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Near-Circularity in Capacity and Maximally Convergent Polynomials

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Abstract

If f is a power series with radius R of convergence, $R > 1$, it is well-known that the method of Carathéodory–Fejér constructs polynomial approximations of f on the closed unit disk which show the typical phenomenon of near-circularity on the unit circle. Let E be compact and connected and let f be holomorphic on E. If $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of polynomials converging maximally to f on E, it is shown that the modulus of the error functions $f - p_n$ is asymptotically constant in capacity on level lines of the Green's function $g_\Omega(z, \infty)$ of the complement Ω of E in $\overline{\mathbb{C}}$ with pole at infinity, thereby reflecting a type of near-circularity, but without gaining knowledge of the winding numbers of the error curves with respect to the point 0.

Keywords Complex approximation · Near-circularity · Maximal convergence · Capacity · Equilibrium measure

Mathematics Subject Classification 30C85 · 30E10 · 31A05 · 31A15 · 41A10

1 Introduction: Carathéodory–Fejér Approximation

Let $f(z)$ be a power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

Dedicated to the memory of Richard Varga

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with radius of convergence R, $1 < R < \infty$, and let $\Lambda \subset \mathbb{N}$ such that

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\in\Lambda, n\to\infty} \sqrt[n]{|a_n|}. \tag{1.1}$$

If $\gamma > 1/R$, we may assume that for $n \in \Lambda$

$$a_{n+1} \neq 0 \quad \text{and} \quad \left|a_{n+1+j}\right| \leq |a_{n+1}|\, \gamma^j \text{ for } j = 1, 2, \dots. \tag{1.2}$$

Let $\mathcal{P}_n$ denote the collection of all algebraic polynomials of degree at most n. Then Carathéodory and Fejér considered the following procedure to construct near-best uniform approximations of f on the closed unit disk: Let $m > 0$ be fixed and define

$$f_{n,m}(z) = \sum_{k=n+1}^{n+m+1} a_k z^k,$$

then there exists a unique function

$$f^*_{n,m}(z) = \sum_{k=-\infty}^{n} c_k z^k + f_{n,m}(z) \tag{1.3}$$

that is analytic on $|z| > 1$ and continuous on $|z| \geq 1$ such that $\left\| f^*_{n,m} \right\|_{|z|=1}$ is minimal among all extensions of $f_{n,m}(z)$ of type (1.3) (cf. Goluzin [3, Ch. XI, §7], Trefethen [10]). Moreover, $f^*_{n,m}(z)$ can be expressed as

$$f^*_{n,m}(z) = \lambda\, z^{n+m+1} \prod_{i=1}^{\nu} \left(\frac{1 - \overline{\alpha}_i z}{z - \alpha_i} \right), \tag{1.4}$$

with $\lambda \in \mathbb{C}$, $|\lambda| \geq |a_{n+1}|$ and $\nu \leq m$ poles α_i in the open unit disk. If γ is sufficiently small (for example $\gamma < (\sqrt{13} - 1)/6 \approx 0.43426\dots$), then Hollenhorst [4, 5] proved that the function $f^*_{n,m}(z)$ has exactly m poles in the interior of the unit disk and $f^*_{n,m}(z)$ describes on $|z| = 1$ exactly $n + 1$ circles. Moreover, let

$$p_{n,m}(z) = \sum_{k=0}^{n} (a_k - c_k) z^k$$

be the **CF-approximation** (Carathéodory–Fejér), then the Blaschke product in (1.4) induces that

$$f(z) - p_{n,m}(z) = f^*_{n,m}(z) + R_{n,m}(z).$$

is nearly circular on $|z| = 1$ for $n \to \infty, n \in \Lambda$, by using asymptotic estimates of $\left\| R_{n,m} \right\|_{|z|=1}$ (cf. Hollenhorst [4, 5], Trefethen [1, 10]). Trefethen was the first to use the notion *near-circularity* for this behavior, namely, for sufficiently small γ and sufficiently big m (for example the standard choice is $m = n + 1$) the results of Hollenhorst ([4], [5]) and Trefethen [10] lead to

$$|\lambda| - O(\gamma^n) \leq \min_{|z|=1} |f(z) - p_{n,m}(z)| \leq \left\| f - p_{n,m} \right\|_{|z|=1} \leq |\lambda| + O(\gamma^n), \tag{1.5}$$

as $n \in \Lambda, n \to \infty$, which reflects the near-circularity of the error curve $\left(f - p_{n,m}\right)(z)$ on the unit circle and moreover,

$$|a_{n+1}| \leq |\lambda| \leq |a_{n+1}| \, (1 + O(1)) \quad \text{as } n \in \Lambda, n \to \infty.$$

Keeping in mind (1.1) and (1.2), we get the coarser inequalities

$$\frac{1}{R} \leq \limsup_{n\in\Lambda, n\to\infty} \min_{|z|=1} |f(z) - p_{n,m}(z)|^{1/n} \leq \limsup_{n\in\Lambda, n\to\infty} \left\| f - p_{n,m} \right\|_{|z|=1}^{1/n} \leq \frac{1}{R}.$$

Hence, in the above inequalities the equality sign always holds and therefore

$$\frac{1}{R} = \lim_{n\in\Lambda, n\to\infty} \min_{|z|=1} |f(z) - p_{n,m}(z)|^{1/n} = \lim_{n\in\Lambda, n\to\infty} \left\| f - p_{n,m} \right\|_{|z|=1}^{1/n} = \frac{1}{R}. \tag{1.6}$$

This is now the starting point of our investigations.

Let E be compact and connected in $\mathbb{C}$ with connected complement $\Omega = \overline{\mathbb{C}} \backslash E$ and let $g_\Omega(z, \infty)$ denote the Green's function of Ω with pole at ∞, and let Γ_σ denote a level line of $g_\Omega(z, \infty)$ and let f be holomorphic inside $\Gamma_{\rho(f)}$, where $\rho(f)$ is the maximal parameter of holomorphy of f. Furthermore, if $\{p_n\}_{n\in\mathbb{N}}$ is a polynomial sequence converging maximally to f, then the objective of this paper is to find $\Lambda \subset \mathbb{N}$ and compact sets $K_n \subset \Gamma_\sigma, n \in \Lambda$, such that analogous to (1.6) we have for $1 < \sigma < \rho(f)$

$$\frac{\sigma}{\rho(f)} = \lim_{n\in\Lambda, n\to\infty} \inf_{z\in\Gamma_\sigma\backslash K_n} |f(z) - p_n(z)|^{1/n} = \lim_{n\in\Lambda, n\to\infty} \|f - p_n\|_{\Gamma_\sigma}^{1/n} = \frac{\sigma}{\rho(f)},$$

where the capacity of the exceptional set K_n tends to 0 as $n \in \Lambda, n \to \infty$.

2 Main Results

For $B \subset \mathbb{C}$, we denote by B° the set of interior points of B, by $\overline{B}$ its closure and by ∂B the boundary of B and we use $\|\cdot\|_B$ for the supremum norm over B. Let $\mathcal{A}(B)$ be the class of functions that are holomorphic (i.e. analytic and single-valued) in a neighborhood of B.

