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Strictly positive definite non-isotropic kernels on two-point homogeneous manifolds: the asymptotic approach

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Abstract
We present sufficient conditions for a family of positive definite kernels on a compact two-point homogeneous space to be strictly positive definite based on their expansion in eigenfunctions of the Laplace–Beltrami operator. We also present a characterisation of this kernel class. The family analyzed is a generalization of the isotropic kernels and the case of a real sphere is analyzed in details.

Keywords Strictly positive definite kernels · Non-isotropic kernels · Covariance functions · Two point compact homogeneous spaces · Spheres

Mathematics Subject Classification 33C45 · 41A05 · 41A58 · 42A82 · 42C10 · 43A85 · 43A90

1 Introduction

During the last five years, there has been a tremendous number of publications stating new results on positive definite kernels on spheres, see for example [1, 2] and reference therein and a smaller number studying other manifolds [3–10] including two-point homogeneous manifolds, tori, Euclidean spaces and products of these. Most of the results focus on isotropic positive definite kernels, which are kernels that only depend on the geodesic distance of their arguments. Isotropic kernels are used in approximation theory, where they are often referred to as spherical or radial basis functions [2] and are for example applied in geostatistics [11]. They are also of importance in statistics where they occur as correlation functions of Gaussian random fields [12].

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There are few results on non-isotropic kernels among them the axially-symmetric kernels discussed in [13].

This publication will characterize (strictly) positive definite kernels with a specific series representation, which are not necessary isotropic but include the isotropic kernels as special case. We prove the result for all two-point homogeneous manifolds but study the specific implications in detail for the case of the d-dimensional sphere. The results are the first, known to the authors, that show that strict positivity definiteness can be obtained with significantly less positive coefficients in the series expansion compared to the isotropic case.

We will briefly summarize necessary definitions in the first section and prove the abstract result for kernels on two-point homogeneous spaces in the second section. For the d-sphere we derive explicit conditions for the strict positive definiteness of convolutional kernels and in the process prove a new estimate for the absolute value of spherical harmonics, these result are given in section three.

1.1 Definitions and notation

We assume that the manifold $\mathbb{M}$ is two-point homogeneous. Therefore it is isomorphic to one of the following five cases as proven in [14],

$$\mathbb{M} = \mathbb{S}^{d-1}, \quad \mathbb{M} = P^{d-1}(\mathbb{R}), \ \mathbb{M} = P^{d-1}(\mathbb{C}), \ \mathbb{M} = P^{d-1}(H), \ \mathbb{M} = P^{16}(Cay).$$

From [15] we take the following well established results. There exists an orthonormal base of $L^2(\mathbb{M})$ such that each function in the basis $f_{j,k}$ is smooth and

$$\triangle f_{j,k} = \lambda_k f_{j,k}, \quad k \in \mathbb{N}, \ j = 1, \ldots, m_k$$

where

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_\ell \leq, \ldots, \quad \lim_{\ell \to \infty} \lambda_\ell = +\infty,$$

are the distinct eigenvalues of the Laplace-Beltrami operator on $\mathbb{M}$, denoted by $\triangle$, and m_k is the dimension of the eigenspace H_k corresponding to λ_k.

The metric in $\mathbb{M}$ is $d(\xi, \zeta) = \arccos \langle \xi, \zeta \rangle$ when $\mathbb{M}$ is a sphere, otherwise $d(\xi, \zeta) = 2 \arccos |\langle \frac{\tilde{\xi}}{|\tilde{\xi}|}, \frac{\tilde{\zeta}}{|\tilde{\zeta}|} \rangle|$, where $\tilde{\xi}, \tilde{\zeta}$ are arbitrary class representatives. The famous addition formula reads

$$\sum_{j=1}^{m_k} f_{j,k}(\xi) \overline{f_{j,k}(\zeta)} = c_k P_k^{(\alpha,\beta)} \left(\cos \left(d(\xi, \zeta) \right) \right), \quad \xi, \zeta \in \mathbb{M}, \tag{1}$$

where

$$c_k = \frac{\Gamma(\beta+1)(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(\alpha+\beta+2)\Gamma(k+\beta+1)}$$

and throughout $P_k^{(\alpha,\beta)}$, denotes the Jacobi polynomials normalized by

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)}. \tag{2}$$

The coefficients satisfy $\alpha = \frac{d-3}{2}$, β takes one of the values $(d-3)/2,\ -1/2,\ 0,\ 1,\ 3,$ in the order of the five manifolds being studied.

Definition 1 A kernel $K : \mathbb{M} \times \mathbb{M} \to \mathbb{C}$ is called positive definite on $\mathbb{M}$ if the matrix

$$K_\Xi = \{K(\xi,\zeta)\}_{\xi,\zeta\in\Xi}$$

is positive semi-definite on $\mathbb{C}^{|\Xi|}$ for arbitrary finite sets of distinct points $\Xi \subset \mathbb{M}$.

