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Affine density, von Neumann dimension and a problem of Perelomov[☆]



Luís Daniel Abreu^{a,c,*}, Michael Speckbacher^{b,c}

^a *Acoustics Research Institute, Austrian Academy of Sciences, Wohllebengasse 12-14, A-1040, Vienna, Austria*

^b *Katholische Universität Eichstätt-Ingolstadt, Mathematisch-Geographische Fakultät, Ostenstraße 26, 85071 Eichstätt, Germany*

^c *Universität Wien, Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria*

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ABSTRACT

We provide a solution to Perelomov's 1972 problem concerning the existence of a phase transition (known in signal analysis as 'Nyquist rate') determining the basis properties of certain affine coherent states labelled by Fuchsian groups. As suggested by Perelomov, the transition is given according to the hyperbolic volume of the fundamental region. The solution is a more general form (in phase space) of the $PSL(2, \mathbb{R})$ variant of a 1989 conjecture of Kristian Seip about wavelet frames, where the same value of 'Nyquist rate' is obtained as the trace of a certain localization operator. The proof consists of first connecting the problem to the theory of von Neumann algebras, by introducing a new class of projective representations of $PSL(2, \mathbb{R})$ acting on non-analytic Bergman-type spaces. We then adapt to this setting a new method for computing von Neumann dimensions, due to Sir Vaughan Jones. Our solution contains necessary conditions in the form of a 'Nyquist rate' dividing frames from Riesz sequences of coherent states and sampling from interpolating sequences. They hold for an infinite sequence of spaces of polyanalytic functions containing the eigenspaces of the Maass

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* Corresponding author.

E-mail address: abreuluisdaniel@gmail.com (L.D. Abreu).

operator and their orthogonal sums. Within mild boundaries, we show that our result is best possible, by characterizing our sequence of function spaces as the only invariant spaces under the non-analytic $PSL(2, \mathbb{R})$ -representations.

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1. Introduction

When the measure of successive dilations of a given compact set displays exponential growth, one often faces severe obstructions in problems involving basis properties or zeros of functions with membership in the associated function spaces. The exponential growth is characteristic for the disc and half-plane hyperbolic space models, for the affine group, and for more general hyperbolic spaces and non-unimodular groups. The obstructions posed by the exponential growth have been removed for Bergman spaces in [35,45], but the methods fully depend on tools for analytic functions.

In the present paper, we study completeness and basis properties of non-analytic functions indexed by Fuchsian groups, naturally associated with Maass forms and wavelet representations (coefficients of the non-unimodular affine group). Our results provide a solution of Perelomov's 1972 problem [36] about basis properties of discrete coherent states for a whole class of $PSL(2, \mathbb{R})$ -invariant coherent states, and include a proof that the solution is best possible within natural assumptions, by characterizing all function spaces which are invariant under the action of a new class of projective $PSL(2, \mathbb{R})$ -representations, introduced in this paper for non-analytic Bergman spaces. The arguments, based on the interplay of ideas from signal analysis, number theory, and von Neumann algebras, build on a recent method of Sir Vaughan Jones [25], to whose memory this paper is dedicated.

1.1. The Nyquist rate

There is a general 'folk theorem' in signal analysis claiming the existence of a critical density, the so called 'Nyquist rate', which must be exceeded for the complete representation of signals. The 'stable' version of the problem, where a lower bound on the ℓ^2 -norm of the discrete coefficients leads to the formulation in terms of frames achieved a notorious mainstream status at the crossroads of pure and applied mathematics. For spaces of functions whose Fourier transform is supported in a general measurable set, a theorem of Landau [30], and a version for radial functions [2] turn these heuristics into a precise result. In Gabor analysis, a similar condition for lattices in the time-frequency plane follows from the work of Rieffel [39] and has been extended to general sets by Ramanathan and Steger [38,9] (interestingly enough, Rieffel's proof depends on arguments using the coupling constant of von Neumann algebras; developments with a more explicit link to signal representations can be found in [11, Section 6], or in [8,19]). An overview

of these and other results is given in [22] as a historical recollection, and in [14] and [32] as a technical survey exploring the limits of existing operator theory methods. It is clear from these contributions that groups of exponential growth like the ‘ $ax + b$ ’-group are beyond the scope of the methods available.

1.2. The affine density problem

No known concept of density has been able to capture the ‘folk theorem’ in the case of wavelets with positive frequencies, defined as the coefficients of the ‘ $ax + b$ ’-group representation acting on a function ψ on the Hardy space $\mathcal{H}^2(\mathbb{C}^+)$ of analytic functions on the upper half-plane \mathbb{C}^+ with finite norm

$$\|f\|_2 = \sup_{s>0} \left(\int_{\mathbb{R}} |f(x + is)|^2 dx \right)^{1/2},$$

and originally introduced in physics as *affine coherent states*. The origin of the problem can be traced back to the work of Perelomov in the early 70’s [35]. In [35], a complete description of the completeness properties of affine coherent states indexed by Fuchsian groups with automorphic cusp forms has been obtained for the vector of least weight. This corresponds to the Cauchy wavelet $\psi_0^\alpha(t) = (t + i)^{-\alpha-1}$, whose phase space is a weighted Bergman space. The question of a more general result has been raised in [36].

For band-limited functions with band-width W , a comparison between the number of eigenvalues of the Toeplitz operator $T_\Omega^I = P_W \chi_I P_W$, where χ_I is the indicator function of a region $I \subset \mathbb{R}$ and P_W is the orthogonal projection onto the Paley-Wiener space of band-limited functions, and the number of sampling and interpolation points on the same region suffices to establish that the Nyquist rate is given by $\frac{\text{trace}(T_\Omega^I)}{|I|}$, see [30]. Let W_ψ denote the wavelet transform with mother wavelet ψ as defined in (3.1) and set $\mathcal{W}_\psi := W_\psi(\mathcal{H}^2(\mathbb{C}^+))$. In [42], reasoning in analogy with the aforementioned situation, Seip conjectured that the same thing should be true for wavelets, suggesting that the quantity $\frac{C_\psi}{\|\psi\|_2^2}$ could stand as a Nyquist rate for wavelet systems. An analysis of the eigenvalue profile of the wavelet Toeplitz operator corresponding to the Cauchy wavelet $T_{\psi_0^\alpha}^I = P_{\mathcal{W}_{\psi_0^\alpha}} \chi_I P_{\mathcal{W}_{\psi_0^\alpha}}$ made by Daubechies and Paul [12] has shown that the analogies with the euclidean problem solved by Landau were misleading, since the error resulting from the difference between the first two moments of $T_{\psi_0^\alpha}^I$ (inherited from the exponential growth of the ‘ $ax + b$ ’-group) has precisely the same asymptotic behaviour as the trace itself, when the region $I \subset \mathbb{C}^+$ is dilated to approach the whole upper half plane \mathbb{C}^+ . The same thing happens for general wavelets [13]. Nevertheless, Seip pursued the problem further in a body of research that culminated in a complete description of sampling and interpolation sequences in Bergman spaces [45], an elegant adaptation of a scheme of Beurling where several ideas from Korenblum’s ‘hyperbolic Nevanlinna theory’ [28] have been put in use and refined. Unfortunately, the methods of [45] seem to fully depend on

the holomorphic structure of the space and, as recently proved [23], the only choice of ψ lending itself to analytic phase spaces is (up to a chirp) the Cauchy wavelet $\psi_0^\alpha(t) = (t + i)^{-\alpha-1}$. After 30 years of research activity on wavelet frames (mostly concentrated in the period 1990-2010), and despite the participation of a significant group of mathematicians who introduced new ideas and found several interesting partial results (we refer to [20, 29, 21] for a sample of this activity), a full solution of the problem (i.e., generic wavelets and sampling sets) has been assigned a status between ‘impossible’ and ‘hopeless’ (see [29] for a detailed explanation of the technical and conceptual obstructions).