Let K be a compact subset of the complex plane $\mathbb{C}$ and let $\mathcal{M}(K)$ be the collection of all probability measures supported on K. Then the logarithmic potential of $\mu \in \mathcal{M}(K)$

is defined by

$$U^{\mu}(z) = \int \log \frac{1}{|z-t|} \, d\mu(t)$$

and the logarithmic energy $I(\mu)$ by

$$I(\mu) := \int \int \log \frac{1}{|z-t|} \, d\mu(t) \, d\mu(z) = \int U^{\mu}(z) \, d\mu(z).$$

Let

$$V(K) := \inf\{I(\mu) : \mu \in \mathcal{M}(K)\},$$

then $V(K)$ is either finite or $V(K) = +\infty$. The quantity

$$\operatorname{cap} K = e^{-V(K)}$$

is called the *logarithmic capacity* or *capacity* of K.

Let K be compact in the complex plane $\mathbb{C}$ with $\operatorname{cap} K > 0$ and connected complement $\Omega(K) = \overline{\mathbb{C}} \setminus K$ in the extended plane $\overline{\mathbb{C}}$. We define by $g_{\Omega(K)}(z, \infty)$ the Green's function of $\Omega(K)$ with pole at ∞, i.e.,

(i) $g_{\Omega(K)}(z, \infty)$ is positive and harmonic in $\Omega(K) \setminus \{\infty\}$,
(ii) $\lim_{|z| \to \infty} \left(g_{\Omega(K)}(z, \infty) - \log |z| \right) = -\log \operatorname{cap}$ K,
(iii) $\lim_{\zeta \in \Omega(K),\, \zeta \to z} g_{\Omega(K)}(\zeta, \infty) = 0$ for quasi-every z $\in \partial \Omega(K)$.

Since $\operatorname{cap} K > 0$, the Green's function $g_{\Omega(K)}(z, \infty)$ is unique and there exists a unique measure $\mu_K \in \mathcal{M}(K)$ such that

$$I(\mu_K) = -\log \operatorname{cap} K = V(K)$$

and we have

$$U^{\mu_K}(z) = -g_{\Omega(K)}(z, \infty) - \log \operatorname{cap} K, \quad z \in \Omega(K).$$

Here, μ_K is called the *equilibrium measure* of K.

In the following, let E be a fixed compact and connected set with $\operatorname{cap} E > 0$ and connected complement $\Omega := \overline{\mathbb{C}} \setminus E$. We denote by $g_{\Omega}(z, \infty)$ the Green's function of the region Ω with pole at ∞. Since E is connected and $\operatorname{cap} E > 0$, the Green's function $g_{\Omega}(z, \infty)$ is unique and $g_{\Omega}(\zeta, \infty)$ tends to 0 as $\zeta \in \Omega$ tends to $z \in \partial\Omega$ for quasi-every $z \in \partial\Omega$.

Now, let us define for $\sigma > 1$ the *Green domains* E_{σ} by

$$E_{\sigma} := \{z \in \Omega : g_{\Omega}(z, \infty) < \log \sigma\} \cup E$$

with boundary $\Gamma_{\sigma} := \partial E_{\sigma}$. Hence, the Green domains E_{σ} are Jordan regions for any $\sigma > 1$.

If $f \in \mathcal{A}(E)$, then there exist $\rho > 1$ and polynomials $p_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} \le \frac{1}{\rho},$$

due to a result of Walsh [13]. If $f \in \mathcal{A}(E)$ is not an entire function and if $\rho(f)$ denotes the maximal parameter $\rho > 1$, $1 < \rho < \infty$, such that f is holomorphic on E_ρ, then there exist polynomials $p_n \in \mathcal{P}_n$ such that

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} = \frac{1}{\rho(f)}.$$

Such a sequence $p_n \in \mathcal{P}_n$ is called *maximally convergent*. Moreover, Walsh [13, (§4.7, Thm. 7, Thm. 8 and its Cor., pp. 79–81)] proved that for such maximally convergent polynomials

$$\limsup_{n\to\infty} \|f - p_n\|_{\Gamma_\sigma}^{1/n} = \frac{\sigma}{\rho(f)}, \quad 1 < \sigma < \rho(f) < \infty. \tag{2.1}$$

For $z \in E_{\rho(f)} \backslash E$ we define the functions

$$F_n(z) := \frac{1}{n} \log |f(z) - p_n(z)| - g_\Omega(z, \infty) + \log \rho(f), \tag{2.2}$$

which are subharmonic and upper semicontinuous in $E_{\rho(f)} \backslash E$ and harmonic outside the zeros of $f - p_n$. Then Walsh [12] has used for results of type (2.1) more generally the notion of *exact harmonic majorant*, namely that the sequence $F_n(z)$, $n \in \mathbb{N}$, of subharmonic functions has on the region $E_{\rho(f)} \backslash E$ the zero function as exact harmonic majorant, i.e.,

$$\limsup_{n\to\infty} \max_{z\in S} F_n(z) = 0$$

for any continuum S in $E_{\rho(f)} \backslash E$, S not a single point.

If S is a compact set in $E_{\rho(f)} \backslash E$ and $\varepsilon > 0$, we define

$$K_n(S; \varepsilon) := \{z \in S : F_n(z) \le -\varepsilon\}\,. \tag{2.3}$$

and introduce for $1 < \kappa_1 \le \kappa_2 < \infty$ the annulus

$$D_{\kappa_1,\kappa_2} := \overline{E}_{\kappa_2} \backslash E_{\kappa_1}$$

between the level lines Γ_{κ_2} and Γ_{κ_1} of the Green's function $g_\Omega(z, \infty)$.

Then our main result is the following

Theorem *Let E be compact and connected, $f \in \mathcal{A}(E)$ with maximal parameter $\rho(f)$ of holomorphy, $1 < \sigma_1 \le \sigma_2 < \rho(f) < \infty$, and let $\{p_n\}_{n\in\mathbb{N}}$ be maximally convergent to f on E. Then the compact sets $K_n(D_{\sigma_1,\sigma_2}; \varepsilon)$ satisfy*

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \operatorname{cap} K_n(D_{\sigma_1,\sigma_2}; \varepsilon) = 0 \tag{2.4}$$

so that

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \inf_{z\in D_{\sigma_1,\sigma_2}\setminus K_n(D_{\sigma_1,\sigma_2};\varepsilon)} F_n(z) = \limsup_{n\to\infty} \max_{z\in D_{\sigma_1,\sigma_2}} F_n(z) = 0. \tag{2.5}$$

Remark (2.5) implies that there exists $\Lambda \subset \mathbb{N}$ and a sequence $\{\varepsilon_n\}_{n\in\Lambda}$, $\varepsilon_n > 0$, with $\lim_{n\in\Lambda,n\to\infty} \varepsilon_n = 0$ such that the compact sets $K_n(D_{\sigma_1,\sigma_2}; \varepsilon_n)$ satisfy $\operatorname{cap} K_n(D_{\sigma_1,\sigma_2}; \varepsilon_n) \le \varepsilon_n$ and for $n \in \Lambda$

$$\begin{aligned} e^{-\varepsilon_n} &\le \inf_{z\in D_{\sigma_1,\sigma_2}\setminus K_n(D_{\sigma_1,\sigma_2};\varepsilon_n)} \left(\frac{\rho(f)}{e^{g_\Omega(z,\infty)}} |f(z) - p_n(z)|^{1/n} \right) \\ &\le \max_{z\in D_{\sigma_1,\sigma_2}} \left(\frac{\rho(f)}{e^{g_\Omega(z,\infty)}} |f(z) - p_n(z)|^{1/n} \right) \le e^{\varepsilon_n}. \end{aligned}$$

We want to connect the theorem with the phenomenon of near-circularity of Carathéodory–Féjer approximations, described in (1.5), resp. (1.6).