The kernel is strictly positive definite if K_Ξ is a positive definite matrix on $\mathbb{C}^{|\Xi|}$ for arbitrary finite sets of distinct points Ξ.

Note that if K is positive definite then it must be Hermitian, that is, $K(\xi,\zeta) = \overline{K(\zeta,\xi)}$ for every $\xi,\zeta \in \mathbb{M}$.

For this paper we focus on kernels possessing a series representation

$$K(\xi,\zeta) = \sum_{k=0}^{\infty}\sum_{j=1}^{m_k} d_{j,k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)}, \quad d_{j,k} \in \mathbb{C}, \tag{3}$$

with the prior defined basis. This kernel class was for example studied in [16] and [17], where error estimates were discussed for kernels where all coefficients are positive. The above defined kernel class include the real valued isotropic kernels having the form

$$K(\xi,\zeta) = \sum_{k=0}^{\infty} b_k \sum_{j=1}^{m_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)}, \quad b_k \in \mathbb{R}, \tag{4}$$

for the case where $\mathbb{M} = \mathbb{S}^{d-1}$ this expansion is referred to as d-Schoenberg series and well studied for example in [18], but also non-isotropic kernels.

2 Characterizing strict positive definiteness

We note that a kernel of the form (3) is positive definite on $\mathbb{M}$ if and only if $d_{j,k} \geq 0$ for all $k \in \mathbb{N}$, $j = 1,\ldots,m_k$ as proven in [17] Theorem 2.1 and assume positive definiteness of the kernel throughout the rest of the paper. We define for any such kernel the set of positive coefficients as

$$\mathcal{F} := \{(j,k) :\ d_{j,k} > 0\}$$

and additionally the sets

$$A_k := \{j : (j,k) \in \mathcal{F}\}, \quad N := \{k : k \in \mathbb{Z}_+ \wedge \exists d_{j,k} \neq 0\}.$$

Theorem 1 *For a continuous positive definite kernel of the form* (3) *the following properties are equivalent:*

1. *K is strictly positive definite on $\mathbb{M}$.*
2. *For any finite set of distinct points Ξ, $\sum_{\xi \in \Xi} c_\xi f_{j,k}(\xi) = 0$, for all $(j,k) \in \mathcal{F}$ implies $c_\xi = 0$ for all $\xi \in \Xi$.*
3. *For any finite set of distinct points $\Xi \subset \mathbb{M}$,*

$$\sum_{\xi \in \Xi} c_\xi \sum_{j \in A_k} f_{j,k}(\zeta)\overline{f_{j,k}(\zeta)} = 0, \ \forall k \in N, \zeta \in \mathbb{M},$$

implies $c_\xi = 0, \ \forall \xi \in \Xi$.

Proof We start with the implication from (1) to (2) and prove by contradiction. Assume (2) does not hold and there exists a set Ξ and coefficients $c_\xi \in \mathbb{C}$ for which

$$\sum_{\xi \in \Xi} c_\xi \sum_{j \in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} = 0, \ \forall k \in N, \zeta \in \mathbb{M}.$$

Then we immediately deduce

$$\sum_{\xi,\zeta \in \Xi} c_\xi \overline{c_\zeta} K(\xi,\zeta) = 0,$$

which contradicts (1).

We prove that (2) implies (1) by contradiction and assume that K is continuous positive definite but not strictly positive definite. If K is not strictly positive definite there exists a nonempty set of distinct point Ξ and coefficients $c_\xi \in \mathbb{C}$ not all zero with

$$\sum_{\xi,\zeta \in \Xi} c_\xi \overline{c_\zeta} K(\xi,\zeta) = 0.$$

This is equivalent to

$$\sum_{k=0}^{\infty} \sum_{j \in A_k} d_{j,k} f_{j,k} \overline{f_{j,k}} = 0, \tag{5}$$

where $f_{j,k} = \sum_{\xi \in \Xi} c_\xi f_{j,k}(\xi)$ and the sums are interchangeable because of the continuity of K.

Since we know that all the summands are non negative since the $d_{j,k}$ are non-negative, the overall sum can only be zero if all summands are. For the indices $(j,k) \in \mathcal{F}$ this implies $f_{j,k} = 0$. We have proven that (2) can not hold because at least one c_ξ was non zero.

Now we prove the equivalence of (2) and (3). We note that

$$\sum_{\xi \in \Xi} c_\xi \sum_{j \in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} = \sum_{j \in A_k} \left(\sum_{\xi \in \Xi} c_\xi f_{j,k}(\xi) \right) \overline{f_{j,k}(\zeta)}, \qquad \forall \zeta \in \mathbb{M}.$$

Since the eigenfunctions are linearly independent the last is zero if and only if

$$\sum_{\xi \in \Xi} c_\xi f_{j,k}(\xi) = 0$$

for all $j \in A_k$, and all $k \in N$. □

The last theorem proves that strict positive definiteness is independent of the precise value of the $d_{j,k}$ but is only depending on the set $\mathcal{F}$. This justifies that we will distinguish between sets which induce strict positive definiteness and sets that do not.