1.3. Von Neumann dimension

Let us define the functions $\psi_n^\alpha \in \mathcal{H}^2(\mathbb{C}^+)$, $n \in \mathbb{N}_0$, via the Fourier transform of generalized Laguerre polynomials

$$(\mathcal{F}\psi_n^\alpha)(\xi) := \xi^{\frac{\alpha}{2}} e^{-\xi} L_n^\alpha(2\xi), \quad \xi > 0, \quad \text{with} \quad L_n^\alpha(t) = \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \frac{t^k}{k!}, \quad (1.1)$$

where we follow the convention

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(t) e^{-i\xi t} dt.$$

In the present paper, we show that, at least for the orbits $\Gamma(z)$, $z \in \mathbb{C}^+$, of a Fuchsian group Γ and mother wavelets chosen from the family $\{\psi_n^\alpha\}_{n \in \mathbb{N}_0}$, there is indeed a Nyquist rate separating frames from Riesz sequences and sampling from interpolating sets for all possible coherent states defined on the phase space of $\mathcal{W}_{\psi_n^\alpha}$. And such a Nyquist rate is provided by the quantity $\frac{C_{\psi_n^\alpha}}{\|\psi_n^\alpha\|_2^2}$, in perfect agreement with Seip’s 1989 conjecture [42, 43]. The technical reason for the accuracy of this prediction is the following. Our main contribution for the theory of von Neumann algebras is a dimension computation associated with a projective unitary group representation of $PSL(2, \mathbb{R})$ on $L^2(\mathbb{C}^+, \mu)$ for an infinite family of non-analytic reproducing kernel Hilbert spaces indexed by n . This relies on Jones’ new method [25] which, according to the author, incorporates a suggestion of McMullen and is inspired by Atiyah’s setting for the covering space index theorem [5] and boils down to the computation of the trace of the same operator that Seip considered [42, 43]. Note that our approach extends Radulescu’s computation [37] to general n .

This new approach seems to be more elementary than Bekka’s [8] (for the analytic case $n = 0$), where the computation of the von Neumann dimension follows a scheme inspired by the derivation of Atiyah-Schmid’s formula [6] and also than Radulescu’s [37]. It allows us to derive a Nyquist rate for phase spaces of the wavelet transform (see Theorem 2.1) and for super wavelet systems (see Corollary 2.7) using mother wavelets associated to the eigenspaces of the Maass operator.

1.4. Outline

This paper is simply organized as follows: a section with the statements of the main results and the important special cases of Bergman and Maass spaces, and a ‘Background’ section with the essential material on wavelets, Fuchsian groups and von Neumann algebras is followed by a section with the proofs of the main results.

2. Results

Let us define the projective unitary representation τ_n^α of $PSL(2, \mathbb{R})$ on $L^2(\mathbb{C}^+, \mu)$ by

$$\tau_n^\alpha(\gamma)F(z) := \left(\frac{|cz+d|}{cz+d} \right)^{2n+\alpha+1} F(\gamma(z)), \quad (2.1)$$

where $d\mu(w) = \text{Im}(w)^{-2}dw$, $F \in L^2(\mathbb{C}^+, \mu)$, $\gamma \in PSL(2, \mathbb{R})$, and $z \in \mathbb{C}^+$. Note that τ_n^α differs from the classical representation of $PSL(2, \mathbb{R})$ as considered for example in [8,15,37] by the phase factor $\left(\frac{|cz+d|}{cz+d} \right)^{2n+\alpha+1}$, and that, as we show later, τ_n^α leaves $\mathcal{W}_{\psi_n^\alpha}$ invariant.

For the definition of frames, Riesz sequences, uniqueness sets and sampling and interpolating sequences, we refer to Section 3.2.

Theorem 2.1. *Let $\Gamma \subset PSL(2, \mathbb{R})$ be a Fuchsian group with fundamental domain $\Omega \subset \mathbb{C}^+$ (of finite volume) and $F \in \mathcal{W}_{\psi_n^\alpha}$. If $\{\tau_n^\alpha(\gamma)F\}_{\gamma \in \Gamma}$ is a frame for $\mathcal{W}_{\psi_n^\alpha}$, then Ω is compact and*

$$|\Omega| \leq \frac{C_{\psi_n^\alpha}}{\|\psi_n^\alpha\|_2^2} = \frac{2}{\alpha}, \quad (2.2)$$

where $|\Omega|$ is calculated via the measure μ .

If $\{\tau_n^\alpha(\gamma)F\}_{\gamma \in \Gamma}$ is a Riesz sequence for $\mathcal{W}_{\psi_n^\alpha}$, then

$$|\Omega| \geq \frac{C_{\psi_n^\alpha}}{\|\psi_n^\alpha\|_2^2} = \frac{2}{\alpha}. \quad (2.3)$$

Remark 2.2. Given a Hilbert space \mathcal{H} and a notion of density of points, a Nyquist rate can be formally defined as a precise value of density which must be exceeded for the existence of frames for \mathcal{H} , and cannot be exceeded for the existence of Riesz sequences for \mathcal{H} . There are several notions of affine density (see those in [29] and [45]), all of them essentially defined as the large r limit of the ratio between the number of points in a hyperbolic ball of radius r and the measure of the hyperbolic ball. For a Fuchsian group Γ one can use the hyperbolic geodesic distance $d(z, w)$ with $z, w \in \mathbb{C}^+$ to define the counting function $n_r(z, w) = \#\{\gamma \in \Gamma : d(w, \gamma(z)) < r\}$ and the hyperbolic ball B_r . Then a result of Patterson [34] gives

$$\lim_{r \rightarrow \infty} \frac{n_r(z, w)}{|B_r|} = \frac{1}{|\Omega|}.$$

Thus, Theorem 2.1 says that $\alpha/2$ is a Nyquist rate for all the Hilbert spaces $\mathcal{W}_{\psi_n^\alpha}$, $n \in \mathbb{N}_0$.

Applying the inverse wavelet transform to the function $\tau_n^\alpha(\gamma)F$, a Nyquist condition for a particular family of vectors in $\mathcal{H}^2(\mathbb{C}^+)$ follows.

Corollary 2.3. *Let $\Gamma \subset PSL(2, \mathbb{R})$ be a Fuchsian group with fundamental domain $\Omega \subset \mathbb{C}^+$ (of finite volume), $\Phi \in L^2(\mathbb{C}^+, \mu)$, and the family of vectors $\{\varphi_\gamma\}_{\gamma \in \Gamma}$ be weakly defined by*

$$\varphi_\gamma := \int_{\mathbb{C}^+} \Phi(z) \left(\frac{cz + d}{|cz + d|} \right)^{2n+\alpha+1} \pi(\gamma(z)) \psi_n^\alpha d\mu(z). \quad (2.4)$$

If $\{\varphi_\gamma\}_{\gamma \in \Gamma}$ is a frame for $\mathcal{H}^2(\mathbb{C}^+)$, then Ω is compact and (2.2) holds, and if $\{\varphi_\gamma\}_{\gamma \in \Gamma}$ is a Riesz sequence for $\mathcal{H}^2(\mathbb{C}^+)$, then (2.3) holds.

If we choose F in Theorem 2.1 to be the reproducing kernel $k_{\psi_n^\alpha}(z, \cdot)$ of the space $\mathcal{W}_{\psi_n^\alpha}$, then we can state the Nyquist condition for wavelet systems generated by Fuchsian groups and wavelets ψ_n^α . As recently observed by Romero and Velthoven in a different formulation [41], the following result can actually be obtained from the analytic case $n = 0$ of Theorem 2.1 above, using Radulescu's extension of the dimension computation in the projective case (non-integer α) for $n = 0$ [37].

Corollary 2.4. *Let $\Gamma \subset PSL(2, \mathbb{R})$ be a Fuchsian group with fundamental domain $\Omega \subset \mathbb{C}^+$ (of finite volume), and $z \in \Omega$. If the family $\{\pi(\gamma(z))\psi_n^\alpha\}_{\gamma \in \Gamma}$ is a wavelet frame for $\mathcal{H}^2(\mathbb{C}^+)$ (equivalently, if $\Gamma(z)$ is a sampling sequence for $\mathcal{W}_{\psi_n^\alpha}$), then Ω is compact and (2.2) holds.*

Condition (2.2) is also necessary if $\Gamma(z)$ is a uniqueness set for the wavelet space $\mathcal{W}_{\psi_n^\alpha}$. Thus, if (2.3) holds, then there exists a function in the wavelet space $\mathcal{W}_{\psi_n^\alpha}$ vanishing on $\Gamma(z)$.