Corollary 1 *There exist $\Lambda \subset \mathbb{N}$ and a sequence*

$$\{\varepsilon_n\}_{n\in\Lambda} \text{ with } \lim_{n\in\Lambda,n\to\infty} \varepsilon_n = 0$$

such that for any σ, $1 < \sigma_1 \le \sigma \le \sigma_2 < \rho(f)$, the compact sets

$$K_n(\Gamma_\sigma; \varepsilon_n) = \Gamma_\sigma \cap K_n(D_{\sigma_1,\sigma_2}; \varepsilon_n)$$

satisfy $\operatorname{cap} K_n(\Gamma_\sigma; \varepsilon_n) \le \varepsilon_n$ *for $n \in \Lambda$ and moreover,*

$$\frac{\sigma}{\rho(f)} e^{-\varepsilon_n} \le \inf_{z\in\Gamma_\sigma\setminus K_n(\Gamma_\sigma;\varepsilon_n)} |f(z) - p_n(z)|^{1/n} \le \|f - p_n\|_{\Gamma_\sigma}^{1/n} \le \frac{\sigma}{\rho(f)} e^{\varepsilon_n}.$$

Corollary 2 *Let $1 < \sigma < \rho(f)$. Then there exist $\Lambda \subset \mathbb{N}$ and a sequence*

$$\{\varepsilon_n\}_{n\in\Lambda}, \ \varepsilon_n > 0, \ \lim_{n\in\Lambda,n\to\infty} \varepsilon_n = 0,$$

together with a sequence

$$\{\sigma_n\}_{n\in\Lambda}, \ 1 < \sigma_n < \rho(f), \ \lim_{n\in\Lambda,n\to\infty} \sigma_n = \sigma,$$

such that for $n \in \Lambda$

$$\frac{\sigma_n}{\rho(f)} e^{-\varepsilon_n} \le \min_{z \in \Gamma_{\sigma_n}} |f(z) - p_n(z)|^{1/n} \le \|f - p_n\|_{\Gamma_{\sigma_n}}^{1/n} \le \frac{\sigma_n}{\rho(f)} e^{\varepsilon_n}.$$

3 Proof of the Theorem

Let us assume that the theorem is false, i.e.,

$$\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \operatorname{cap} K_n(D_{\sigma_1,\sigma_2}; \varepsilon) > 0. \tag{3.1}$$

Then our final goal will be to prove for some $\theta,\ 0 < \theta < 1$, and $\tau,\ 1 < \tau < \rho(f)$,

$$\|f - p_n\|_{\Gamma_\tau} \le \left(\frac{\theta\tau}{\rho(f)}\right)^n \quad \text{for all sufficiently big n.}$$

This would imply that

$$\|p_{n+1} - p_n\|_{\Gamma_\tau} \le 2 \left(\frac{\theta\tau}{\rho(f)}\right)^n$$

and finally the telescoping series

$$f = \sum_{n=0}^{\infty} (p_{n+1} - p_n)$$

shows, using the Bernstein–Walsh Lemma (cf. [13, §4.5, Thm. 5)]), that f is holomorphic in a neighborhood of $\overline{E}_{\rho(f)}$, contradicting the definition of $\rho(f)$.

Starting from the definition in (2.3), we note that for $\varepsilon < \varepsilon'$ we obtain

$$K_n(D_{\sigma_1,\sigma_2}; \varepsilon') \subset K_n(D_{\sigma_1,\sigma_2}; \varepsilon).$$

Therefore the function

$$h(\varepsilon) := \liminf_{n \to \infty} \operatorname{cap} K_n(D_{\sigma_1,\sigma_2}; \varepsilon)$$

is monotonically decreasing with $\epsilon,\ \varepsilon > 0$. Hence (3.1) implies that there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that

$$h(\varepsilon) \ge 2\,\delta \quad \text{for all } 0 < \varepsilon \le \varepsilon_0.$$

Due to the definition of $h(\varepsilon_0)$, there exists $n_0 = n_0(\varepsilon_0)$ such that

$$\operatorname{cap} K_n(D_{\sigma_1,\sigma_2}; \varepsilon) \ge \operatorname{cap} K_n(D_{\sigma_1,\sigma_2}; \varepsilon_0) \ge \delta > 0 \tag{3.2}$$

for all $n \geq n_0(\varepsilon_0)$ and $0 < \varepsilon \leq \varepsilon_0$.

Next, let us introduce the conformal mapping

$$\Phi : \Omega = \overline{\mathbb{C}} \backslash E \longrightarrow \{z : |z| > 1\}, \tag{3.3}$$

normalized by $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. In Ω we define the subsets

$$\Omega^+ := \{z \in \Omega : \operatorname{Im}(\Phi(z)) \geq 0\},$$
$$\Omega^- := \{z \in \Omega : \operatorname{Im}(\Phi(z)) \leq 0\}.$$

Let us define for any compact set $K \subset E_{\rho(f)} \backslash E$

$$K^+ := \left\{z \in K : z \in \Omega^+\right\}, \tag{3.4}$$
$$K^- := \left\{z \in K : z \in \Omega^-\right\}. \tag{3.5}$$

Then K^+ and K^- are compact sets and, applied to $K = K_n := K_n(D_{\sigma_1,\sigma_2}; \varepsilon)$, we obtain

$$K_n := K_n(D_{\sigma_1,\sigma_2}; \varepsilon) = K_n^+ \cup K_n^-.$$

According to a Theorem of Nevanlinna [6] (cf. [7, Thm. 11.4] or [8, Thm. 5.1.4]) we have

$$\frac{1}{\log \frac{d}{\operatorname{cap} K_n}} \leq \frac{1}{\log \frac{d}{\operatorname{cap} K_n^+}} + \frac{1}{\log \frac{d}{\operatorname{cap} K_n^-}}, \tag{3.6}$$

where d is the diameter of $\overline{E}_{\rho(f)}$. Let us define

$$\widetilde{K}_n := \begin{cases} K_n^+ & \text{if } \operatorname{cap} K_n^+ \geq \operatorname{cap} K_n^-, \\ K_n^- & \text{if } \operatorname{cap} K_n^+ < \operatorname{cap} K_n^-. \end{cases}$$

Then (3.6) leads to

$$\operatorname{cap} \widetilde{K}_n \geq \frac{(\operatorname{cap} K_n)^2}{d} \geq \frac{\delta^2}{d},$$

where $K_n = K_n(D_{\sigma_1,\sigma_2}; \varepsilon)$ $(0 < \varepsilon \leq \varepsilon_0)$ satisfies (3.2). Hence, replacing K_n by $\widetilde{K}_n$ we may assume in the following that the sets $K_n = K_n(D_{\sigma_1,\sigma_2}; \varepsilon)$ satisfy for all sufficiently large n the properties:

(i) $\operatorname{cap} K_n = \operatorname{cap} K_n(\sigma_1, \sigma_2; \varepsilon) \geq \delta$,
(ii) K_n is of type K_n^+ or of type K_n^-,
(iii) $0 < \delta < 1$.