The existing results for isotropic kernels on compact two point homogeneous manifolds allow us to derive the following conditions for sets that induce strict positive definiteness:

Corollary 2 *Let $\mathbb{M} \neq \mathbb{S}^{d-1}$, with $\alpha > \beta$ be a two-point homogeneous manifold and K a continuous kernel of the form* (3)

- *For K to be strictly positive definite it is necessary that N includes infinitely many integers.*
- *For K to be strictly positive definite it is sufficient that*

$$\mathcal{L} = \{k : d_{j,k} > 0,\ \forall j = 1, \ldots, m_k\}$$

includes infinitely many integers.

Proof The statements follow from Theorem 1 together with the characterisation of strictly positive definite isotropic kernels on two-point homogeneous spaces in Theorem 3.1 and Theorem 3.3 of [3]. □

The matching result for the case of the d-sphere has already been established in [19] Theorem 4. Since the original proofs of the results for isotropic kernels used only the limiting behaviour of the Jacobi-polynomials for $k \to \infty$, it is easy to deduce that removing a finite number of coefficients from $\mathcal{F}$ will not change whether $\mathcal{F}$ induces strictly positive definite. In the next section we prove that removing infinite numbers of coefficients is also possible under certain conditions.

3 The asymptotic approach

To establish condition which require less positive coefficients, we summarize these known properties of the Jacobi-polynomials from Lemma 2.2 of [3].

Lemma 3 *1.* $P_k^{\alpha,\beta}(-t) = (-1)^k P_k^{\beta,\alpha}(t)$,
2. $\lim_{k\to\infty} P_k^{\alpha,\beta}(t)[P_k^{\alpha,\beta}(1)]^{-1} = 0, \quad \forall t \in (-1, 1)$,
3. $\lim_{k\to\infty} P_k^{\beta,\alpha}(1)[P_k^{\alpha,\beta}(1)]^{-1} = 0$ *if* $\alpha > \beta$.

Theorem 4 *Let $N \subset \mathbb{Z}_+$ be an infinite set and $A_k \subset \{1, \ldots, m_k\}$, $k \in N$, such that*

$$\lim_{k\in N} \frac{c_k^{-1} \sum_{j\in A_k^c} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)}}{P_k^{\alpha,\beta}(1)} = 0, \quad \forall \xi, \zeta \in \mathbb{M}, \tag{6}$$

where $A_k^c := \{1, \ldots, m_k\} \backslash A_k$. Then $\mathcal{F} := \{(j,k) : k \in N, j \in A_k\}$ induces strict positive definiteness whenever $\mathbb{M} \neq \mathbb{S}^{d-1}$. In the case $\mathbb{M} = \mathbb{S}^{d-1}$ the same relation is valid under the additional requirement that $E := N \cap 2\mathbb{Z}_+$ and $O := N \cap (2\mathbb{Z}_+ + 1)$ are infinite.

Proof We consider the kernel

$$K(\xi, \zeta) = \sum_{k=0}^{\infty} a_k \sum_{j\in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)}, \quad a_k > 0. \tag{7}$$

By Theorem 1, proving that K is strictly positive definite is sufficient for the proof of the theorem. We see that K is not strictly positive definite if and only if there exists a set of distinct points $\Xi \in \mathbb{M}$ and coefficients $c_\xi \in \mathbb{C}$ such that

$$\sum_{\xi\in\Xi} c_\xi \sum_{j\in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} = 0, \quad \forall k \in N, \ j \in A_k, \ \zeta \in \mathbb{M} \tag{8}$$

and at least one $c_\xi \neq 0$, this follows from (3) of Theorem 1. If $\mathbb{M} \neq \mathbb{S}^{d-1}$, by Theorem 3 we obtain that for $\xi, \zeta \in \mathbb{M}$

$$\begin{aligned} &\lim_{k\in N} \frac{c_k^{-1}}{P_k^{\alpha,\beta}(1)} \sum_{j\in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} \\ &= \lim_{k\in N} = \frac{c_k^{-1}}{P_k^{\alpha,\beta}(1)} \left(P_k^{\alpha,\beta}(\cos(d(\xi,\zeta)) - \sum_{j\in A_k^c} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} \right) = \delta_{\xi,\zeta}. \end{aligned}$$

Hence, if we apply this relation to Eq. 8, we conclude that

$$0 = \lim_{k\in N} \sum_{\xi\in\Xi} c_\xi \sum_{j\in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} c_\zeta, \quad \zeta \in \Xi.$$

For the case of the sphere, the same arguments hold for all points $\xi \in \Xi$ with $-\xi \notin \Xi$. We assume without loss of generalization that a point $\xi \in \Xi$ if and only if $-\xi \in \Xi$, as it does not affect the result we aim to prove and simplifies the terminology.

