}\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)}
\end{aligned}$$

are due to Lemma 2.1, Equation (4.2), and $\sin(\tau) \leq \tau$ for all $\tau \in \mathbb{R}$ with $\tau > 0$.

Now, if $\tau - \|\mathcal{Z}\| \leq t \leq \tau + \|\mathcal{Z}\|$ and $2\|\mathcal{Z}\| \leq \tau \leq \pi - 2\|\mathcal{Z}\|$, we have $\|\mathcal{Z}\| \leq t \leq \pi - \|\mathcal{Z}\|$, and thus

$$\begin{aligned}\sin(\tau) &= \sin(\tau - t + t) = \sin(t)\cos(\tau - t) + \sin(\tau - t)\cos(t)\\ &\leq \sin(t) + \sin(\|\mathcal{Z}\|) \leq 2\sin(t).\end{aligned}$$

This yields the following upper bound for the third summand

$$\begin{aligned}B_3 &:= \int_{2\|\mathcal{Z}\|}^{\pi-2\|\mathcal{Z}\|} \sin(\tau)^{q-1} \int_{\tau-\|\mathcal{Z}\|}^{\tau+\|\mathcal{Z}\|} \left|(v_n \circ \cos)'(t)\right| \mathrm{d}t\mathrm{d}\tau\\
&\leq 2 \int_{2\|\mathcal{Z}\|}^{\pi-2\|\mathcal{Z}\|} \int_{\tau-\|\mathcal{Z}\|}^{\tau+\|\mathcal{Z}\|} \left|(v_n \circ \cos)'(t)\right| \sin(t)^{q-1} \mathrm{d}t\mathrm{d}\tau\\
&= 2 \int_{2\|\mathcal{Z}\|}^{\pi-2\|\mathcal{Z}\|} \int_{-\|\mathcal{Z}\|}^{\|\mathcal{Z}\|} \left|v_n'(\cos(t+\tau))\right| \sin(t+\tau)^{q} \mathrm{d}t\mathrm{d}\tau\\
&= 2 \int_{-\|\mathcal{Z}\|}^{\|\mathcal{Z}\|} \int_{2\|\mathcal{Z}\|}^{\pi-2\|\mathcal{Z}\|} \left|v_n'(\cos(t+\tau))\right| \sin(t+\tau)^{q} \mathrm{d}\tau\mathrm{d}t.\end{aligned}$$

Another change of variables and enlarging the integration interval yields

$$\begin{aligned}&= 2 \int_{-\|\mathcal{Z}\|}^{\|\mathcal{Z}\|} \int_{2\|\mathcal{Z}\|+t}^{\pi-2\|\mathcal{Z}\|+t} \left|v_n'(\cos(\tau))\right| \sin(\tau)^{q} \mathrm{d}\tau\mathrm{d}t\\
&\leq 2 \int_{-\|\mathcal{Z}\|}^{\|\mathcal{Z}\|} \int_{0}^{\pi} \left|v_n'(\cos(\tau))\right| \sin(\tau)^{q} \mathrm{d}\tau\mathrm{d}t.\\
&= 4\|\mathcal{Z}\| \int_{0}^{\pi} \left|v_n'(\cos(\tau))\right| \sin(\tau)^{q} \mathrm{d}\tau\\
&\leq 4\|\mathcal{Z}\| C_q \frac{n+q^2}{\omega_{q-1}},\end{aligned}$$

where we used Lemma 4.2 for the last step. Using $(n+q^2)\|\mathcal{Z}\| < 1$, we obtain

$$\begin{aligned}&\left\|T_{\Xi,\mathcal{Z},n}(f)\right\|_{\mu_q,1}\\
&\leq \|f\|_{\mu_q,1}\,\omega_{q-1}\left(2(n+q^2)^{q+1}\|\mathcal{Z}\|^{q+1}\frac{2^{q+3}}{q\omega_{q-1}\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)}\right.\\
&\qquad\left. + 4\|\mathcal{Z}\| C_q \frac{n+q^2}{\omega_{q-1}}\right)\\
&= \|f\|_{\mu_q,1}(n+q^2)\|\mathcal{Z}\|\left(\frac{2^{q+4}}{q\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)} + 4C_q\right)\end{aligned}$$

and the proof is finished. □

We now turn to the other boundary case $p = \infty$.