If $\{\pi(\gamma(z))\psi_n^\alpha\}_{\gamma \in \Gamma}$ is a wavelet Riesz sequence (equivalently, if $\Gamma(z)$ is an interpolating sequence for $\mathcal{W}_{\psi_n^\alpha}$), then (2.3) holds.

While Ω needs to be compact for the existence of frames, this is not the case for uniqueness sets (see the discussion in [27, Section 5]).

Observe that the above results leave $|\Omega| = C_{\psi_n^\alpha} / \|\psi_n^\alpha\|_2^2$ as the only possibility for $\{\pi(\gamma(z))\psi_n^\alpha\}_{\gamma \in \Gamma}$ to be a wavelet Riesz basis for $\mathcal{H}^2(\mathbb{C}^+)$. In the case $n = 0$, we are led to analytic Bergman spaces, which contain no sampling and interpolation sequences: given Γ interpolating, pick f vanishing on the orbit of $\Gamma - \{\gamma_0\}$ for some $\gamma_0 \in \Gamma$ with $f(\gamma_0(z)) = i$, and then turn Γ into a zero sequence setting $F(w) = \frac{w-i}{w+i}f(w)$; thus interpolating sequences cannot be sampling. For other values of n , the spaces do not

seem to be closed under shift multipliers and nothing is known about the existence of such sequences. The perturbation argument used in [44] also does not seem to work here since, at least for subgroups of $PSL(2, \mathbb{Z})$, a ‘small’ perturbation of the parameters would change the group structure. We leave this as an interesting research question. So far, the only space known to have sampling and interpolating sequences is the Paley-Wiener space, where the multiplier algebra is trivial (see [46] for an engrossing discussion about this).

We show that the choice (1.1) is, within some boundaries, the only family leading to a well-defined problem. This can be seen as a converse of the first step in the proof of the above results, where we show that the specific wavelet spaces are invariant under the action of Fuchsian groups which is not the case for general wavelets. In particular, we show that these are the only wavelet spaces invariant under representations of the form (2.1) if we assume reasonable restrictions on ψ . To be precise, let $\mathcal{A} \subset \mathcal{H}^2(\mathbb{C}^+)$ be defined as

$$\mathcal{A} := \{f \in \mathcal{H}^2(\mathbb{C}^+) : \widehat{f}(\xi) \in \mathbb{R}, \widehat{f}\widehat{f}', \xi\widehat{f}\widehat{f}'' \in L^1(\mathbb{R}^+)\}. \quad (2.5)$$

The situation is depicted in the next result, which may be of independent interest.

Theorem 2.5. *If $F \in \mathcal{W}_{\psi_n^\alpha}$, then $\tau_n^\alpha(g)F \in \mathcal{W}_{\psi_n^\alpha}$ for every $g \in PSL(2, \mathbb{R})$. Conversely, let σ_s be a projective unitary group representation of $PSL(2, \mathbb{R})$ on $L^2(\mathbb{C}^+, \mu)$ of the form*

$$\sigma_s(g)F(z) = \left(\frac{|cz + d|}{cz + d} \right)^s F(g(z)).$$

If $\psi \in \mathcal{A}$ and σ_s leaves \mathcal{W}_ψ invariant, then for some $n \in \mathbb{N}_0$ and $C \in \mathbb{C} \setminus \{0\}$ we have

$$\psi = C\psi_n^\alpha,$$

where $\alpha = s - 2n - 1$.

The first statement of Theorem 2.5 is crucial for the proof of Theorem 2.1 and its corollaries as well as for Theorem 2.8.

Remark 2.6. Theorem 2.1 is an extension of Perelomov’s result [35] for general $PSL(2, \mathcal{R})$ coherent states in terms of $|\Omega|$, as suggested in [36], but it still lacks a sufficient condition for a given $F \in \mathcal{W}_{\psi_n^\alpha}$ to be complete, since the classes of incomplete and Riesz sequences may or may not be disjoint. However, the observations in the proof of Theorem 2.1 combined with the other implication of Theorem 3.1 allow to state a sufficient condition in the following sense: if $|\Omega| \leq \frac{C_{\psi_n^\alpha}}{\|\psi_n^\alpha\|_2^2} = \frac{2}{\alpha}$ then, for every $n \geq 0$, there exists $F \in \mathcal{W}_{\psi_n^\alpha}$, such that $\{\tau_n^\alpha(\gamma)F\}_{\gamma \in \Gamma}$ is complete for $\mathcal{W}_{\psi_n^\alpha}$.

An interesting special case of the above result follows from the choice $\alpha = 2(B - n) - 1$ (this leads to a finite sequence of orthogonal spaces $\mathcal{W}_{\psi_n^{2(B-n)-1}}$, while the spaces $\mathcal{W}_{\psi_n^\alpha}$

are in general not orthogonal) considered by Mouayn [33] to attach coherent states to hyperbolic Landau levels. Let us consider the Maass operator

$$H_B := -(z - \bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} - B(z - \bar{z}) \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) = -s^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} \right) + 2iBs \frac{\partial}{\partial x}.$$

The spectrum of H_B consists of a continuous part $[1/4, +\infty[$ and a finite number of eigenvalues. To each eigenvalue $\lambda_n^B = (B - n)(B - n - 1)$, for $n = 0, 1, \dots, \lfloor B - \frac{1}{2} \rfloor$, corresponds an infinite dimensional reproducing kernel Hilbert eigenspace $A_n^B(\mathbb{C}^+, \mu) \subset L^2(\mathbb{C}^+, \mu)$. By computing the reproducing kernels (see Section 3.3) we can identify the spaces $\mathcal{W}_{\psi_n^{2(B-n)-1}} = A_n^B(\mathbb{C}^+, \mu)$. From Corollary 2.4 it then follows:

Corollary 2.7. *Let $n \leq \lfloor B - \frac{1}{2} \rfloor$. If $\Gamma(z)$ is a sampling sequence for the space $A_n^B(\mathbb{C}^+, \mu)$, then Ω is compact and*

$$|\Omega| \leq \frac{1}{B - n - 1/2}.$$

If $\Gamma(z)$ is an interpolating sequence for the space $A_n^B(\mathbb{C}^+, \mu)$, then

$$|\Omega| \geq \frac{1}{B - n - 1/2}.$$

It follows from the result above that, if $|\Omega| > \frac{1}{B-n-1/2}$, then one can find $F \in A_n^B(\mathbb{C}^+, \mu)$ vanishing on $\Gamma(z)$. While in [25] explicit examples of functions in $A_0^B(\mathbb{C}^+, \mu)$ vanishing on $\Gamma(z)$ are provided by (analytic) automorphic forms, explicit examples of functions in $A_n^B(\mathbb{C}^+, \mu)$ (which are only real analytic and polyanalytic) are provided by the theory of Maass forms [31,40]. Thus, it may be possible to use Maass cusp forms to adapt the construction in [25, Section 7], but we will not pursue this topic in the present paper.

There is more to be said here. If we define

$$\mathbf{A}_N^B(\mathbb{C}^+, \mu) := \bigoplus_{n=0}^{N-1} A_n^B(\mathbb{C}^+, \mu) = \bigoplus_{n=0}^{N-1} \mathcal{W}_{\psi_n^{2(B-n)-1}}, \quad (2.6)$$

then there also exist a Nyquist rate for $\mathbf{A}_N^B(\mathbb{C}^+, \mu)$ separating sampling and interpolation sequences:

Theorem 2.8. *Let $N \in \{1, \dots, \lfloor B - \frac{1}{2} \rfloor\}$. If $\Gamma(z)$ is a sampling sequence for the space $\mathbf{A}_N^B(\mathbb{C}^+, \mu)$, then Ω is compact and*

$$|\Omega| \leq \frac{2}{N(2B - N)}. \quad (2.7)$$

If $\Gamma(z)$ is an interpolating sequence for the space $\mathbf{A}_N^B(\mathbb{C}^+, \mu)$, then

$$|\Omega| \geq \frac{2}{N(2B - N)}. \quad (2.8)$$

An equivalent interpretation of Theorem 2.8 (in the spirit of the second identity of (2.6)) is that (2.7) is a necessary condition for the vector valued system $\{\pi(\gamma(z))\phi_N^B\}_{\gamma \in \Gamma} \subset \mathcal{H}^2(\mathbb{C}^+, \mathbb{C}^N)$, where

$$\phi_N^B := (\psi_0^{2B-1}, \psi_1^{2B-3}, \dots, \psi_{N-1}^{2(B-N)+1}),$$

to be a *wavelet superframe* for $\mathcal{H}^2(\mathbb{C}^+, \mathbb{C}^N)$, see e.g. [1] for further information on superframes. In particular, we get the following result.