Let B_n denote the complement of K_n, $B_n = \overline{\mathbb{C}} \backslash K_n$. Then B_n is connected, since the functions $F_n(z)$ of (2.2) are subharmonic in $E_{\rho(f)} \backslash E$. Because K_n satisfies (ii), we obtain

$$E_{\sigma_1} \subset B_n \quad \text{and} \quad \overline{\mathbb{C}} \backslash \overline{E}_{\sigma_2} \subset B_n.$$

Let μ_n denote the equilibrium measure of K_n, so the logarithmic potential U^{μ_n} is superharmonic and lower semicontinuous in $\mathbb{C}$ (cf. [11, Thm. II.23, p. 45)]) and

$$U^{\mu_n}(z) = -g_{B_n}(z, \infty) - \log \ \text{cap}\ K_n, \ z \in B_n,$$

where $g_{B_n}(z, \infty)$ is Green's function of B_n with pole at ∞ (cf. [9, Ch. I, Sect. 1.4, Eq. (4.8), p. 53)]. According to a theorem of Frostman (cf. [11, Thm. III.12, p. 60] or [8, (Thm. 3.3.4, p. 59]),

$$U^{\mu_n}(z) \leq -\log \ \text{cap}\ \mathrm{K_n}, \ z \in \mathbb{C},$$

and

$$U^{\mu_n}(z) = -\log \ \text{cap}\ K_n \quad \text{for q.e. } \ z \in K_n,$$

or more precisely, $U^{\mu_n}(z) = -\log \ \text{cap}\ K_n$ for all $z \in K_n$ except on a F_σ-set of ∂K_n with capacity 0.

For the following we choose 4 additional auxiliary parameters r, R and τ_1, τ_2 such that

$$1 < r < \tau_1 < \sigma_1 \leq \sigma_2 < \tau_2 < R < \rho(f),$$

and we define for $\mu \in \mathcal{M}(D_{\sigma_1,\sigma_2})$

$$M_{r,R}(\mu) := \max_{\Gamma_r \cup \Gamma_R} U^\mu(z), \quad M_{\tau_1,\tau_2}(\mu) := \max_{\Gamma_{\tau_1} \cup \Gamma_{\tau_2}} U^\mu(z).$$

Lemma 1 *Let $\mu \in \mathcal{M}(D_{\sigma_1,\sigma_2})$ with* $\text{supp}(\mu) \subset \Omega^+$ *or* $\text{supp}(\mu) \subset \Omega^-$*, so*

$$M_{\tau_1,\tau_2}(\mu) > M_{r,R}(\mu). \tag{3.7}$$

If μ_n is the equilibrium measure of K_n, then

$$-\log \ \text{cap}\ K_n = \max_{z \in K_n} U^{\mu_n}(z) > M_{\tau_1,\tau_2}(\mu_n) > M_{r,R}(\mu_n). \tag{3.8}$$

Proof The logarithmic potential $U^\mu(z)$ is harmonic outside of $\text{supp}(\mu)$, hence in $\mathbb{C} \backslash \left(D_{\sigma_1,\sigma_2} \cap \Omega^+\right)$ or in $\mathbb{C} \backslash \left(D_{\sigma_1,\sigma_2} \cap \Omega^-\right)$. Since

$$E_r \subset E_{\tau_1} \subset \mathbb{C} \backslash \left(D_{\sigma_1,\sigma_2} \cap \Omega^+\right) \quad \text{or} \quad E_r \subset E_{\tau_1} \subset \mathbb{C} \backslash \left(D_{\sigma_1,\sigma_2} \cap \Omega^-\right),$$

we get by the maximum principle of harmonic functions

$$\max_{z\in\Gamma_r} U^{\mu}(z) < \max_{z\in\Gamma_{\tau_1}} U^{\mu}(z). \tag{3.9}$$

Moreover,

$$\Gamma_{\tau_2} \subset \mathbb{C}\backslash D_{\sigma_1,\sigma_2} \quad \text{and} \quad \lim_{z\to\infty} U^{\mu}(z) = -\infty$$

and, again by the maximum principle,

$$\max_{z\in\Gamma_{\tau_2}} U^{\mu}(z) > \max_{z\in\Gamma_R} U^{\mu}(z). \tag{3.10}$$

Then (3.9) and (3.10) yield

$$M_{\tau_1,\tau_2}(\mu) > M_{r,R}(\mu).$$

Concerning (3.8), the theorem of Frostman implies that

$$-\log \operatorname{cap}\, K_n = \max_{z\in K_n} U^{\mu_n}(z).$$

If $z_0 \in \mathbb{C}\backslash K_n$, then

$$U^{\mu_n}(z_0) < -\log \operatorname{cap}\, K_n,$$

otherwise, the theorem of Frostman yields

$$-\log \operatorname{cap}\, K_n \geq \max_{z\in W} U^{\mu_n}(z) \geq U^{\mu_n}(z_0) \geq -\log \operatorname{cap}\, K_n,$$

where W is some neighborhood of z_0. Then $U^{\mu_n}(z) = -\log \operatorname{cap}\, K_n$ for $z \in \mathbb{C}\backslash K_n$, contradicting

$$\lim_{z\to\infty} U^{\mu_n}(z) = -\infty.$$

Hence, the first inequality in (3.8) holds, the second is a special case of (3.7). □

We define

$$H_n(z) := \frac{U^{\mu_n}(z) - M_{r,R}(\mu_n)}{-\log\ \operatorname{cap}\ K_n - M_{r,R}(\mu_n)}$$

and the domain

$$D_{r,R}^{(n)} := B_n \cap D_{r,R}^{\circ}.$$

Then $H_n(z)$ is harmonic in $D^{(n)}_{r,R}$ and satisfies the boundary conditions

$$H_n(z) \leq 0 \quad \text{for} \quad z \in \Gamma_r \cup \Gamma_R, \tag{3.11}$$

$$\lim_{\xi \in D^{(n)}_{r,R},\ \xi \to z} H_n(\xi) = 1 \quad \text{for q.e. } z \in \partial D^{(n)}_{r,R} \cap K_n. \tag{3.12}$$