Again, by Theorem 3 we obtain that for $\xi, \zeta \in \mathbb{S}^{d-1}$

$$\lim_{k\in E} \frac{c_k^{-1}}{P_k^{\alpha,\beta}(1)} \sum_{j\in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)}$$
$$= \lim_{k\in E} \frac{c_k^{-1}}{P_k^{\alpha,\beta}(1)} \left(P_k^{\alpha,\beta}(\cos(d(\xi,\zeta)) - \sum_{j\in A_k^c} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} \right) = \delta_{\xi,\zeta} + \delta_{\xi,-\zeta}.$$

Similarly,

$$\lim_{k\in O} \frac{c_k^{-1}}{P_k^{\alpha,\beta}(1)} \sum_{j\in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} = \delta_{\xi,\zeta} - \delta_{\xi,-\zeta}.$$

If we apply both relations to Eq. 8, we conclude that for every $\zeta \in \Xi$

$$0 = \lim_{k\in E} \sum_{\xi\in\Xi} c_\xi \sum_{j\in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} = c_\zeta + c_{-\zeta},$$
$$0 = \lim_{k\in O} \sum_{\xi\in\Xi} c_\xi \sum_{j\in A_k} f_{j,k}(\xi)\overline{f_{j,k}(\zeta)} = c_\zeta - c_{-\zeta},$$

which implies that $c_\zeta = c_{-\zeta} = 0$. □

The theorem shows that precise implications for the structure of A_k will depend on the eigenfunctions of the manifold. The derivation of practical conditions is non trivial but possible, as will be evident from the discussion of the sphere.

4 The asymptotic approach for the *d*-sphere

Now we focus on the d-sphere, which is the case where $\alpha = \beta = (d-3)/2$ and we present sufficient conditions for which the key assumptions of Theorem 4 are valid. The sufficient condition will be given in terms of the asymptotic behaviour of $|A_k^c|$ for $k \in E$ and O.

Therefore the main challenge of this section is to derive and estimates for

$$|f_{j,k}(\xi)\overline{f_{j,k}(\zeta)}|, \quad \forall \xi, \zeta \in \mathbb{S}^{d-1}$$

which holds for all $(j,k) \in A_k^c$.

4.1 The eigenfunctions on the sphere

On the sphere the eigenfunctions corresponding to the eigenvalues $\lambda_k = k\,(k+d-1)$ are spherical harmonics and the number of eigenfunctions corresponding to the eigenvalue λ_k is denoted by $N_{k,d}$. The numbers are given by $N_{0,d} = 1$,

$$N_{k,d} = \frac{(2k+d-2)\,(k+d-3)!}{k!\,(d-2)!}.$$

For $\xi \in \mathbb{S}^{d-1}$, with polar coordinate representation $(\theta_1, \ldots, \theta_{d-1})^T$ satisfying

$$\begin{aligned}
\xi_1 &= \cos(\theta_{d-1}) \\
\xi_2 &= \sin(\theta_{d-1})\cos(\theta_{d-2}) \\
&\vdots \\
\xi_{d-1} &= \sin(\theta_{d-1})\sin(\theta_{d-2})\cdots\cos(\theta_1) \\
\xi_d &= \sin(\theta_{d-1})\sin(\theta_{d-2})\cdots\sin(\theta_1),
\end{aligned}$$

where $\theta_1 \in [0, 2\pi)$ and the others $\theta_j \in [0, \pi]$. The spherical harmonics of degree α_{d-1} can explicitly be given by

$$Y_{\alpha_1,\ldots,\alpha_{d-1}}(\xi) = \frac{1}{\sqrt{2\pi}} e^{i\alpha_1\theta_1} \prod_{j=2}^{d-1} {}_j\tilde{\mathsf{P}}_{\alpha_j}^{\alpha_{j-1}}\left(\theta_j\right), \tag{9}$$

where $\alpha_1, \ldots, \alpha_{d-1}$ are integers satisfying

$$\alpha_{d-1} \geq \cdots \geq |\alpha_1|$$

and

$${}_j\tilde{\mathsf{P}}_L^{\ell}(\theta) = \left(\frac{\pi\Gamma(j/2)}{\Gamma((j+1)/2)}\right)^{1/2} {}_jc_L^{\ell}\,(\sin(\theta))^{-(2-j)/2}\,\mathsf{P}_{L+(j-2)/2}^{-(\ell+(j-2)/2)}(\cos(\theta)), \tag{10}$$

where P_ν^μ are the associated Legendre functions and

$${}_jc_L^{\ell} := \left(\frac{2L+j-1}{2}\frac{(L+\ell+j-2)!}{(L-\ell)!}\right)^{1/2}.$$