Corollary 2.9. *Let $N \in \{1, \dots, \lfloor B - \frac{1}{2} \rfloor\}$. If $\{\pi(\gamma(z))\phi_N^B\}_{\gamma \in \Gamma}$ is a wavelet superframe for $\mathcal{H}^2(\mathbb{C}^+, \mathbb{C}^N)$, then Ω is compact and*

$$|\Omega| \leq \frac{2}{N(2B - N)}. \quad (2.9)$$

Note that a similar necessary condition for super-wavelet frames generated by mother wavelets connected to the true polyanalytic Bergman spaces and the hyperbolic lattice was shown in [1]. We also refer to [18] for a result by Gröchenig and Lyubarskii in the context of Gabor superframes with Hermite windows and of Balan [7] providing necessary conditions for general Gabor superframes.

As a final remark, we outline the physical relevance of the results. The Maass operator has been considered [3,10] as an alternative model for the formation of Landau levels, an important model in the physics of the Quantum-Hall effect. Here, a constant magnetic field of intensity B acts on the open hyperbolic plane realized as the Poincaré upper half-plane \mathbb{C}^+ , leading to the formation of a mixed spectrum with a discrete part corresponding to *bound states* (*hyperbolic Landau levels*), and a continuous part corresponding to *scattering states*. The finite number of the levels, which depends on the strength of the magnetic field is, in principle, a physical advantage of such a model. In [3] the first bounds on the size of the fundamental region, which correspond to a ‘cell’ (Pauli exclusion principle allows only one electron per cell), have been obtained. Note that the above improves the main result of [3] by a factor of $n + 1$. Given a sample modelled by a bounded region $\Delta \subset \mathbb{C}^+$ in a hyperbolic Landau level, assuming the Pauli exclusion principle, Corollary 2.4 provides estimates for the number of particles distributed by the model on such a region.

3. Background

3.1. Wavelets on the Hardy space

The affine group $G_a = \mathbb{R} \times \mathbb{R}^+$ is defined by the group law [17]

$$(b, a) \cdot (x, s) = (ax + b, as), \quad (b, a), (x, s) \in G_a.$$

Here, we identify G_a with the upper half plane \mathbb{C}^+ and note that the left-invariant Haar measure on the affine group is given by

$$d\mu(z) := \frac{dx dy}{s^2}, \quad \text{where } z = x + is \in \mathbb{C}^+.$$

Let us define a unitary representation of G_a on the Hardy space $\mathcal{H}^2(\mathbb{C}^+)$ by

$$\pi(z)\psi(t) := T_x D_s \psi(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{t-x}{s}\right), \quad z = x + is \in \mathbb{C}^+.$$

The *wavelet transform* of a function $f \in \mathcal{H}^2(\mathbb{C}^+)$ using the *mother wavelet* $\psi \in \mathcal{H}^2(\mathbb{C}^+)$ is then defined as

$$W_\psi f(z) := \langle f, \pi(z)\psi \rangle = \int_{\mathbb{R}} f(t) \overline{T_x D_s \psi(t)} dt = \sqrt{s} \int_0^\infty \widehat{f}(\xi) \overline{\widehat{g}(s\xi)} e^{ix\xi} d\xi, \quad (3.1)$$

where ψ satisfies the *admissibility condition*

$$C_\psi := \int_{\mathbb{R}^+} |\widehat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} < \infty.$$

For $f_1, f_2 \in \mathcal{H}^2(\mathbb{C}^+)$ and ψ admissible, the following orthogonality relation holds

$$\int_{\mathbb{C}^+} W_\psi f_1(z) \overline{W_\psi f_2(z)} d\mu(z) = C_\psi \langle f_1, f_2 \rangle. \quad (3.2)$$

In particular, if we set $f_2 = \pi(z)\psi$, then for every $f \in \mathcal{H}^2(\mathbb{C}^+)$ one has

$$W_\psi f(z) = \frac{1}{C_\psi} \int_{\mathbb{C}^+} W_\psi f(w) \langle \pi(w)\psi, \pi(z)\psi \rangle d\mu(z), \quad z \in \mathbb{C}^+. \quad (3.3)$$

This reproducing formula then implies that the range of the wavelet transform

$$\mathcal{W}_\psi := \{F \in L^2(\mathbb{C}^+, \mu) : F = W_\psi f, f \in \mathcal{H}^2(\mathbb{C}^+)\}$$

is a reproducing kernel subspace of $L^2(\mathbb{C}^+, \mu)$ with kernel

$$k_\psi(z, w) = \frac{1}{C_\psi} \langle \pi(w)\psi, \pi(z)\psi \rangle, \quad \text{and} \quad k_\psi(z, z) = \frac{\|\psi\|_2^2}{C_\psi}.$$

Moreover, k_ψ is the integral kernel of the projection operator $P_\psi : L^2(\mathbb{C}^+, \mu) \rightarrow \mathcal{W}_\psi$. The reproducing kernel of \mathcal{W}_ψ can easily be computed using the Fourier transform $\mathcal{F} : \mathcal{H}^2(\mathbb{C}^+) \rightarrow L^2(0, \infty)$ as follows

$$\begin{aligned}
\langle \pi(w)g, \pi(z)g \rangle &= \langle \mathcal{F}(\pi(w)g), \mathcal{F}(\pi(z)g) \rangle_{L^2(0, \infty)} \\
&= (ss')^{\frac{1}{2}} \int_0^\infty \widehat{g}(s'\xi) \overline{\widehat{g}(s\xi)} e^{i(x-x')\xi} d\xi.
\end{aligned} \tag{3.4}$$

3.2. Wavelet frames and Riesz sequences

Let \mathcal{H} be a separable Hilbert space and \mathcal{I} be a discrete index set. The family of vectors $\{\psi_k\}_{k \in \mathcal{I}}$ is called a *frame* if there exist constants $A, B > 0$ such that, for every $f \in \mathcal{H}$, one has

$$A\|f\|^2 \leq \sum_{k \in \mathcal{I}} |\langle f, \psi_k \rangle|^2 \leq B\|f\|^2. \tag{3.5}$$

If, on the other hand, there exist constants $A, B > 0$ such that, for every $c \in \ell^2(\mathcal{I})$

$$A\|c\|_{\ell^2(\mathcal{I})}^2 \leq \left\| \sum_{k \in \mathcal{I}} c_k \psi_k \right\|^2 \leq B\|c\|_{\ell^2(\mathcal{I})}^2, \tag{3.6}$$

then $\{\psi_k\}_{k \in \mathcal{I}}$ is called a *Riesz sequence*. A Riesz sequence that is also a frame is called a *Riesz basis*.

Let $\Gamma \subset \mathbb{C}^+$ be a discrete set and $\psi \in \mathcal{H}^2(\mathbb{C}^+)$ be admissible. We call $\{\pi(\gamma)\psi\}_{\gamma \in \Gamma}$ a *wavelet frame* (resp. *wavelet Riesz sequence*) if it is a frame (resp. Riesz sequence) for $\mathcal{H}^2(\mathbb{C}^+)$. Using (3.3) we can observe that $\{\pi(\gamma)\psi\}_{\gamma \in \Gamma}$ is a frame for $\mathcal{H}^2(\mathbb{C}^+)$ if and only if $\{\overline{k_\psi(\gamma, \cdot)}\}_{\gamma \in \Gamma}$ is a frame for \mathcal{W}_ψ . In particular, $\{\overline{k_\psi(\gamma, \cdot)}\}_{\gamma \in \Gamma}$ is *dense* in \mathcal{W}_ψ .