Lemma 4.4. *The mapping $T_{\Xi,\mathcal{Z},n}$ is a bounded linear operator from $L_\infty(\mathbb{S}^q)$ to $L_\infty(\mathbb{S}^q)$ with norm*

$$\left\|T_{\Xi,\mathcal{Z},n}\right\|_{\infty\to\infty} \leq 4C_q(n+q^2)\|\mathcal{Z}\|.$$

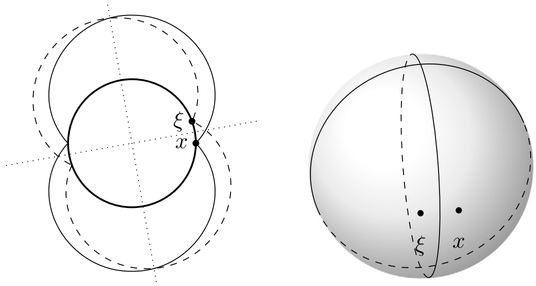

Fig. 3. For non-antipodal points x and ξ on $\mathbb{S}^q$, the sets U_1 and U_2 each take two opposite quarters of the sphere. In the left panel the dashed thin line shows the sine of the geodesic distance to ξ and the solid thin one depicts the sine of the geodesic distance to x. In the right panel, the dashed and solid thin lines show the boundary of U_1.

Proof. The triangle inequality yields

$$\begin{aligned}
&\left\|T_{\Xi,\mathcal{Z},n}(f)\right\|_{\mu_q,\infty} \\
&= \mu_q\text{-}\operatorname*{ess\,sup}_{x\in\mathbb{S}^q} \Big| \int_{\mathbb{S}^q} \sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x)(v_n(x\cdot y) - v_n(\xi\cdot y)) f(y)\mathrm{d}\mu_q(y)\Big| \\
&\le \mu_q\text{-}\operatorname*{ess\,sup}_{x\in\mathbb{S}^q} \Big| \int_{\mathbb{S}^q} \sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x)(v_n(x\cdot y) - v_n(\xi\cdot y))\mathrm{d}\mu_q(y)\Big| \;\|f\|_{\mu_q,\infty} \\
&\le \mu_q\text{-}\operatorname*{ess\,sup}_{x\in\mathbb{S}^q} \int_{\mathbb{S}^q} \sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x) \left|(v_n(x\cdot y) - v_n(\xi\cdot y))\right| \mathrm{d}\mu_q(y) \;\|f\|_{\mu_q,\infty} .
\end{aligned}$$

For μ_q-almost all $x \in \mathbb{S}^q$, there exists a unique element $\xi \in \Xi$ with $x \in Z_\xi$. For such pairs (x,ξ), the integral $\int_{\mathbb{S}^q} \sum_{\xi\in\Xi} \mathbf{1}_{Z_\xi}(x) \left|(v_n(x\cdot y) - v_n(\xi\cdot y))\right| \mathrm{d}\mu_q(y)$ reduces to

$$\int_{\mathbb{S}^q} \left|v_n(x\cdot y) - v_n(\xi\cdot y)\right| \mathrm{d}\mu_q(y) = \sum_{j=1}^{2} \int_{\mathbb{S}^q} \mathbf{1}_{U_j}(y) \left|v_n(x\cdot y) - v_n(\xi\cdot y)\right| \mathrm{d}\mu_q(y) \tag{4.4}$$

where we split $\mathbb{S}^q$ into the two sets $U_1 := \{y\in\mathbb{S}^q \;:\; \sin(d(x,y)) \le \sin(d(\xi,y))\}$ and $U_2 := \mathbb{S}^q \setminus U_1$, see Fig. 3 for an illustration.