The formula is taken from [20], Equation (2.5) and α_{d-1} is the degree of the spherical harmonic, also, it is after a reparametrization a consequence of Theorem 1.5.1 in [21]. A small difference is that we use

$$\frac{1}{Vol(\mathbb{S}^{d-1})}\int_{\mathbb{S}^{d-1}} Y_\alpha(x)Y_\beta(x)d\sigma(x) = \delta_{\alpha,\beta}$$

while [20] uses $1/2\pi$ instead of $1/Vol(\mathbb{S}^{d-1})$, which is solved by adding the first constant in the definition of ${}_j\tilde{\mathsf{P}}_L^{\ell}$. The spherical harmonic is an eigenfunction corresponding to eigenvalue $\lambda_{\alpha_{d-1}}$. We define the index set corresponding to the order k as

$$\tau_k^{d-1} := \{\alpha = (\alpha_1, \ldots, \alpha_{d-1}) \in \mathbb{Z}^{d-1},\ |\alpha_1| \leq \alpha_2 \leq \cdots \leq \alpha_{d-1} = k\}.$$

The spherical harmonics Y_α with $\alpha \in \tau_k^{d-1}$ form an orthonormal basis of H_k and therefore $|\tau_k^{d-1}| = N_{k,d}$.

Since ${}_j\tilde{\mathsf{P}}^{\ell}_L(0) = {}_j\tilde{\mathsf{P}}^{\ell}_L(\pi) = 0$ whenever $\ell > 0$, we have that if $x \in \mathbb{S}^{d-j-2}$ where $1 \le j \le d-2$

$$Y_\alpha((x,0)) = 0, \quad \alpha \in \tau_k^{d-1}, \text{ when } \alpha_j \neq 0.$$

The complement of the set $\{\alpha_j \neq 0\}$ in the case $j = d-2$ only includes the index $\alpha = (0, \ldots, 0, k)$. More generally, we define

$${}_j\tau_k^{d-1} := \{\alpha \in \tau_k^{d-1}, \quad \alpha_j = 0\}.$$

4.2 Estimating the absolute value of the eigenfunctions

In order to use $|A_k^c|$ in (6) we will derive an estimate of

$$|Y_\alpha(\xi)\overline{Y_\alpha(\zeta)}|, \quad \forall \alpha \in \tau_k^{d-1} \backslash {}_j\tau_k^{d-1}$$

which only depends on k and j but as a first step we will prove a new estimate of

$$|Y_\alpha(\xi)|, \quad \forall \xi \in \mathbb{S}^{d-1}, \ \forall \alpha \in \tau_k^{d-1} \backslash {}_j\tau_k^{d-1}.$$

To determine an upper bound for (10) we note that in Theorem 2 in [22] it is proved the following inequality

$$|\mathsf{P}_m^n(\cos\theta)| \le \frac{\Gamma(1/4)(\sin(\theta))^{-1/4}}{\pi} \sqrt{\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}} \frac{1}{m^{1/4}}, \quad m \ge |n|. \tag{11}$$

We further need an inequality for the case of half integer coefficients, $|\mathsf{P}_{m+1/2}^{n+1/2}(\cos\theta)|$. For this we use an estimate for Jacobi polynomials obtained in [23]. We recall the relation between Legendre polynomials and Gegenbauer polynomials (Equation 14.3.21 in [24])

$$\mathsf{P}_\nu^\mu(x) = \frac{2^\mu \Gamma(1-2\mu)\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)\Gamma(1-\mu)(1-x^2)^{\mu/2}} C_{\nu+\mu}^{(\frac{1}{2}-\mu)}(x) \tag{12}$$

and the relation between Gegenbauer polynomials and Jacobi polynomials (Equation 18.7.1 in [24])

$$C_n^\lambda(x) = \frac{\Gamma(2\lambda+n)\Gamma(\lambda+1/2)}{\Gamma(2\lambda)\Gamma(\lambda+1/2+n)} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x). \tag{13}$$

Thus obtaining that for $\nu \ge \mu \ge 0$ with $\nu - \mu \in \mathbb{N}$:

$$\mathsf{P}_\nu^{-\mu}(x) = \frac{2^{-\mu}\Gamma(1+2\mu)\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)\Gamma(1+\mu)(1-x^2)^{-\mu/2}} C_{\nu-\mu}^{(\frac{1}{2}+\mu)}(x)$$

$$= \frac{2^{-\mu}\Gamma(1+2\mu)\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)\Gamma(1+\mu)(1-x^2)^{-\mu/2}} \frac{\Gamma(1+\mu+\nu)\Gamma(\mu+1)}{\Gamma(2\mu+1)\Gamma(\nu+1)} P_{\nu-\mu}^{(\mu,\mu)}(x)$$
$$= (1-x^2)^{\mu/2} \frac{\Gamma(\nu-\mu+1)}{2^{\mu}\Gamma(\nu+1)} P_{\nu-\mu}^{(\mu,\mu)}(x).$$