We say that Γ is a *sampling sequence* for the wavelet space \mathcal{W}_ψ if there exist constants $A, B > 0$ such that for every $F \in \mathcal{W}_\psi$

$$A\|F\|^2 \leq \sum_{\gamma \in \Gamma} |F(\gamma)|^2 \leq B\|F\|^2,$$

and Γ is called a *uniqueness set* for \mathcal{W}_ψ if $F(\gamma) = 0$, for every $\gamma \in \Gamma$ implies $F = 0$. Likewise, we say that Γ is an *interpolating sequence* if $\{\overline{k_\psi(\gamma, \cdot)}\}_{\gamma \in \Gamma}$ is a Riesz sequence for \mathcal{W}_ψ .

For $\mathbf{f} = (f_0, f_1, \dots, f_{N-1})$, $\boldsymbol{\psi} = (\psi_0, \psi_1, \dots, \psi_{N-1}) \in \mathcal{H}^2(\mathbb{C}^+, \mathbb{C}^N)$ we define

$$\mathbf{W}_\boldsymbol{\psi} \mathbf{f}(z) := \sum_{n=0}^{N-1} W_\psi f_n(z), \quad z \in \mathbb{C}^+,$$

and say that $\{\pi(\gamma(z))\boldsymbol{\psi}\}_{\gamma \in \Gamma}$ is a *wavelet superframe* if there exist $A, B > 0$ such that for every $\mathbf{f} \in \mathcal{H}^2(\mathbb{C}^+, \mathbb{C}^N)$

$$A\|\mathbf{f}\|_{\mathcal{H}^2(\mathbb{C}^+, \mathbb{C}^N)}^2 \leq \sum_{\gamma \in \Gamma} |\mathbf{W}_\boldsymbol{\psi} \mathbf{f}(\gamma)|^2 \leq B\|\mathbf{f}\|_{\mathcal{H}^2(\mathbb{C}^+, \mathbb{C}^N)}^2.$$

3.3. The Laguerre functions

Throughout this contribution we are interested in analyzing wavelets ψ_n^α which are defined via their Fourier transform in terms of the *generalized Laguerre polynomials* L_n^α as

$$(\mathcal{F}\psi_n^\alpha)(\xi) := \xi^{\frac{\alpha}{2}} e^{-\xi} L_n^\alpha(2\xi), \quad \text{with} \quad L_n^\alpha(\xi) = \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \frac{\xi^k}{k!}.$$

The orthogonality relation for generalized Laguerre polynomials reads

$$\int_0^\infty t^\alpha e^{-t} L_n^\alpha(t) L_m^\alpha(t) dt = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}.$$

From [47, Equation (4)], it follows that

$$\int_0^\infty t^{\alpha-1} e^{-t} L_n^\alpha(t)^2 dt = \frac{\Gamma(n+\alpha+1)}{n! \alpha}. \quad (3.7)$$

Consequently,

$$\frac{C_{\psi_n^\alpha}}{\|\psi_n^\alpha\|_2^2} = \frac{2}{\alpha}. \quad (3.8)$$

For $z = x + is, w = y + iv \in \mathbb{C}^+$, the reproducing kernel of $\mathcal{W}_{\psi_n^\alpha}$ is given by

$$k_{\psi_n^\alpha}(z, w) = 2^\alpha \alpha \left(\frac{\bar{z} - w}{z - \bar{w}} \right)^n \left(\frac{\sqrt{sv}}{-i(z - \bar{w})} \right)^{\alpha+1} P_n^{(\alpha,0)} \left(1 - \frac{8sv}{|z - \bar{w}|^2} \right), \quad (3.9)$$

where $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial

$$P_n^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+n+1)}{n! \Gamma(\alpha+\beta+n+1)} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+k+1)} \left(\frac{t-1}{2} \right)^k.$$

Note that (3.9) can be derived using (3.4) and [16, p. 809, 7.414 (4)].

3.4. A particular representation of $\mathrm{PSL}(2, \mathbb{R})$ and Fuchsian groups

Let I be the identity matrix in \mathbb{R}^2 . The quotient group

$$\mathrm{PSL}(2, \mathbb{R}) := \mathrm{SL}(2, \mathbb{R}) / \{\pm I\}$$

is also known as the group of *Moebius transformations* and can be identified with the *Poincaré half plane* \mathbb{C}^+ . The element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R})$ acts on \mathbb{C}^+ as

$$g(z) := \frac{az + b}{cz + d}.$$

Note that the measure μ is invariant under the action of $g \in PSL(2, \mathbb{R})$. Let us define $j : PSL(2, \mathbb{R}) \times \mathbb{C}^+ \rightarrow \mathbb{C} \setminus \{0\}$ by

$$j(g, z) = (cz + d)^{-1}.$$

It is easy to check that j satisfies the *cocycle condition*

$$j(gh, z) = j(g, h(z))j(h, z).$$

Hence, if we use the principal branch of the argument (i.e. $\arg(z) \in (-\pi, \pi]$) to define z^ν , we have

$$j(gh, z)^\nu = \lambda(g, h, \nu)j(g, h(z))^\nu j(h, z)^\nu,$$

for a unimodular function λ . It then immediately follows that $\tau_n^\alpha : PSL(2, \mathbb{R}) \rightarrow \mathcal{U}(L^2(\mathbb{C}^+, \mu))$ given by

$$\tau_n^\alpha(g)F(z) := \left(\frac{|cz + d|}{cz + d} \right)^{2n+\alpha+1} F(g(z)), \quad (3.10)$$

defines a projective unitary group representation of $PSL(2, \mathbb{R})$ on $L^2(\mathbb{C}^+, \mu)$. Note that τ_n^α reduces to the standard representation in the analytic case ($n = 0$).

Following Jones [25, Sect. 4] (the finite area condition is not included by all authors), we define a *Fuchsian group* Γ as a discrete subgroup of $PSL(2, \mathbb{R})$ for which there exists a fundamental domain with finite hyperbolic area (finite co-volume).

A *fundamental domain* Ω for Γ is a closed subset of \mathbb{C}^+ such that the images of Ω under Γ cover \mathbb{C}^+ , Ω is the closure of its interior Ω^0 , and for each two distinct $\gamma_1, \gamma_2 \in \Gamma$ one has $\gamma_1(\Omega)^0 \cap \gamma_2(\Omega)^0 = \emptyset$.

3.5. Von Neumann algebras

In this section, we closely follow the brief exposition of the topic in [25] and repeat the results that are necessary for our arguments. For a thorough introduction to von Neumann algebras we refer for example to [26].

A *von Neumann algebra* M is a $*$ -closed unital algebra of bounded operators on a (complex) Hilbert space \mathcal{H} which is closed under the topology of pointwise convergence on \mathcal{H} . A vector $\psi \in \mathcal{H}$ is called *cyclic* if $M\psi$ is dense in \mathcal{H} and ψ is called *separating* if

$T \mapsto T\psi$, $T \in M$, is injective. If there exists a trace functional $\text{tr} : M \rightarrow \mathbb{C}$, such that $\text{tr}(AB) = \text{tr}(BA)$, $A, B \in M$, and $\text{tr}(I) = 1$, then M is called *finite* and we define the space $L^2(M)$ as the completion of M with respect to the pre-Hilbert space inner product $\langle A, B \rangle = \text{tr}(B^*A)$.

It is a standard result on von Neumann algebras that if \mathcal{H} is any Hilbert space on which M acts, then there is an M -linear isometry

$$U : \mathcal{H} \rightarrow \bigoplus_{n=1}^{\infty} L^2(M).$$

On $\bigoplus_{n=1}^{\infty} L^2(M)$, the commutant M' of M admits a trace Tr_{L^2} which is canonically normalized such that the trace of any projection onto one of the $L^2(M)$'s is equal to 1. With that convention, we define the *von Neumann dimension* by

$$\dim_M \mathcal{H} = \text{Tr}_{L^2}(UU^*).$$

The following elementary results on the von Neumann dimension will be the key component in our arguments.

Theorem 3.1. *There is a cyclic vector for M if and only if $\dim_M \mathcal{H} \leq 1$.*

Theorem 3.2. *There is a separating vector for M if and only if $\dim_M \mathcal{H} \geq 1$.*

Definition 3.3. Let Γ be a (countable) discrete group. The *von Neumann algebra* of Γ , which we denote by $vN(\Gamma)$, is the von Neumann algebra on $\ell^2(\Gamma)$ generated by the left regular representation $\gamma \mapsto \lambda_\gamma$, where $\lambda_\gamma(f)(\gamma') = f(\gamma^{-1}\gamma')$.