Denoting by $y(\tau,\tilde{y}) = \cos(\tau)x + \sin(\tau)\tilde{y}$, $\tau\in[0,\pi]$, $\tilde{y}\in\{z\in\mathbb{S}^q \;:\; x\cdot z = 0\} \cong \mathbb{S}^{q-1}$ the polar coordinates of $y\in\mathbb{S}^q$ with respect to x as the north pole we obtain

$$\begin{aligned}
&\int_{\mathbb{S}^q} \mathbf{1}_{U_1}(y) \left|v_n(x\cdot y) - v_n(\xi\cdot y)\right| \mathrm{d}\mu_q(y) \\
&= \int_{\mathbb{S}^{q-1}} \int_0^\pi \mathbf{1}_{U_1}(y(\tau,\tilde{y})) \left|v_n(x\cdot y(\tau,\tilde{y})) - v_n(\xi\cdot y(\tau,\tilde{y}))\right| \sin(\tau)^{q-1}\mathrm{d}\tau\,\mathrm{d}\mu_{q-1}(\tilde{y}) \\
&= \int_{\mathbb{S}^{q-1}} \int_0^\pi \mathbf{1}_{U_1}(y(\tau,\tilde{y})) \int_{[d(x,y(\tau,\tilde{y})),d(\xi,y(\tau,\tilde{y}))]} \left|v_n'(\cos(t))\sin(t)\right| \mathrm{d}t \;\sin(\tau)^{q-1}\mathrm{d}\tau\,\mathrm{d}\mu_{q-1}(\tilde{y}).
\end{aligned}$$

As the sine function is concave on $[d(x,y(\tau,\tilde{y})), d(\xi,y(\tau,\tilde{y}))] \subseteq [0,\pi]$, it attains its minimum on the boundary, which is $d(x,y(\tau,\tilde{y})) = \tau$ in the case of $y(\tau,\tilde{y})\in U_1$, and thus $\sin(\tau) \le \sin(t)$ for all $t\in[d(x,y(\tau,\tilde{y})), d(\xi,y(\tau,\tilde{y}))]$. This leads to the upper bound

$$\begin{aligned}
&\int_{\mathbb{S}^{q-1}} \int_0^\pi \int_{[d(x,y(\tau,\tilde{y})),d(\xi,y(\tau,\tilde{y}))]} \left|v_n'(\cos(t))\sin(t)^q\right| \mathrm{d}t\mathrm{d}\tau\mathrm{d}\mu_{q-1}(\tilde{y}) \\
&\le \int_{\mathbb{S}^{q-1}} \int_0^\pi \int_0^{d(x,\xi)} \left|v_n'(\cos(t+\tau))\sin(t+\tau)^q\right| \mathrm{d}t\mathrm{d}\tau\mathrm{d}\mu_{q-1}(\tilde{y}) \\
&\le \omega_{q-1} \int_0^\pi \int_0^{d(x,\xi)} \left|v_n'(\cos(t+\tau))\sin(t+\tau)^q\right| \mathrm{d}t\mathrm{d}\tau
\end{aligned}$$

$$= \omega_{q-1} \int_0^{d(x,\xi)} \int_0^{\pi} \left| v_n'(\cos(t+\tau))\sin(t+\tau)^q \right| \mathrm{d}\tau \mathrm{d}t.$$

Utilizing the periodicity of $(v_n' \circ \cos)\sin^q$ and Lemma 4.2, we obtain

$$\begin{aligned}
&= \omega_{q-1} \int_0^{d(x,\xi)} \int_t^{\pi+t} \left| v_n'(\cos(\tau))\sin(\tau)^q \right| \mathrm{d}\tau \mathrm{d}t \\
&\le \omega_{q-1} \int_0^{d(x,\xi)} \int_0^{2\pi} \left| v_n'(\cos(\tau))\sin(\tau)^q \right| \mathrm{d}\tau \mathrm{d}t \\
&= \omega_{q-1} \cdot d(x,\xi) \cdot 2 \int_0^{\pi} \left| v_n'(\cos(\tau))\sin(\tau)^q \right| \mathrm{d}\tau \\
&\le 2d(x,\xi) C_q(n+q^2).
\end{aligned}$$