For (3.12) we have used the theorem of Frostman. Next, let us define

$$\alpha_n := \max_{z \in \Gamma_{\tau_1} \cup \Gamma_{\tau_2}} H_n(z)$$

so

$$\alpha_n = \frac{M_{\tau_1,\tau_2}(\mu_n) - M_{r,R}(\mu_n)}{-\log\ \text{cap}\ K_n - M_{r,R}(\mu_n)}.$$

Lemma 2 *Let*

$$\beta_n := M_{\tau_1,\tau_2}(\mu_n) - M_{r,R}(\mu_n),$$

so

$$\liminf_{n \to \infty} \beta_n > 0 \tag{3.13}$$

and

$$\liminf_{n \to \infty} \alpha_n \geq \alpha > 0. \tag{3.14}$$

Proof Let us assume that (3.13) is false, i.e., there exists, because of (3.7), a subset $\Lambda \subset \mathbb{N}$ such that

$$\lim_{n \in \Lambda, n \to \infty} \left(M_{\tau_1,\tau_2}(\mu_n) - M_{r,R}(\mu_n)\right) = 0. \tag{3.15}$$

Let

$$D^{+}_{\sigma_1,\sigma_2} = \left\{z \in D_{\sigma_1,\sigma_2} : \phi(z) \in \Omega^{+}\right\}$$

and

$$D^{-}_{\sigma_1,\sigma_2} = \left\{z \in D_{\sigma_1,\sigma_2} : \phi(z) \in \Omega^{-}\right\}.$$

according to the definitions in (3.4) and (3.5). Since K_n is either of type K_n^{+} or of type K_n^{-}, there exists an infinite set $\Lambda_1 \subset \Lambda$ such that

$$\mu_n \in \mathcal{M}(D^{+}_{\sigma_1,\sigma_2}) \text{ (resp. } \mu_n \in \mathcal{M}(D^{-}_{\sigma_1,\sigma_2})) \text{ for } n \in \Lambda_1.$$

Then by Helly's Selection Theorem, there exists $\Lambda^* \subset \Lambda_1$ and $\mu \in \mathcal{M}(D^+_{\sigma_1,\sigma_2})$ (resp. $\mu \in \mathcal{M}(D^-_{\sigma_1,\sigma_2})$ such that

$$\lim_{n\in\Lambda^*, n\to\infty} U^{\mu_n}(z) = U^{\mu}(z) \text{ for } z \in \mathbb{C}\setminus D^+_{\sigma_1,\sigma_2} \text{ (resp. } z \in \mathbb{C}\setminus D^-_{\sigma_1,\sigma_2})$$

and the functions U^{μ_n}, $n \in \Lambda^*$, are uniformly bounded on compact sets of $\mathbb{C}\setminus D^+_{\sigma_1,\sigma_2}$ (resp. $\mathbb{C}\setminus D^-_{\sigma_1,\sigma_2}$). Then $\{U^{\mu_n}\}_{n\in\Lambda^*}$ converges uniformly on compact sets of $\mathbb{C}\setminus D^+_{\sigma_1,\sigma_2}$ (resp. $\mathbb{C}\setminus D^-_{\sigma_1,\sigma_2}$) (cf. Goluzin [3, Ch. 1, §1, Thm. 3, p. 20)]).

Now,

$$\Gamma_r \cup \Gamma_{\tau_1} \cup \Gamma_{\tau_2} \cup \Gamma_R$$

is a compact subset of $\mathbb{C}\setminus D^+_{\sigma_1,\sigma_2}$ and of $\mathbb{C}\setminus D^-_{\sigma_1,\sigma_2}$ as well. Therefore, the functions U^{μ_n}, $n \in \Lambda^*$, converge uniformly to U^{μ} on $\Gamma_r \cup \Gamma_{\tau_1} \cup \Gamma_{\tau_2} \cup \Gamma_R$.

Hence, (3.15) implies that

$$0 = \lim_{n\in\Lambda^*, n\to\infty} \big(M_{\tau_1,\tau_2}(\mu_n) - M_{r,R}(\mu_n)\big) = M_{\tau_1,\tau_2}(\mu) - M_{r,R}(\mu). \tag{3.16}$$

Then (3.16) contradicts Lemma 1 and (3.13) is true.

Concerning (3.14), we consider the denominator

$$-\log \operatorname{cap} K_n - M^{(n)}_{r,R}.$$

Because of (3.2), for all $0 < \varepsilon \leq \varepsilon_0$

$$\operatorname{cap} K_n = \operatorname{cap} K_n(D_{\sigma_1,\sigma_2}; \varepsilon) \geq \delta > 0$$

for $n \geq n_0(\varepsilon_0)$ and we have assumed that $0 < \delta < 1$. Therefore

$$-\log \operatorname{cap} K_n \leq \log \frac{1}{\delta}, \quad n \geq n_0(\varepsilon_0), \tag{3.17}$$

Define

$$m := \max \left\{ 1, \max_{z\in\Gamma_r\cup\Gamma_R, t\in D_{\sigma_1,\sigma_2}} |z-t| \right\},$$

so $m \geq 1$ and

$$-U^{\mu_n}(z) = \int \log|z-t| d\mu_n(t) \leq \log m \quad \text{for } z \in \Gamma_r \cup \Gamma_R, \tag{3.18}$$

and consequently (3.17) and (3.18) lead to

$$-\log \operatorname{cap} K_n - M^{(n)}_{r,R} \;\leq\; \log \frac{1}{\delta} + \log m > 0 \tag{3.19}$$

for all $n \geq n_0(\varepsilon_0)$. Hence, by (3.19) and (3.13) we obtain the inequality (3.14) and Lemma 2 is proven. □

Next, we consider the harmonic measures

$$H_n^*(z) = \omega(z, \partial K_n, D_{r,R}^{(n)}), \tag{3.20}$$

i.e., $H_n^*(z)$ is harmonic in the domain $D_{r,R}^{(n)}$ and satisfies the boundary conditions

$$H_n^*(z) = 0 \quad \text{for} \quad z \in \Gamma_r \cup \Gamma_R \tag{3.21}$$

and

$$\lim_{\xi \in D_{r,R}^{[n)},\ \xi \to z} H_n^*(\xi) = 1 \quad \text{for q.e.} \quad z \in \partial K_n. \tag{3.22}$$