More generally, if $\omega : \Gamma \times \Gamma \rightarrow \mathbb{T}$ is a unit circle valued 2-cocycle, $vN_\omega(\Gamma)$ is generated on $\ell^2(\Gamma)$ by the unitaries λ_γ^ω where $\lambda_\gamma^\omega f(\gamma') = \omega(\gamma, \gamma')f(\gamma^{-1}\gamma')$.

The subsequent proposition allows us to explicitly calculate the von Neumann dimension in the particular case that the von Neumann algebra is generated by a group representation of a discrete, infinite conjugacy class (ICC) group. The results can be found in [25, Proposition 3.6 & Corollary 3.7], but similar calculations have already appeared in the proof of [15, Theorem 3.3.2].

Proposition 3.4. *Let Γ be a discrete ICC group and $\gamma \mapsto \tau(\gamma)$ a projective group representation on \mathcal{H} with 2-cocycle ω . Suppose there is a projection Q on \mathcal{H} such that*

$$\tau(\gamma^{-1})Q\tau(\gamma) \perp Q, \text{ for } \gamma \in \Gamma - \{e\}, \text{ and } \sum_{\gamma \in \Gamma} \tau(\gamma^{-1})Q\tau(\gamma) = I.$$

Then the following statements hold:

- (i) There is a Γ -linear unitary map $U : \mathcal{H} \rightarrow \ell^2(\Gamma) \otimes Q\mathcal{H}$ with $U\tau(\gamma)U^{-1} = \lambda_\gamma^\omega \otimes Id$, for $\gamma \in \Gamma$.
- (ii) The action of Γ on \mathcal{H} makes it into a $vN_\omega(\Gamma)$ -module.
- (iii) If P is another projection on \mathcal{H} that commutes with $\tau(\gamma)$ for every $\gamma \in \Gamma$, then

$$\dim_{vN_\omega(\Gamma)} P\mathcal{H} = \text{trace}_{B(\mathcal{H})}(QPQ),$$

where $\text{trace}_{B(\mathcal{H})}$ denotes the usual trace.

Note that (iii) is not explicitly formulated in this form in [25, Corollary 3.7] but it appears in the last line of the proof.

4. Proof of the main results

4.1. Proof of Theorem 2.5

First, we observe that the wavelet space $\mathcal{W}_{\psi_n^\alpha}$ is invariant under the action of τ_n^α , the projective unitary group representation of $PSL(2, \mathbb{R})$ on $L^2(\mathbb{C}^+, \mu)$ defined in Section 3.4.

Lemma 4.1. *If $F \in \mathcal{W}_{\psi_n^\alpha}$, then $\tau_n^\alpha(g)F \in \mathcal{W}_{\psi_n^\alpha}$ for every $g \in PSL(2, \mathbb{R})$.*

Proof. Let $F \in \mathcal{W}_{\psi_n^\alpha}$. As $\mathcal{W}_{\psi_n^\alpha}$ is a reproducing kernel Hilbert space, it follows that $\tau_n^\alpha(\gamma)F \in \mathcal{W}_{\psi_n^\alpha}$ if and only if for every $z \in \mathbb{C}^+$

$$\tau_n^\alpha(g)F(z) = \int_{\mathbb{C}^+} \tau_n^\alpha(g)F(w)k_{\psi_n^\alpha}(z, w)d\mu(w).$$

Let us now establish an identity for $k_{\psi_n^\alpha}$ that allows us to verify the previous condition. Note that

$$\text{Im}(g(z)) = \frac{\text{Im}(z)}{|cz + d|^2}, \quad \text{and} \quad g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

It is then straightforward to show the following relations

$$\frac{\text{Im}(g(z))\text{Im}(w)}{|g(z) - \bar{w}|^2} = \frac{\text{Im}(z)\text{Im}(g^{-1}(w))}{\left|z - \overline{g^{-1}(w)}\right|^2}, \quad (4.1)$$

$$\frac{|cz + d|}{cz + d} \frac{\sqrt{\text{Im}(g(z))\text{Im}(w)}}{-i(g(z) - \bar{w})} = \frac{-cw + a}{|-cw + a|} \frac{\sqrt{\text{Im}(z)\text{Im}(g^{-1}(w))}}{-i(z - \overline{g^{-1}(w)})}, \quad (4.2)$$

$$\left(\frac{|cz + d|}{cz + d}\right)^2 \frac{\overline{g(z)} - w}{g(z) - \bar{w}} = \left(\frac{-cw + a}{|-cw + a|}\right)^2 \frac{\bar{z} - g^{-1}(w)}{z - \overline{g^{-1}(w)}}. \quad (4.3)$$

Plugging equations (4.1)-(4.3) into the explicit expression for $k_{\psi_n^\alpha}$ from (3.9) yields

$$\left(\frac{|cz+d|}{cz+d}\right)^{2n+\alpha+1} k_{\psi_n^\alpha}(g(z), w) = \left(\frac{-cw+a}{|-cw+a|}\right)^{2n+\alpha+1} k_{\psi_n^\alpha}(z, g^{-1}(w)). \quad (4.4)$$

Since we assumed that $F \in \mathcal{W}_{\psi_n^\alpha}$, this shows

$$\begin{aligned} \tau_n^\alpha(g)F(z) &= \left(\frac{|cz+d|}{cz+d}\right)^{2n+\alpha+1} F(g(z)) \\ &= \left(\frac{|cz+d|}{cz+d}\right)^{2n+\alpha+1} \int_{\mathbb{C}^+} F(w) k_{\psi_n^\alpha}(g(z), w) d\mu(w) \\ &= \int_{\mathbb{C}^+} F(w) \left(\frac{-cw+a}{|-cw+a|}\right)^{2n+\alpha+1} k_{\psi_n^\alpha}(z, g^{-1}(w)) d\mu(w) \\ &= \int_{\mathbb{C}^+} F(g(w)) \left(\frac{-cg(w)+a}{|-cg(w)+a|}\right)^{2n+\alpha+1} k_{\psi_n^\alpha}(z, w) d\mu(w) \\ &= \int_{\mathbb{C}^+} F(g(w)) \left(\frac{|cw+d|}{cw+d}\right)^{2n+\alpha+1} k_{\psi_n^\alpha}(z, w) d\mu(w) \\ &= \int_{\mathbb{C}^+} \tau_n^\alpha(g)F(w) k_{\psi_n^\alpha}(z, w) d\mu(w), \end{aligned} \quad (4.5)$$

where we used the cocycle relation $j(g^{-1}, g(w)) = j(g, w)^{-1}$ to derive the second to last identity. \square

We now prove the second statement of Theorem 2.5, which characterizes the functions invariant under the kind of projective unitary group representation of $PSL(2, \mathbb{R})$ on $L^2(\mathbb{C}^+, \mu)$ we are using.

Lemma 4.2. *Let \mathcal{A} be given by (2.5) and σ_s be a projective unitary group representation of $PSL(2, \mathbb{R})$ on $L^2(\mathbb{C}^+, \mu)$ of the form*

$$\sigma_s(g)F(z) = \left(\frac{|cz+d|}{cz+d}\right)^s F(g(z)).$$

If $\psi \in \mathcal{A}$ and σ_s leaves \mathcal{W}_ψ invariant, then for some $n \in \mathbb{N}_0$ and $C \in \mathbb{C} \setminus \{0\}$ we have

$$\psi = C\psi_n^\alpha,$$

where $\alpha = s - 2n - 1$.