The same manipulations can be applied to the second summand in (4.4) but with ξ as the north pole. We obtain

$$\begin{aligned}
&\int_{\mathbb{S}^q} \left| v_n(x\cdot y) - v_n(\xi \cdot y) \right| \mathrm{d}\mu_q(y) \\
&= \sum_{j=1}^{2} \int_{\mathbb{S}^q} \mathbf{1}_{U_j}(y) \left| v_n(x\cdot y) - v_n(\xi \cdot y) \right| \mathrm{d}\mu_q(y) \\
&\le 4 \, \|\mathcal{Z}\| \, C_q(n+q^2).
\end{aligned}$$

This means that

$$\left\| T_{\Xi,\mathcal{Z},n}(f) \right\|_{\mu_q,\infty} \le \|f\|_{\mu_q,\infty} \, 4C_q(n+q^2) \, \|\mathcal{Z}\|$$

which finishes the proof. □

Remark 4.5. A modification of the technique used for the case $p = 1$ in the second step of the preceding proof can also be applied to the case $p = \infty$. In contrast to the above strategy we would generate an additional summand of order $(n \, \|\mathcal{Z}\|)^{q+1}$.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We first show that

$$\left| \|P\|_{\mu_q,p} - \|P\|_{\mu_q(\Xi,\mathcal{Z}),p} \right| \le \left\| T_{\Xi,\mathcal{Z},n}(P) \right\|_{\mu_q,p} \tag{4.5}$$

holds for all $1 \le p \le \infty$ and every $P \in \Pi_n^q$. For $1 \le p < \infty$, the triangle inequality, the reproducing property of v_n, and Hölder's inequality give

$$\begin{aligned}
&\left| \|P\|_{\mu_q,p} - \|P\|_{\mu_q(\Xi,\mathcal{Z}),p} \right| \\
&= \left| \left(\sum_{\xi\in\Xi} \int_{Z_\xi} |P(x)|^p \, \mathrm{d}\mu_q(x) \right)^{\frac{1}{p}} - \left(\sum_{\xi\in\Xi} \int_{Z_\xi} |P(\xi)|^p \, \mathrm{d}\mu_q(x) \right)^{\frac{1}{p}} \right| \\
&\le \left(\sum_{\xi\in\Xi} \int_{Z_\xi} |P(x) - P(\xi)|^p \, \mathrm{d}\mu_q(x) \right)^{\frac{1}{p}} \\
&= \left(\sum_{\xi\in\Xi} \int_{Z_\xi} \left| \int_{\mathbb{S}^q} v_n(x\cdot y) P(y) \, \mathrm{d}\mu_q(y) - \int_{\mathbb{S}^q} v_n(\xi\cdot y) P(y) \, \mathrm{d}\mu_q(y) \right|^p \mathrm{d}\mu_q(x) \right)^{\frac{1}{p}}
\end{aligned}$$

$$= \left(\sum_{\xi \in \Xi} \int_{Z_\xi} \left| \int_{\mathbb{S}^q} (v_n(x \cdot y) - v_n(\xi \cdot y)) P(y) \mathrm{d}\mu_q(y) \right|^p \mathrm{d}\mu_q(x) \right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{S}^q} \left| \int_{\mathbb{S}^q} \sum_{\xi \in \Xi} \mathbf{1}_{Z_\xi}(x) (v_n(x \cdot y) - v_n(\xi \cdot y)) P(y) \, \mathrm{d}\mu_q(y) \right|^p \mathrm{d}\mu_q(x) \right)^{\frac{1}{p}}$$