It is known that H_n^* exists and is unique, (3.21) holds because all points of Γ_r and Γ_R are regular points, (3.22) is a consequence of cap $K_n > 0$ (cf. Ransford [8, Cor. 4.2.6, p. 95)]). Because of (3.21) and (3.22), the extended maximum principle, resp. minimum principle, yields

$$0 \leq H_n^*(z) \leq 1 \quad \text{for} \quad z \in D_{r,R}^{(n)}.$$

But since H_n^* is not constant, the function H_n^* cannot attain a local maximum or minimum in $D_{r,R}^{(n)}$. Hence

$$0 < H_n^*(z) < 1 \quad \text{for} \quad z \in D_{r,R}^{(n)}. \tag{3.23}$$

Lemma 3 *Let*

$$\gamma_n = \min_{z \in \Gamma_{\tau_1} \cup \Gamma_{\tau_2}} H_n^*(z),$$

then

$$\liminf_{n \to \infty} \gamma_n = \gamma > 0.$$

Proof Let us assume that Lemma 3 is false, i.e., (3.23) implies that

$$\liminf_{n \to \infty} \gamma_n = 0.$$

We choose a subset $\Lambda \subset \mathbb{N}$ such that

$$\liminf_{n \in \Lambda, n \to \infty} \gamma_n = 0. \tag{3.24}$$

Concerning the harmonic measures H_n^*, $n \in \Lambda$, there exists $\Lambda_1 \subset \Lambda$ such that the functions $H_n^*(z)$, $n \in \Lambda_1$, converge to a harmonic function $H^*(z)$ locally uniformly in $D_{r,R}^{(n)}$, especially on the compact set $\Gamma_{\tau_1} \cup \Gamma_{\tau_2}$ (cf. Goluzin [3, Ch. 1, §1, Thm. 2, p. 20)]). Because of (3.23) and (3.24), we get by the maximum principle

$$H^*(z) \equiv 0, \; z \in D_{r,R}^{(n)}.$$

Applied to $\Gamma_{\tau_1} \cup \Gamma_{\tau_2}$, we obtain for

$$\varepsilon_n := \max_{z \in \Gamma_{\tau_1} \cup \Gamma_{\tau_2}} H_n^*(z), \quad n \in \Lambda_1,$$

that

$$\lim_{n \in \Lambda_1, n \to \infty} \varepsilon_n = 0.$$

By (3.11), (3.12), (3.21), (3.22) we have for $n \in \mathbb{N}$

$$H_n^*(z) - H_n(z) \geq 0, \quad z \in \Gamma_r \cup \Gamma_R,$$

and

$$\lim_{\xi \in D_{r,R}^{(n)}, \xi \to z} (H_n^*(z) - H_n(z)) = 0 \quad \text{for q.e. } \; z \in \partial K_n.$$

Then the extended maximum principle yields

$$H_n^*(z) - H_n(z) \geq 0 \quad \text{for } \; z \in D_{r,R}^{(n)}, \; n \in \mathbb{N},$$

(cf. Ransford [8, Thm. 3.6.9, p. 70]). Hence

$$\liminf_{n \in \Lambda_1, n \to \infty} \min_{z \in \Gamma_{\tau_1} \cup \Gamma_{\tau_2}} (H_n^*(z) - H_n(z)) \geq 0. \tag{3.25}$$

Let $\xi_n \in \Gamma_{\tau_1} \cup \Gamma_{\tau_2}$ with

$$\alpha_n = \max_{\Gamma_{\tau_1} \cup \Gamma_{\tau_2}} H_n(z) = H_n(\xi_n),$$

then for $n \in \Lambda_1$

$$H_n^*(\xi_n) - H_n(\xi_n) \leq \varepsilon_n - H_n(\xi_n) = \varepsilon_n - \alpha_n$$

and

$$\min_{z \in \Gamma_{\tau_1} \cup \Gamma_{\tau_2}} (H_n^*(z) - H_n(z)) \leq H_n^*(\xi_n) - H_n(\xi_n) \leq \varepsilon_n - \alpha_n.$$

Consequently, by Lemma 2,

$$\liminf_{z\in\Lambda_1, n\to\infty} \min_{\Gamma_{\tau_1}\cup\Gamma_{\tau_1}} (H_n^*(z) - H_n(z)) \leq \liminf_{n\in\Lambda_1, n\to\infty} (-\alpha_n) \leq -\alpha < 0,$$

in contrast to (3.25). Hence, the assumption that Lemma 3 is false, is refuted. □

In the following we will use the functions $F_n(z)$ of (2.2). $F_n(z)$ is subharmonic in $E_{\rho(f)}\backslash E$ and the compact sets $K_n(\sigma_1, \sigma_2; \varepsilon)$ are

$$K_n(D_{\sigma_1,\sigma_2}; \varepsilon) = \left\{z \in D_{\sigma_1,\sigma_2} : F_n(z) \leq -\varepsilon\right\}.$$

We will compare $F_n(z)$ with

$$F_n^*(z) := -a\ H_n^*(z) + b \quad \text{with} \quad a > 0,\ b > 0, \tag{3.26}$$

where $H_n^*(z) = \omega(z, \partial K_n, D_{r,R}^{(n)})$ is the harmonic measure defined in (3.20)–(3.22).

Lemma 4 *There exist parameters $a > 0$, $b > 0$ and $n_0 \in \mathbb{N}$ such that F_n^*, defined in (3.26), is a harmonic majorant of the subharmonic function F_n in $D_{r,R}^{(n)}$ for $n \geq n_0$ and moreover, there exists $\gamma^* > 0$ such that*

$$\max_{z\in\Gamma_{\tau_1}\cup\Gamma_{\tau_2}} F_n^*(z) \leq -\gamma^* < 0 \quad \textit{for} \quad n \geq n_0, n \in \mathbb{N}.$$

Proof Let $\widetilde{\varepsilon} > 0$ be arbitrary. Because of the maximal convergence of p_n to f, there exists $n_1 = n_1(\widetilde{\varepsilon})$ such that

$$F_n(z) \leq \widetilde{\varepsilon} \quad \text{for} \quad z \in \Gamma_r \cup \Gamma_R \quad \text{and} \quad n \geq n_1(\widetilde{\varepsilon}).$$

Due to the definition of $K_n = K_n(D_{\sigma_1,\sigma_2}; \varepsilon)$,

$$F_n(z) = -\varepsilon, \quad z \in \partial K_n.$$

The parameter ε is always fixed and $0 < \varepsilon \leq \varepsilon_0$, where ε_0 satisfies (3.2).

We will define a and b constructively:

The function $F_n^*(z)$ of (3.26) satisfies

$$F_n^*(z) = b \quad \text{for} \quad z \in \Gamma_r \cup \Gamma_R,$$
$$F_n^*(z) = -a + b \quad \text{for q.e.} \quad z \in \partial K_n.$$

Hence, F_n^* is a harmonic majorant of F_n in $D_{r,R}^{(n)}$ if

$$b \geq \widetilde{\varepsilon} \quad \text{and} \quad -a + b \geq -\varepsilon.$$

First, we choose

$$b = \widetilde{\varepsilon} \quad \text{and} \quad a = b + \varepsilon. \tag{3.27}$$