Proof. If σ_s leaves \mathcal{W}_ψ invariant, then

$$\begin{aligned}\sigma_s(g)F(z) &= \int_{\mathbb{C}^+} \left(\frac{|cw+d|}{cw+d} \right)^s F(g(w))k_\psi(z, w)d\mu(w) \\ &= \int_{\mathbb{C}^+} F(w) \left(\frac{-cw+a}{|-cw+a|} \right)^s k_\psi(z, g^{-1}(w))d\mu(w).\end{aligned}$$

On the other hand, one has

$$\sigma_s(g)F(z) = \int_{\mathbb{C}^+} F(w) \left(\frac{|cz+d|}{cz+d} \right)^s k_\psi(g(z), w)d\mu(w).$$

Combining the two preceding identities, it follows that

$$\left(\frac{|cz+d|}{cz+d} \right)^s k_\psi(g(z), w) = \left(\frac{-cw+a}{|-cw+a|} \right)^s k_\psi(z, g^{-1}(w)), \quad (4.6)$$

for every $g \in PSL(2, \mathbb{R})$ and every $z, w \in \mathbb{C}^+$. Consequently, setting $g = g_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, $\theta \in [0, 2\pi)$, and $w = i$ yields the necessary condition

$$\left(\frac{|z \sin \theta + \cos \theta|}{z \sin \theta + \cos \theta} \right)^s W_\psi \psi(g_\theta(z)) = e^{i\theta s} W_\psi \psi(z).$$

Now, multiply both sides by $e^{i\theta s} \text{Im}(z)^{-s/2}$ to show

$$\left(\frac{e^{i\theta}}{z \sin \theta + \cos \theta} \right)^s W_\psi^s \psi(g_\theta(z)) = W_\psi^s \psi(z),$$

where $W_\psi^s f(z) := \text{Im}(z)^{-s/2} W_\psi f(z)$. Thus, $W_\psi^s f(z)$ is $SO(2)$ -invariant in the sense of [24]. From [24, Theorem 2.1] it then follows that ψ is given in the Fourier domain by

$$\widehat{\psi}(\xi) = C \xi^{\frac{s-1-2n}{2}} e^{-\xi} L_n^{s-1-2n}(2\xi), \quad \xi > 0,$$

for some $C \in \mathbb{C} \setminus \{0\}$, and $n \in \mathbb{N}_0$ with $n < \frac{s-1}{2}$. If we set $\alpha = s - 1 - 2n$ and solve for s , then we get $\psi = C\psi_n^\alpha$ together with $\sigma_s = \tau_n^\alpha$. \square

4.2. Some preparatory results

Let now Γ be a Fuchsian group with fundamental domain $\Omega \subset \mathbb{C}^+$, and ψ_n^α as defined in Section 3.3. Note that every Fuchsian group is ICC [4].

With a slight abuse of notation we now consider $\tau_n^\alpha : \Gamma \rightarrow \mathcal{U}(\mathcal{W}_{\psi_n^\alpha})$ to be the restriction of τ_n^α to Γ and $\mathcal{W}_{\psi_n^\alpha}$ which is a projective representation of Γ .

In the next Lemma, we verify the conditions of Proposition 3.4.

Lemma 4.3. Let $\Gamma \subset PSL(2, \mathbb{R})$ be a Fuchsian group with fundamental domain $\Omega \subset \mathbb{C}^+$, $P_{\mathcal{W}_{\psi_n^\alpha}} : L^2(\mathbb{C}^+, \mu) \rightarrow \mathcal{W}_{\psi_n^\alpha}$ be the projection onto $\mathcal{W}_{\psi_n^\alpha}$, and

$$Q_\Omega : L^2(\mathbb{C}^+, \mu) \rightarrow L^2(\mathbb{C}^+, \mu), \quad Q_\Omega f(z) := \chi_\Omega(z)f(z),$$

where χ_Ω denotes the characteristic function of Ω . Then $\tau_n^\alpha(\gamma)Q_\Omega\tau_n^\alpha(\gamma^{-1}) \perp Q_\Omega$, for every $\gamma \in \Gamma - \{e\}$,

$$\sum_{\gamma \in \Gamma} \tau_n^\alpha(\gamma)Q_\Omega\tau_n^\alpha(\gamma^{-1}) = I_{L^2(\mathbb{C}^+, \mu)},$$

and $P_{\mathcal{W}_{\psi_n^\alpha}}$ commutes with $\tau_n^\alpha(\gamma)$ for every $\gamma \in \Gamma$.

Proof. First, we have that $\tau_n^\alpha(\gamma)Q_\Omega\tau_n^\alpha(\gamma^{-1}) = Q_{\gamma^{-1}(\Omega)} \perp Q_\Omega$, for $\gamma \neq e$, since

$$\begin{aligned} \tau_n^\alpha(\gamma)Q_\Omega\tau_n^\alpha(\gamma^{-1})F(z) &= \tau_n^\alpha(\gamma) \left(\chi_\Omega(z) \cdot \left(\frac{|-cz + a|}{-cz + a} \right)^{2n+\alpha+1} F(\gamma^{-1}(z)) \right) \\ &= \chi_\Omega(\gamma(z)) \cdot F(z) \\ &= \chi_{\gamma^{-1}(\Omega)}(z)F(z) = Q_{\gamma^{-1}(\Omega)}F(z), \end{aligned}$$

and $\Omega^0 \cap \gamma^{-1}(\Omega)^0 = \emptyset$. The second property then immediately follows as

$$\sum_{\gamma \in \Gamma} \tau_n^\alpha(\gamma)Q_\Omega\tau_n^\alpha(\gamma^{-1}) = \sum_{\gamma \in \Gamma} Q_{\gamma^{-1}(\Omega)} = \sum_{\gamma \in \Gamma} \chi_{\gamma^{-1}(\Omega)} = \chi_{\mathbb{C}^+} = I_{L^2(\mathbb{C}^+, \mu)}.$$

Repeating the arguments that we used to derive (4.5) for $F = P_{\mathcal{W}_\psi}G$, $G \in L^2(\mathbb{C}^+, \mu)$, then shows that $P_{\mathcal{W}_{\psi_n^\alpha}}\tau_n^\alpha(\gamma) = \tau_n^\alpha(\gamma)P_{\mathcal{W}_{\psi_n^\alpha}}$. \square

Combining this with Proposition 3.4, we obtain

$$\dim_{v_{N_\omega}(\Gamma)} \mathcal{W}_{\psi_n^\alpha} = \dim_{v_{N_\omega}(\Gamma)} P_{\mathcal{W}_{\psi_n^\alpha}} L^2(\mathbb{C}^+, \mu) = \text{trace}_{B(L^2(\mathbb{C}^+, \mu))}(Q_\Omega P_{\mathcal{W}_{\psi_n^\alpha}} Q_\Omega),$$

which enables us to compute the von Neumann dimension of the wavelet spaces in terms of the trace of the trace class integral operator $Q_\Omega P_{\mathcal{W}_{\psi_n^\alpha}} Q_\Omega$.

Proposition 4.4. Let Γ, Ω, Q_Ω and $P_{\mathcal{W}_{\psi_n^\alpha}}$ as in Lemma 4.3. Then

$$\text{trace}_{B(L^2(\mathbb{C}^+, \mu))}(Q_\Omega P_{\mathcal{W}_{\psi_n^\alpha}} Q_\Omega) = |\Omega| \frac{\|\psi_n^\alpha\|_2^2}{C_{\psi_n^\alpha}}.$$

Proof. First, observe that $\tilde{P} := Q_\Omega P_{\mathcal{W}_{\psi_n^\alpha}} Q_\Omega$ is an integral operator with integral kernel

$$K_n^\alpha(z, w) := \chi_\Omega(z)k_{\psi_n^\alpha}(z, w)\chi_\Omega(w).$$

If we show that \tilde{P} is trace class, then the result follows from

$$\text{trace}_{B(\mathcal{H})}(\tilde{P}) = \int_{\mathbb{C}^+} K_n^\alpha(z, z) d\mu(z) = \int_{\Omega} k_{\psi_n^\alpha}(z, z) d\mu(z) = |\Omega| \frac{\|\psi_n^\alpha\|_2^2}{C_{\psi_n^\alpha}}.$$

Note that the operator $Q_\Omega P_{\mathcal{W}_{\psi_n^\alpha}}$ is Hilbert-Schmidt since

$$\begin{aligned} \|Q_\Omega P_{\mathcal{W}_{\psi_n^\alpha}}\|_{\mathcal{HS}}^2 &= \int_{\mathbb{C}^+} \int_{\mathbb{C}^+} |\chi_\Omega(z) k_{\psi_n^\alpha}(z, w)|^2 d\mu(w) d\mu(z) \\ &= \int_{\Omega} \int_{\mathbb{C}^+} \frac{1}{C_{\psi_n^\alpha}} |\langle \pi(z) \psi_n^\alpha, \pi(w) \psi_n^\alpha \rangle|^2 d\mu(w) d\mu(z) \\ &= \int_{\Omega} \|\pi(z) \psi_n^\alpha\|_2^2 d\mu(z) = |\Omega| \|\psi_n^\alpha\|_2^2 < \infty. \end{aligned}$$

It thus follows, that \tilde{P} is trace class as the product of two Hilbert-Schmidt operators. \square

With everything in place, the proofs of our main results take only a few lines.