$$= \left\| T_{\Xi,\mathcal{Z},n}(P) \right\|_{\mu_q,p}.$$

If $p = \infty$, the same arguments yield

$$\left| \|P\|_{\mu_q,\infty} - \|P\|_{\mu_q(\Xi,\mathcal{Z}),\infty} \right|$$

$$\leq \sup_{\xi \in \Xi} \mu_q\text{-} \operatorname*{ess\,sup}_{x \in Z_\xi} |P(x) - P(\xi)|$$

$$\leq \sup_{\xi \in \Xi} \mu_q\text{-} \operatorname*{ess\,sup}_{x \in Z_\xi} \left| \int_{\mathbb{S}^q} (v_n(x \cdot y) - v_n(\xi \cdot y)) P(y) \mathrm{d}\mu_q(y) \right|$$

$$= \mu_q\text{-} \operatorname*{ess\,sup}_{x \in \mathbb{S}^q} \left| \int_{\mathbb{S}^q} \sum_{\xi \in \Xi} \mathbf{1}_{Z_\xi}(x) (v_n(x \cdot y) - v_n(\xi \cdot y)) P(y) \mathrm{d}\mu_q(y) \right|$$

$$= \left\| T_{\Xi,\mathcal{Z},n}(P) \right\|_{\mu_q,1}$$

which proves (4.5).

In order to show that the linear operator $T_{\Xi,\mathcal{Z},n} : L_p(\mathbb{S}^q, \mu_q) \to L_p(\mathbb{S}^q, \mu_q)$ is bounded for every $p \in [1, \infty]$ with operator norm less or equal to η we first note that this follows for $p = 1$ and $p = \infty$ from Lemmas 4.3 and 4.4 and $6C_q(n + q^2) \|\mathcal{Z}\| \leq \eta$. (Note that $\frac{2^{q+4}}{q\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)} + 4C_q < 6C_q$ for $q \in \mathbb{N}$.) For $1 < p < \infty$, the statement follows by the Riesz–Thorin interpolation theorem. We conclude that

$$\left| \|P\|_{\mu_q,p} - \|P\|_{\mu_q(\Xi,\mathcal{Z}),p} \right| \leq \eta \, \|P\|_{\mu_q,p}$$

for all $P \in \Pi_n^q$ whenever $6C_q(n + q^2) \|\mathcal{Z}\| \leq \eta$. This is equivalent to the assertion. □

The condition $6C_q(n+q^2) \|\mathcal{Z}\| \leq \eta$ appearing in Theorem 4.1 gives a lower bound on the number N of samples through volumetric arguments of the partition. Namely, if $\|\mathcal{Z}\| \leq \frac{\eta}{6C_q(n+q^2)} =: r$, then $\mu(Z) \leq \omega_{q-1} \int_0^r \sin(t)^{q-1} \mathrm{d}t$ for each $Z \in \mathcal{Z}$, and as $\mathcal{Z}$ is a partition of $\mathbb{S}^q$, the cardinality of $\mathcal{Z}$ is

$$N \geq \frac{\omega_q}{\left(\omega_{q-1} \int_0^r \sin(t)^{q-1} \mathrm{d}t\right)} \gtrsim_q r^{-q}.$$

As a corollary, we obtain a seemingly partition-free variant of Theorem 4.1 with the upper bound on the partition norm $\|\mathcal{Z}\|$ replaced by an upper bound on the mesh norm δ_Ξ. It relies on the construction of a partition $\mathcal{Z}$ from the sample set Ξ such that the partition norm and the mesh norm satisfy a two-sided inequality, and hiding the partition in the weights of the discretized norm.