Then we want to fix $\widetilde{\varepsilon}$ such that

$$0 > \max_{z \in \Gamma_{\tau_1} \cup \Gamma_{\tau_2}} F_n^*(z) = -a \min_{z \in \Gamma_{\tau_1} \cup \Gamma_{\tau_2}} H_n^*(z) + b = -a\, \gamma_n + b$$

for sufficiently big n. If we choose $n_2 = n_2(\gamma) \in \mathbb{N}$ such that by Lemma 3

$$\gamma_n \geq \frac{\gamma}{2} \quad \text{for} \quad n \geq n_2(\gamma),$$

then

$$\max_{z \in \Gamma_{\tau_1} \cup \Gamma_{\tau_2}} F_n^*(z) \leq -a \frac{\gamma}{2} + b < 0 \tag{3.28}$$

for $n \geq n_2(\gamma)$ if

$$-a \frac{\gamma}{2} + b = -(b + \varepsilon) \frac{\gamma}{2} + b < 0$$

or

$$b \left(1 - \frac{\gamma}{2}\right) < \varepsilon \frac{\gamma}{2}$$

or

$$b < \varepsilon \frac{\gamma}{2 - \gamma}, \tag{3.29}$$

where we have used $a = b + \varepsilon$ of (3.27), keeping in mind that $0 < \gamma \leq 1$. Therefore, defining

$$b := \frac{\varepsilon}{2} \frac{\gamma}{2 - \gamma}, \tag{3.30}$$

then (3.29) holds and (3.27) yields

$$\widetilde{\varepsilon} = \frac{\varepsilon}{2} \frac{\gamma}{2 - \gamma} \tag{3.31}$$

and

$$a = b + \varepsilon = \frac{\varepsilon}{2} \frac{4 - \gamma}{2 - \gamma}. \tag{3.32}$$

With

$$\gamma^* := \frac{\varepsilon}{4} \gamma$$

and the parameters a of (3.32) and b of (3.30) we obtain in (3.28)

$$\max_{z\in\Gamma_{\tau_1}\cup\Gamma_{\tau_2}} F_n^*(z) \leq -\gamma^* = -\frac{\varepsilon}{4}\gamma < 0$$

for

$$n \geq n_0 := \max\left(n_1(\widetilde{\varepsilon}), n_2(\gamma)\right),$$

where $\widetilde{\varepsilon}$ is defined in (3.31) and γ is the parameter from Lemma 3. Hence the proof of Lemma 4 is complete. □

Now, we are in position for the final step of the proof: Because of Lemma 4, we have

$$\max_{z\in\Gamma_{\tau_1}\cup\Gamma_{\tau_2}} F_n^*(z) \leq -\gamma^* < 0 \quad \text{for all } n \geq n_0. \tag{3.33}$$

Since $F_n^*(z)$ is a harmonic majorant of the subharmonic function $F_n(z)$ in $D_{r,R}^{(n)}$, we may restrict (3.33) for the further arguments either to Γ_{τ_1} or to Γ_{τ_2}. Let us choose Γ_{τ_1}, i.e., we consider

$$\max_{z\in\Gamma_{\tau_1}} \left(\frac{1}{n}\log|f(z) - p_n(z)| - g_\Omega(z,\infty) + \log\rho(f)\right) \leq -\gamma^*$$

for all $n \geq n_0$, or

$$\|f - p_n\|_{\Gamma_{\tau_1}} \leq \left(\frac{\tau_1}{\rho(f)}e^{-\gamma^*}\right)^n, \quad n \geq n_0.$$

Thus,

$$\|p_{n+1} - p_n\|_{\Gamma_{\tau_1}} \leq 2\left(\frac{\tau_1}{\rho(f)}e^{-\gamma^*}\right)^n, \quad n \geq n_0.$$

Then the telescoping series

$$f = p_{n_0} + \sum_{n=n_0}^{\infty}(p_{n+1} - p_n)$$

converges to a holomorphic function in a neighborhood of $\overline{E}_{\rho(f)}$, using well-known arguments and the Bernstein–Walsh-Lemma (cf. Walsh [13, Sect. 4.6)]). Hence, $\rho(f)$ is not the maximal parameter of holomorphy of f, which is a contradiction. Hence, (3.1) is not true and (2.4) is proven.

Concerning (2.5): Because F_n is subharmonic in D_{σ_1,σ_2}, the maximum principle yields

$$\max_{z\in D_{\sigma_1,\sigma_2}} F_n(z) = \max\left(\max_{z\in\Gamma_{\sigma_1}} F_n(z),\ \max_{z\in\Gamma_{\sigma_2}} F_n(z)\right) = \max_{z\in\Gamma_{\sigma_1}\cup\Gamma_{\sigma_2}} F_n(z). \tag{3.34}$$

Then the maximal convergence of the polynomials $p_n \in \mathcal{P}_n$ to f implies

$$\limsup_{n\to\infty} \max_{z\in D_{\sigma_1,\sigma_2}} F_n(z) = \limsup_{n\to\infty} \max_{z\in\Gamma_{\sigma_1}\cup\Gamma_{\sigma_2}} F_n(z) = 0. \tag{3.35}$$

On the other hand, the definition of $K_n(D_{\sigma_1,\sigma_2};\varepsilon)$ yields

$$\inf_{D_{\sigma_1,\sigma_2}\setminus K_n(D_{\sigma_1,\sigma_2};\varepsilon)} F_n(z) \geq -\varepsilon \quad \text{for any} \quad \varepsilon > 0 \text{ and } n \in \mathbb{N}. \tag{3.36}$$

Let $\varepsilon \to 0$, then by (3.36)

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \inf_{z\in D_{\sigma_1,\sigma_2}\setminus K_n(D_{\sigma_1,\sigma_2};\varepsilon)} F_n(z) \geq 0$$

and, together with (3.35),

$$0 = \limsup_{n\to\infty} \max_{z\in D_{\sigma_1,\sigma_2}} F_n(z) \geq \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \inf_{z\in D_{\sigma_1,\sigma_2}\setminus K_n(D_{\sigma_1,\sigma_2};\varepsilon)} F_n(z) \geq 0.$$

Hence, (2.5) and the Theorem is proven. □

4 Proof of the Corollaries

Proof of Corollary 1 Because of (3.34) and (3.35),

$$\limsup_{n\to\infty} \max_{z\in D_{\sigma_1,\sigma_2}} F_n(z) = 0. \tag{4.1}$$

Hence, there exists a sequence $\{\varepsilon_n^*\}_{n\in\mathbb{N}}$, $\varepsilon_n^* > 0$, with $\lim_{n\to\infty} \varepsilon_n^* = 0$ and $m_n^* \in \mathbb{N}$ such that

$$F_m(z) \leq \varepsilon_n^* \quad \text{for} \quad z \in D_{\sigma_1,\sigma_2} \quad \text{and} \quad m \geq m_n^*.$$

Now let us define

$$\delta(\varepsilon_n^*) := \liminf_{n\to\infty} \operatorname{cap} K_n(D_{\sigma_1,\sigma_2};\varepsilon_n^*),$$

so $\delta(\epsilon_n^*) \to 0$ as $n \to \infty$. Then there exists $m_n \in \mathbb{N},\ m_n \geq m_n^*$, such that

$$\operatorname{cap} K_{m_n}(D_{\sigma_1,\sigma_2}; \varepsilon_n^*) \leq 2\,\delta(\varepsilon_n^*). \tag{4.2}$$