By settling $\alpha = \beta$ in Theorem 1.1 in [23], we have that there exists a constant $C \leq 12$ for which

$$(1-x^2)^{\alpha/2+1/4}|P_n^{\alpha,\alpha}(x)| \leq C \frac{2^{\alpha}\Gamma(n+\alpha+1)}{\Gamma(n+1)^{1/2}\Gamma(n+2\alpha+1)^{1/2}}(2n+2\alpha+1)^{-1/4}.$$

Combining the last equations we find

$$|\mathsf{P}_{\nu}^{-\mu}(x)| \leq C(1-x^2)^{-1/4} \frac{\Gamma(\nu-\mu+1)^{1/2}}{\Gamma(\nu+\mu+1)^{1/2}}(2\nu+1)^{-1/4}. \tag{14}$$

Note that this estimate implies an almost similar inequality to the one in Eq. 11 when $\nu = m$ and $\mu = n$. Now, we estimate ${}_j\tilde{\mathsf{P}}_L^{\ell}(\theta)$ defined in (10) based on P_m^{-n} and $\mathsf{P}_{m+1/2}^{-(n+1/2)}$, where $j \geq 2$ and $L \geq \ell \geq 0$. When $j = 2$, we also include the cases $L \geq |\ell|$.

Corollary 5 *For each $j \geq 2$ there exists a function $C_j : (0, \pi) \to \mathbb{R}$ such that*

$$|{}_j\tilde{\mathsf{P}}_L^{\ell}(\cos(\theta))| \leq C_j(\theta)(2L+j-1)^{1/4}, \quad L \geq \ell \geq 0.$$

Further, we can include the points 0 and π if $\ell > 0$.

Proof Indeed, by equation (14) we have that

$$\frac{\Gamma((j+1)/2)^{1/2}}{\pi^{1/2}\Gamma(j/2)^{1/2}}|{}_j\tilde{\mathsf{P}}_L^{\ell}(\cos(\theta))|$$
$$\leq \left(\frac{2L+j-1}{2}\right)^{1/2}(\sin(\theta))^{-(j-2)/2}C\sin(\theta)^{-1/2}(2L+j-1)^{-1/4}$$
$$= \frac{C(\sin(\theta))^{-(j-1)/2}}{\sqrt{2}}(2L+j-1)^{1/4}.$$

Assume $\ell > 0$, by (12) we have that the function ${}_j\tilde{\mathsf{P}}_L^{\ell}(\cos(\theta))$ is a multiple of $(\sin(\theta))^{\ell}C_{L-\ell}^{\ell+(j-1)/2}(\cos(\theta))$, hence ${}_j\tilde{\mathsf{P}}_L^{\ell}(\pm 1) = 0$. □

Lemma 6 *Let $\alpha \in \tau_k^{d-1} \backslash_j \tau_k^{d-1}$ and $j \in \{1, \ldots, d-2\}$, then there exist a function $D : \mathbb{S}^{d-1} \to \mathbb{R}$ independent of α such that for any $\xi \in \mathbb{S}^{d-1}$:*

$$|Y_{\alpha}(\xi)| \leq D(\xi)\sqrt{N_{\alpha_j, j+1}} \prod_{\ell=j+1}^{d-1}(2\alpha_{\ell}+\ell-1)^{1/4}.$$

Proof As a result of the representation of the spherical harmonics in Eq. 9, we find that

$$Y_{\alpha_1,\ldots,\alpha_{d-1}}(\xi) = Y_{\alpha_1,\ldots,\alpha_j}(\xi') \prod_{\ell=j+1}^{d-1} {}_\ell\tilde{P}_{\alpha_\ell}^{\alpha_{\ell-1}}(\theta_\ell)\,, \tag{15}$$

and ξ' is the point on $\mathbb{S}^j$ with polar coordinates $(\theta_1,\ldots,\theta_j)$ and $Y_{\alpha_1,\ldots,\alpha_j}$ is a spherical harmonic in $\mathbb{S}^j$ of degree α_j. By applying the universal estimate

$$|Y_{\alpha_1,\ldots,\alpha_j}(\xi')| \le \sqrt{N_{\alpha_j,j+1}}$$

to the first part, the estimate is a consequence of the summation formula of spherical harmonics [21] Equation 1.2.8 and applying Theorem 5 to the second part of the spherical harmonics we conclude that

$$|Y_\alpha(x)| \le \sqrt{N_{\alpha_j,j+1}} \prod_{\ell=j+1}^{d-1} C_\ell(\theta_\ell)(2\alpha_\ell+\ell-1)^{1/4}.$$

□

We can restate the last theorem to include all indices and expressing the restrictions in terms of the point.