Corollary 4.6. *Let $n, q \in \mathbb{N}$ and $\eta \in (0, 1)$. Let further $\Xi \subset \mathbb{S}^q$ be a finite set satisfying*

$$48C_q q\sqrt{2q(q+1)}(n + q^2)\delta_\Xi \leq \eta.$$

Then there exist non-negative numbers a_ξ, $\xi \in \Xi$, such that

$$(1 - \eta) \|P\|_{\mu_q,p} \leq \left(\sum_{\xi \in \Xi} a_\xi \, |P(\xi)|^p \right)^{\frac{1}{p}} \leq (1 + \eta) \|P\|_{\mu_q,p}$$

for all $p \in [1, \infty]$ and $P \in \Pi_n^q$.

Proof. Let $\Xi = \{\xi_1, \dots, \xi_N\} \subseteq \mathbb{S}^q$. Then [27, Proposition 3.2] gives a partition $\mathcal{Z} = \{Z_1, \dots, Z_M\}$ of $\mathbb{S}^q$ for some $M \leq N$ such that there exists an M-element subset Ξ_0 of Ξ for which the pair $(\Xi_0, \mathcal{Z})$ is compatible, and the inequality

$$\delta_\Xi \leq \|\mathcal{Z}\| \leq 8q\sqrt{2q(q+1)}\delta_\Xi$$

is satisfied. We plug this into $48C_q q\sqrt{2q(q+1)}(n+q^2)\delta_\Xi \leq \eta$ to obtain $6C_q \|\mathcal{Z}\|(n+q^2) \leq \eta$. Set

$$a_\xi = \begin{cases} \mu_q(Z_\xi) & \text{if } \xi \in \Xi_0, \\ 0 & \text{else,} \end{cases}$$

and apply Theorem 4.1. □

Via equal-area partitions of the sphere, we can also get an equal-weight version of Theorem 4.1.

Corollary 4.7. *Let $n, q \in \mathbb{N}$ and $\eta \in (0,1)$. For $\alpha_q := 8\left(\frac{\omega_q q}{\omega_{q-1}}\right)^{\frac{1}{q}}$ and*

$$N \geq \left(\frac{6C_q(n+q^2)\alpha_q}{\eta}\right)^q,$$

there exists a finite subset $\Xi = \{\xi_1, \dots, \xi_N\}$ of $\mathbb{S}^q$ with

$$(1-\eta)\|P\|_{\mu_q,p} \leq \left(\frac{\omega_q}{N}\sum_{j=1}^N \left|P(\xi_j)\right|^p\right)^{\frac{1}{p}} \leq (1+\eta)\|P\|_{\mu_q,p}$$

for all $p \in [1,\infty]$ and $P \in \Pi_n^q$.

Proof. Using [22, Theorem 3.1.3], there exists a number α_q (which may only depend on q) and a partition $\mathcal{Z} = \{Z_1, \dots, Z_N\}$ of $\mathbb{S}^q$ such that $\mu_q(Z_j) = \frac{\omega_q}{N}$ and $\|\mathcal{Z}\| \leq \alpha_q N^{-\frac{1}{q}}$. According to [5, Теорема 6], one may have $\alpha = 8\left(\frac{\omega_q q}{\omega_{q-1}}\right)^{\frac{1}{q}}$. Plugging this into $N \geq \left(\frac{6C_q(n+q^2)\alpha_q}{\eta}\right)^q$, we get $6C_q(n+q^2)\|\mathcal{Z}\| \leq \eta$. For assembling the set Ξ, just take one interior point of each $Z \in \mathcal{Z}$. It remains to apply Theorem 4.1. □

5. How good are random points?

In the previous section we have seen that the performance of sample points (regarding the distortion parameter $\eta > 0$) improves with smaller partition norm $\|\mathcal{Z}\|$ of the corresponding partition, or, differently, with a smaller mesh norm δ_Ξ. By [29, Corollary 3.3], the expected mesh norm of N points on the sphere $\mathbb{S}^q$ drawn independently and identically distributed according to σ_q is asymptotically equivalent to $\left(\frac{N}{\log(N)}\right)^{-1/q}$. In the present section, we show that, when drawing enough points, we obtain a good Marcinkiewicz–Zygmund inequality for all $1 \leq p \leq \infty$ simultaneously, with the parameter η depending on $\left(\frac{N}{\log(N)}\right)^{-1/q}$. Compared to the consideration in Section 3 we obtain a worse scaling of the number of points with respect to η. Note, that in Section 3 we considered only $p = 2$ and observed a scaling of η^{-2} independently of the dimension q and explicit constants at the price of an additional logarithm in the dimension.