Define

$$\Lambda := \{m_n\}_{n\in\mathbb{N}}$$

and

$$\varepsilon_n := \max(\varepsilon_n^*, 2\,\delta(\varepsilon_n^*)), \quad n \in \mathbb{N}, \tag{4.3}$$

then we get

$$F_{m_n}(z) \leq \varepsilon_{m_n}^* \leq \varepsilon_{m_n}, \quad z \in D_{\sigma_1,\sigma_2}.$$

Moreover, since $\varepsilon_{m_n} \geq \varepsilon_{m_n}^*$ we have

$$K_{m_n}(D_{\sigma_1,\sigma_2}; \varepsilon_{m_n}) \subset K_{m_n}(D_{\sigma_1,\sigma_2}; \varepsilon_{m_n}^*)$$

and, together with (4.2) and (4.3), we obtain

$$\operatorname{cap} K_{m_n}(D_{\sigma_1,\sigma_2}; \varepsilon_{m_n}) \leq \operatorname{cap} K_{m_n}(D_{\sigma_1,\sigma_2}; \varepsilon_{m_n}^*) \leq 2\ \delta(\varepsilon_{m_n}^*) \leq \varepsilon_{m_n}.$$

Then,

$$-\varepsilon_{m_n} \leq \inf_{z\in D_{\sigma_1,\sigma_2}\setminus K_{m_n}(D_{\sigma_1,\sigma_2};\varepsilon_{m_n})} F_{m_n}(z) \ \leq \ \max_{z\in D_{\sigma_1,\sigma_2}} F_{m_n}(z) \leq \varepsilon_{m_n}.$$

Consequently, for any σ, $\sigma_1 \leq \sigma \leq \sigma_2$,

$$\begin{aligned}\frac{\sigma}{\rho(f)} e^{-\varepsilon_{m_n}} &\leq \inf_{z\in\Gamma_\sigma\setminus K_{m_n}(\Gamma_\sigma;\varepsilon_{m_n})} \left| f(z) - p_{m_n}(z)\right|^{1/m_n} \\ &\leq \left\| f - p_{m_n}\right\|_{\Gamma_\sigma}^{1/m_n} \\ &\leq \frac{\sigma}{\rho(f)} e^{\varepsilon_{m_n}}.\end{aligned}$$

Hence, $\Lambda = \{m_n\}_{n\in\mathbb{N}}$ and $\{\varepsilon_n\}_{n\in\Lambda}$ satisfy the inequalities of Corollary 1. □

Proof of Corollary 2 We recall the contraction property of the capacity: If K is a compact set in $\mathbb{C}$ and let $T : K \to \mathbb{C}$ be a mapping satisfying

$$|T(z) - T(w)| \leq \alpha\,|z - w|\,, \quad z, w \in \mathbb{C},$$

where α is a positive constant, then

$$\operatorname{cap} T(K) \leq \alpha\ \operatorname{cap} K$$

(cf. Pommerenke [7] or Ransford [8]).

For the conformal mapping $\Phi = \overline{\mathbb{C}} \backslash E \longrightarrow \{z : |z| > 1\}$ of (3.3) it is known that

$$|\Phi(z) - \Phi(w)| \le c(\rho)\ \ |z - w|$$

for compact sets $K \subset \mathbb{C} \backslash E_\rho$, where

$$c(\rho) = \max_{z \in \Gamma \rho} \left| \Phi'(z) \right|$$

(cf. [2, Lem. 5.1]). Define

$$\sigma_1 := \frac{1 + \sigma}{2} \quad \text{and} \quad \sigma_2 := \frac{\sigma + \rho(f)}{2}.$$

Then the Theorem yields

$$\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \operatorname{cap} K_n(D_{\sigma_1, \sigma_2}; \varepsilon) = 0.$$

Defining for $\varepsilon > 0$

$$\delta(\varepsilon) := \liminf_{n \to \infty} \operatorname{cap} K_n(D_{\sigma_1, \sigma_2}; \varepsilon),$$

we get $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$. Then there exists a sequence $\left\{\varepsilon_n^*\right\}_{n \in \mathbb{N}}$ such that $0 < \varepsilon_n^* \le 1/n$ and

$$\delta(\varepsilon_n^*) \le \frac{1}{c(\sigma_1)} \frac{1}{4n}. \tag{4.4}$$

Set

$$D_n := D_{\sigma - 1/n, \sigma + 1/n},$$

then there exists $m_0 \in \mathbb{N}$ such that $D_n \subset D_{\sigma_1, \sigma_2}$ for $n \ge m_0$.

Because of (4.4) and (4.1), we can choose a subsequence $\{m_n\}_{n \in \mathbb{N}}$, $m_{n+1} > m_n$, such that $m_1 \ge m_0$ and

$$\operatorname{cap} K_{m_n}(D_{\sigma_1, \sigma_2}; \varepsilon_n^*) \le \frac{1}{c(\sigma_1)} \frac{1}{2n}, \tag{4.5}$$

and

$$\max_{z \in D_{\sigma_1, \sigma_2}} F_{m_n}(z) \le \varepsilon_n^*. \tag{4.6}$$

Let p_1 denote the projection $p_1 : \mathbb{C} \backslash \{0\} \to \mathbb{R}_+$,

$$p_1(z) = r = |z| \quad \text{for} \ \ z = re^{i\phi},$$

where we have used polar coordinates (r, ϕ) in $\mathbb{C}\setminus\{0\}$. Then the contraction principle of the capacity, together with (4.5), yields

$$\begin{aligned} \operatorname{cap} p_1(\Phi(K_{m_n}(D_{\sigma_1,\sigma_2};\varepsilon_n^*))) &\le \operatorname{cap} \Phi(K_{m_n}(D_{\sigma_1,\sigma_2};\varepsilon_n^*)) \\ &\le c(\sigma_1) \operatorname{cap} K_{m_n}(D_{\sigma_1,\sigma_2};\varepsilon_n^*) \\ &\le \frac{1}{2n}. \end{aligned} \tag{4.7}$$

On the other hand

$$\operatorname{cap} p_1(\Phi(D_n)) = \operatorname{cap}\left(\left[\sigma - \frac{1}{n}, \sigma + \frac{1}{n}\right]\right) = \frac{1}{n}. \tag{4.8}$$

Comparing (4.7) and (4.8), we conclude that there exists

$$\sigma_{m_n} \in \left[\sigma - \frac{1}{n}, \sigma + \frac{1}{n}\right]$$

such that

$$\Gamma_{\sigma_{m_n}} \cap K_{m_n}(D_{\sigma_1,\sigma_2};\varepsilon_n^*) = \emptyset$$

for all $m_n \ge m_0$. Using (4.6), we can summarize

$$-\varepsilon_n^* \le \min_{z\in\Gamma_{\sigma_{m_n}}} F_{m_n}(z) \le \max_{z\in\Gamma_{\sigma_{m_n}}} F_{m_n}(z) \le \varepsilon_n^*.$$

Consequently, the subset

$$\Lambda = \{m_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$$

and the sequences $\{\sigma_n\}_{n\in\Lambda}$ and $\{\varepsilon_n\}_{n\in\Lambda}$ with $\varepsilon_{m_n} := \varepsilon_n^*$ satisfy the properties of Corollary 2. □

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