Proposition 7 *Let $\xi \in \mathbb{S}^{d-1}$ and $j_\xi \in \{2,\ldots,d-1\}$ defined as $j_\xi = \max(\{2\} \cup \{j \mid \theta_j \in \{0,\pi\}\})$, then there exist a function $D : \mathbb{S}^{d-1} \to \mathbb{R}$ independent of α such that for any $\xi \in \mathbb{S}^{d-1}$:*

$$|Y_\alpha(\xi)| \le D(\xi)\sqrt{N_{\alpha_{j_\xi},j_\xi+1}} \prod_{\ell=j_\xi+1}^{d-1} (2\alpha_\ell+\ell-1)^{1/4}.$$

4.3 The condition for strict positive definiteness

Now we use the result of Theorem 6 together with Theorem 4. We assume ${}_j\tau_k^{d-1} \subset A_k$ for all $k \in E \cup O$ and deduce

$$|Y_\alpha(\xi)\overline{Y_\alpha(\zeta)}| \le D(\xi)D(\zeta)N_{\alpha_j,j+1} \prod_{\ell=j+1}^{d-1} (2\alpha_\ell+\ell-1)^{1/2}, \quad \forall \alpha \in A_k^c. \tag{16}$$

Thereby we find that Theorem 4 is satisfied if

$$\lim_{k\in E} \sum_{\alpha\in A_k^c} \frac{N_{\alpha_j,j+1}\prod_{\ell=j+1}^{d-1}(2\alpha_\ell+\ell-1)^{1/2}}{c_k P_k^{(d-3)/2,(d-3)/2}(1)} = 0$$

$$\lim_{k\in O}\sum_{\alpha\in A_k^c}\frac{N_{\alpha_j,j+1}\prod_{\ell=j+1}^{d-1}(2\alpha_\ell+\ell-1)^{1/2}}{c_k P_k^{(d-3)/2,(d-3)/2}(1)}=0.$$

For a precise description we use

$$\begin{aligned}N_{k,j+1}&=\frac{(2k+j-1)(k+j-2)!}{k!(j-1)!},\\ c_k&=\frac{\Gamma(\frac{d-3}{2}+1)(2k+d-2)\Gamma(k+d-2)}{\Gamma(d-1)\Gamma(k+\frac{d-1}{2})}\end{aligned}$$

and

$$P_k^{(d-3)/2,(d-3)/2}(1)=\frac{\Gamma((d-3)/2+k+1)}{\Gamma((d-3)/2+1)k!}$$

where the last equation is taken from [24], 18.6.1. Combining these definitions and using $N_{\alpha_j,j+1}\leq N_{k,j+1}$ we can show that

$$\begin{aligned}\frac{N_{k,j+1}}{c_k P_k^{(d-3)/2,(d-3)/2}(1)}&=\frac{\Gamma(d-1)(2k+j-1)\Gamma(k+j-1)}{\Gamma(j)(2k+d-2)\Gamma(k+d-2)}\\ &\leq\frac{\Gamma(d-1)}{\Gamma(j)}3^{d-j-1}\frac{\Gamma(2k+j)}{\Gamma(2k+d-1)}\\ &=\frac{3^{d-j-1}\Gamma(d-1)}{\Gamma(j)}\prod_{\ell=j+1}^{d-1}(2k+\ell-1)^{-1}.\end{aligned}$$

Hence, the conditions of the Theorem 4 are satisfied when

$$\begin{aligned}\lim_{k\in E}\sum_{\alpha\in A_k^c}\prod_{\ell=j+1}^{d-1}\frac{(2\alpha_\ell+\ell-1)^{1/2}}{2k+\ell-1}&=0\\ \lim_{k\in O}\sum_{\alpha\in A_k^c}\prod_{\ell=j+1}^{d-1}\frac{(2\alpha_\ell+\ell-1)^{1/2}}{2k+\ell-1}&=0.\end{aligned}$$

We have proven that:

Corollary 8 *Any continuous kernel*

$$K(\xi,\zeta)=\sum_{k\in E\cup O}^{\infty}\sum_{\alpha\in A_k}d_\alpha Y_\alpha(\xi)\overline{Y_\alpha(\zeta)},\quad d_\alpha>0$$

where $E \subset 2\mathbb{Z}_+$, $O \subset 2\mathbb{Z}_+ + 1$ *are infinite sets and* ${}_j\tau_k^{d-1} \subset A_k \subset \tau_k^{d-1}$ *for a* $j \in 1, \ldots, d-1$ *is strictly positive definite when*

$$\lim_{k \in O} \frac{|A_k^c|}{k^{(d-j-1)/2}} = \lim_{k \in E} \frac{|A_k^c|}{k^{(d-j-1)/2}} = 0. \tag{17}$$

It is obvious that a smaller value of j allows for a larger number of indices to be left out of the set A_k. One should note that, on the other hand, the number of fixed indices that need to be included in A_k, $|{}_j\tau_k^{d-1}|$, is larger for small j. For example $|{}_1\tau_k^{d-1}| = \frac{(k+d-3)!}{(d-3)!k!}$, $|{}_{d-3}\tau_k^{d-1}| = k+1$ and $|{}_{d-2}\tau_k^{d-1}| = 1$.