The main result in this section (Theorem 5.2) utilizes insights about a classical problem of probability theory: the coupon collector problem. At each time step, the eponymous coupon collector receives a coupon, chosen at random among M different types. Unsurprisingly, the more coupons the collector receives, the higher the probability that the collection contains each type at least once. The time step after which the collector possesses each type at least once can be modeled as a random variable $T_M : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is some probability space, see [11, p. 36]. As we do not know the number of draws a priori, we should therefore start with an infinite product of the uniform probability space over the set $\{1, \dots, M\}$ of coupon types. To circumvent this, we raise the probability space to a sufficiently high power t, and model the event of not having all M types of coupons after t draws directly as a subset of a *finite* probability space.

Proposition 5.1. *Let $\varepsilon \in (0,1)$, $M, t \in \mathbb{N}$, and $t > M\log\left(\frac{M}{\varepsilon}\right)$. On the finite set $\Omega := \{1, \dots, M\}^t$, a probability measure $\mathbb{P}$ is given by the t-fold product measure on the uniform probability measure on $\{1, \dots, M\}$. Set*

$$A_{t,M} := \left\{(x_1, \dots, x_t) \in \Omega \;:\; \{x_1, \dots, x_t\} = \{1, \dots, M\}\right\}.$$

Then $\mathbb{P}(\Omega \setminus A_{t,M}) < \varepsilon$.

Proof. The assumption $t > M\log\left(\frac{M}{\varepsilon}\right)$ is equivalent to $M\exp(-\frac{t}{M}) < \varepsilon$. Since $(1+a)^t \leq \exp(at)$ for all $a, t \in \mathbb{R}$ with $1 + a \geq 0$, we have $M\left(1 - \frac{1}{M}\right)^t \leq M\exp(-\frac{t}{M}) < \varepsilon$. Taking [11, p. 36] into account, we have $\mathbb{P}(\Omega \setminus A_{t,M}) \leq M\left(1 - \frac{1}{M}\right)^t < \varepsilon$. □

Now a probabilistic L_p-version of the Marcinkiewicz–Zygmund inequality can be given as follows.

Theorem 5.2. *Let $n, q \in \mathbb{N}$, $\eta \in (0,1)$, and set $\alpha_q := 8\left(\frac{\omega_q q}{\omega_{q-1}}\right)^{\frac{1}{q}}$. Choose $N \in \mathbb{N}$ large enough such that*

$$6C_q\alpha_q\left(\frac{N}{4\log(N)}\right)^{-\frac{1}{q}}(n+q^2) < \eta. \tag{5.1}$$

Draw points $\xi_1, \ldots, \xi_N \in \mathbb{S}^q$ independently and identically distributed according to σ_q. Then with probability $\geq 1 - \frac{1}{N}$ with respect to the product measure $\mathbb{P} = \sigma_q^{\otimes N}$, there exists weights $w_1, \ldots, w_N > 0$ such that $\sum_{j=1}^N w_j = 1$ and

$$(1-\eta)\,\|P\|_{\sigma_q,p} \leq \left(\sum_{j=1}^N w_j \left|P(\xi_j)\right|^p\right)^{\frac{1}{p}} \leq (1+\eta)\,\|P\|_{\sigma_q,p}$$

for all $p \in [1,\infty]$ and all $P \in \Pi_n^q$.