4.3. Proof of Theorem 2.1

If $\{\tau_n^\alpha(\gamma)F\}_{\gamma \in \Gamma}$ is a frame for $\mathcal{W}_{\psi_n^\alpha}$, then, as shown in [27, Section 5], this requires Ω to be compact. Moreover, as $\{\tau_n^\alpha(\gamma)F\}_{\gamma \in \Gamma}$ is in particular complete, we have that $vN_\omega(\Gamma)F$ is dense in $\mathcal{W}_{\psi_n^\alpha}$. In other words, F is a cyclic vector for $vN_\omega(\Gamma)$. Theorem 3.1 thus shows that

$$\dim_{vN_\omega(\Gamma)} \mathcal{W}_{\psi_n^\alpha} \leq 1.$$

On the other hand, Proposition 4.4 yields

$$\dim_{vN_\omega(\Gamma)} \mathcal{W}_{\psi_n^\alpha} = \frac{\|\psi_n^\alpha\|_2^2}{C_{\psi_n^\alpha}} |\Omega|.$$

We conclude that

$$|\Omega| \leq \frac{C_{\psi_n^\alpha}}{\|\psi_n^\alpha\|_2^2}.$$

If $\{\tau_n^\alpha(\gamma)F\}_{\gamma \in \Gamma}$ is a Riesz sequence for $\mathcal{W}_{\psi_n^\alpha}$, then by [41, Proposition 5.2] (the statement assumes Kleppner's condition, which is implied by the ICC condition), F is separating for $vN_\omega(\Gamma)$. Theorem 3.2 now shows that

$$\dim_{vN_\omega(\Gamma)} \mathcal{W}_{\psi_n^\alpha} \geq 1.$$

We then conclude as before. \square

4.4. Proof of Corollary 2.3

By the orthogonality relation (3.2) we have that $\{\varphi_\gamma\}_{\gamma \in \Gamma}$ is a frame (resp. Riesz sequence) for $\mathcal{H}^2(\mathbb{C}^+)$ if and only if $\{W_{\psi_n^\alpha} \varphi_\gamma\}_{\gamma \in \Gamma}$ is a frame (resp. Riesz sequence) for $\mathcal{W}_{\psi_n^\alpha}$. Using the definition of φ_λ in (2.4) to calculate $W_{\psi_n^\alpha} \varphi_\gamma$ we find

$$\begin{aligned} W_{\psi_n^\alpha} \varphi_\gamma(z) &= \int_{\mathbb{C}^+} \Phi(w) \left(\frac{cw + d}{|cw + d|} \right)^{2n+\alpha+1} \langle \pi(\gamma(w)) \psi_n^\alpha, \pi(z) \psi_n^\alpha \rangle d\mu(w) \\ &= \left(\frac{|-cz + a|}{-cz + a} \right)^{2n+\alpha+1} \int_{\mathbb{C}^+} \Phi(w) \langle \pi(w) \psi_n^\alpha, \pi(\gamma^{-1}(z)) \psi_n^\alpha \rangle d\mu(w) \\ &= \left(\frac{|-cz + a|}{-cz + a} \right)^{2n+\alpha+1} P_{\mathcal{W}_{\psi_n^\alpha}} \Phi(\gamma^{-1}(z)) \\ &= \tau_n^\alpha(\gamma^{-1}) P_{\mathcal{W}_{\psi_n^\alpha}} \Phi(z), \end{aligned}$$

by (4.4). Hence, $P_{\mathcal{W}_{\psi_n^\alpha}} \Phi$ is a cyclic (resp. separating) vector for $\mathcal{W}_{\psi_n^\alpha}$ and the result follows from Theorem 2.1. \square

4.5. Proof of Corollary 2.4

If $\{\pi(\gamma(z)) \psi_n^\alpha\}_{\gamma \in \Gamma}$ is a wavelet frame (resp. Riesz sequence) for $\mathcal{H}^2(\mathbb{C}^+)$, then $\left\{ \overline{k_{\psi_n^\alpha}(\gamma(z), \cdot)} \right\}_{\gamma \in \Gamma}$ generates a frame (resp. Riesz sequence) for $\mathcal{W}_{\psi_n^\alpha}$. Moreover, since Γ is a subgroup of $PSL(2, \mathbb{R})$, it follows from (4.4) that

$$\left\{ \tau_n^\alpha(\gamma) \overline{k_{\psi_n^\alpha}(z, \cdot)} \right\}_{\gamma \in \Gamma} = \left\{ \left(\frac{cz + d}{|cz + d|} \right)^{2n+\alpha+1} \overline{k_{\psi_n^\alpha}(\gamma(z), \cdot)} \right\}_{\gamma \in \Gamma}.$$

As the factor in front of $\overline{k_{\psi_n^\alpha}(\gamma(z), w)}$ is unimodular, and independent of w , it follows that $\overline{k_{\psi_n^\alpha}(z, \cdot)}$ is a cyclic (resp. separating) vector for $vN_\omega(\Gamma)$ and the first and third statements follow from Theorem 2.1. For the second statement, assume that $|\Omega| > C_{\psi_n^\alpha} / \|\psi_n^\alpha\|_2^2$, then there exist no cyclic vector in $\mathcal{W}_{\psi_n^\alpha}$. In particular, there exist $F \in \mathcal{W}_{\psi_n^\alpha}$ that is orthogonal to $vN_\omega(\Gamma) \overline{k_{\psi_n^\alpha}(z, \cdot)}$, i.e. for every $\gamma \in \Gamma$

$$0 = \langle F, \tau_n^\alpha(\gamma) \overline{k_{\psi_n^\alpha}(z, \cdot)} \rangle = \langle \tau_n^\alpha(\gamma^{-1}) F, \overline{k_{\psi_n^\alpha}(z, \cdot)} \rangle = \tau_n^\alpha(\gamma^{-1}) F(z),$$

which in turn implies $F(\gamma(z)) = 0$ for every $\gamma \in \Gamma$. \square

4.6. Proof of Theorem 2.8

For the choice $\alpha = 2(B - n) - 1$ we have that the exponent of the phase factor in (3.10) is given by $2B$ (i.e., it is independent of $n = 0, 1, \dots, \lfloor B - \frac{1}{2} \rfloor$). By Lemma 4.1, each

space $A_n^B(\mathbb{C}^+, \mu)$ is invariant under the projective representation σ_{2B} . Consequently, the orthogonal sum $\mathbf{A}_N^B(\mathbb{C}^+, \mu) = \bigoplus_{n=0}^{N-1} A_n^B(\mathbb{C}^+, \mu)$ is also invariant under σ_{2B} . One can then proceed in a similar fashion as in the proofs of Theorem 2.1. In particular, the reproducing kernel of $\mathbf{A}_N^B(\mathbb{C}^+, \mu)$ is given by $\mathbf{k}_N^B = \sum_{k=0}^{N-1} k_{\psi_n^{2(B-n)-1}}$ with diagonal

$$\mathbf{k}_N^B(z, z) = \sum_{n=0}^{N-1} k_{\psi_n^{2(B-n)-1}}(z, z) = \sum_{n=0}^{N-1} \frac{2(B-n)-1}{2} = \frac{N(2B-N)}{2}, \quad z \in \mathbb{C}^+.$$

It remains to note that (as in Lemma 4.3) the projection $P_{\mathbf{A}_N^B(\mathbb{C}^+, \mu)}$ commutes with $\sigma_{2B}(\gamma)$, for every $\gamma \in \Gamma$, which implies (using the same steps as in the proof of Proposition 4.4) that

$$\dim_{v_{N_\omega(\Gamma)} \mathbf{A}_N^B(\mathbb{C}^+, \mu)} = |\Omega| \mathbf{k}_N^B(i, i) = |\Omega| \frac{N(2B-N)}{2}. \quad \square$$

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