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References

1. Gneiting, T.: Strictly and non-strictly positive definite functions on spheres. Bernoulli **19**, 1327–1349 (2013)
2. Hubbert, S., Lê Gia, Q.T., Morton, T.M.: Spherical Radial Basis Functions, Theory and Applications. Springer, Berlin (2015)
3. Barbosa, V.S., Menegatto, V.A.: Strictly positive definite kernels on two-point compact homogeneous space. Math. Inequal. Appl. **19**, 743–756 (2016)
4. Barbosa, V.S., Menegatto, V.A.: Strict positive definiteness on products of compact two-point homogeneous spaces. Integral Transform. Spec. Funct. **28**(1), 56–73 (2017)
5. Barbosa, V.S., Gregori, P., Peron, A.P., Porcu, E.: Series expansions among weighted Lebesgue function spaces and applications to positive definite functions on compact two-point homogeneous spaces. J. Math. Anal. Appl. **516**(1), 126487 (2022)
6. Cleanthous, G., Lang, A., Georgiadis, A.G., Porcu, E.: Regularity, continuity and approximation of isotropic Gaussian random fields on compact two-point homogeneous spaces. Stoch. Process. Appl. **130**(8), 4873–4891 (2020)
7. Porcu, E., Cleanthous, G., White, P.: Regularity and approximation of Gaussian random fields evolving temporally over compact two-point homogeneous spaces. TEST **30**(4), 836–860 (2021)
8. Guella, J.C., Menegatto, V.A.: Strictly positive definite kernels on a product of spheres. J. Math. Anal. Appl. **435**, 286–301 (2016)
9. Guella, J.C., Menegatto, V.A.: Strictly positive definite kernels on the torus. Constr. Approx. **46**, 271–284 (2017)
10. Guella, J.C., Menegatto, V.A., Peron, A.P.: Strictly positive definite kernels on a product of spheres II. SIGMA. Symmetry Integr. Geom. Methods Appl. **12**, 103 (2016)

11. Fornberg, B., Flyer, N.: A Primer on Radial Basis Functions with Applications to the Geosciences. SIAM, Philadelphia (2015)
12. Lang, A., Schwab, C.: Isotropic Gaussian random fields on the sphere: regularity, fast simulation and stochastic partial differential equations. Ann. Appl. Probab. **25**, 3047–3094 (2015)
13. Emery, X., Porcu, E., Bissiri, P.G.: A semiparametric class of axially symmetric random fields on the sphere. Stoch. Env. Res. Risk Assess. **33**, 1863–1874 (2019)
14. Wang, H.-C.: Two-point homogeneous spaces. Ann. Math., pp. 177–191 (1952)
15. Giné, E.M.: The addition formula for the eigenfunctions of the Laplacian. Adv. Math. **18**(1), 102–107 (1975)
16. Narcowich, F.J., Schaback, R., Ward, J.D.: Approximation in Sobolev spaces by kernel expansions. J. Approx. Theory **114**(1), 70–83 (2002)
17. Narcowich, F.J.: Generalized Hermite interpolation and positive definite kernels on a Riemannian manifold. J. Math. Anal. Appl. **190**, 165–193 (1995)
18. Berg, C., Porcu, E.: From Schoenberg coefficients to Schoenberg functions. Constr. Approx. **45**, 217–241 (2017)
19. Buhmann, M., Jäger, J.: Strict positive definiteness of convolutional and axially symmetric kernels on d-dimensional spheres. J. Fourier Anal. Appl. **28**(3), 1–25 (2022)
20. Higuchi, A.: Symmetric tensor spherical harmonics on the n-sphere and their application to the de Sitter group SO(N, 1). J. Math. Phys. **28**(7), 1553–1566 (1987)
21. Dai, F., Xu, Y.: Approximation Theory And Harmonic Analysis on Spheres and Balls. Springer, Berlin (2013)
22. Lohöfer, G.: Inequalities for Legendre functions and Gegenbauer functions. J. Approx. Theory **64**(2), 226–234 (1991)
23. Haagerup, U., Schlichtkrull, H.: Inequalities for Jacobi polynomials. Ramanujan J. **33**(2), 227–246 (2014)
24. Olver, F. W. J., Daalhuis, A. B. Olde, Lozier, D. W., Schneider, B. I., Boisvert, R. F., Clark, C. W., Miller, B. R., Saunders. B. V. (eds).: NIST digital library of mathematical functions. http://dlmf.nist.gov/, Release 1.0.15 of 2017-06-01 (2018)