Proof. Let $M := \left\lfloor \frac{N}{2\log(N)} \right\rfloor$. From $M \leq \frac{N}{2\log(N)}$, we infer $N > 2M\log(N) > 2M\log(M)$. Furthermore, we have $\lfloor z \rfloor \geq \frac{1}{2}z$ for all $z \in \mathbb{R}$ with $z > 1$. For $z = \frac{N}{2\log(N)}$, we obtain $M = \left\lfloor \frac{N}{2\log(N)} \right\rfloor \geq \frac{N}{4\log(N)}$. Thus Equation (5.1) implies

$$6C_q\alpha_q M^{-\frac{1}{q}}(n+q^2) \leq \eta.$$

Using [22, Theorem 3.1.3] and [5, Теорема 2.3], there exists a partition $\mathcal{Z} = \{Z_1, \ldots, Z_M\}$ of $\mathbb{S}^q$ such that $\sigma_q(Z_j) = \frac{1}{M}$ and $\|\mathcal{Z}\| \leq \alpha_q M^{-\frac{1}{q}}$. It follows that

$$6C_q(n+q^2)\,\|\mathcal{Z}\| < \eta.$$

Thus the conditions of Theorem 4.1 are met if there is an M-element subset $\Xi \subseteq \{\xi_1, \ldots, \xi_N\}$ such that $(\Xi, \mathcal{Z})$ is compatible. For this, we use Proposition 5.1. It implies that after drawing $N > 2M\log(M)$ points $\xi_1, \ldots, \xi_N \in \mathbb{S}^q$ independently and identically distributed according to σ_q, the probability that each patch $Z_j \in \mathcal{Z}$ contains a non-zero number m_j of the points $\xi_1, \ldots, \xi_N$ in its interior is $\geq 1 - \frac{1}{N}$. Collect one of the points in each patch in a set Ξ, and apply Theorem 4.1. This implies that, with probability $\geq 1 - \frac{1}{N}$, we have

$$(1-\eta)\,\|P\|_{\sigma_q,p} \leq \left(\frac{1}{M}\sum_{\xi\in\Xi}|P(\xi)|^p\right)^{\frac{1}{p}} \leq (1+\eta)\,\|P\|_{\sigma_q,p} \tag{5.2}$$

for all $p \in [1,\infty]$ and all $P \in \Pi_n^q$. This result is independent of which point from a given patch Z_j is put into the set Ξ. In particular, (5.2) holds true if $\xi \in Z_j$ is selected such that $|P(\xi)|^p$ is smallest possible or largest possible. Therefore (5.2) will also hold if we replace the contribution $|P(\xi)|^p$ from each patch by the average $\frac{1}{m_j}\sum_{\xi\in Z_j\cap\{\xi_1,\ldots,\xi_N\}}|P(\xi)|^p$, i.e.,

$$(1-\eta)\,\|P\|_{\sigma_q,p} \leq \left(\frac{1}{M}\sum_{j=1}^M \frac{1}{m_j}\sum_{\xi\in Z_j\cap\{\xi_1,\ldots,\xi_N\}}|P(\xi)|^p\right)^{\frac{1}{p}} \leq (1+\eta)\,\|P\|_{\sigma_q,p}$$

or, equivalently,

$$(1-\eta)\,\|P\|_{\sigma_q,p} \leq \left(\sum_{j=1}^M \sum_{\xi\in Z_j\cap\{\xi_1,\ldots,\xi_N\}}\frac{1}{Mm_j}|P(\xi)|^p\right)^{\frac{1}{p}} \leq (1+\eta)\,\|P\|_{\sigma_q,p}.$$

Now set $w_j := \frac{1}{Mm_j} > 0$ and observe that

$$\sum_{j=1}^N w_j = \sum_{j=1}^M \sum_{\xi\in Z_j\cap\{\xi_1,\ldots,\xi_N\}}\frac{1}{Mm_j} = \sum_{j=1}^M \frac{m_j}{Mm_j} = 1. \quad \square$$

Data availability

Data will be made available on request.

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