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ORIGINAL PAPER

Spectral properties of generalized Cesàro operators in sequence spaces

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Abstract
The generalized Cesàro operators C_t, for $t \in [0, 1]$, were first investigated in the 1980s. They act continuously in many classical Banach sequence spaces contained in $\mathbb{C}^{\mathbb{N}_0}$, such as ℓ^p, c_0, c, bv_0, bv and, as recently shown in Curbera et al. (J Math Anal Appl 507:31, 2022) [26], also in the discrete Cesàro spaces $ces(p)$ and their (isomorphic) dual spaces d_p. In most cases C_t $(t \neq 1)$ is compact and its spectra and point spectrum, together with the corresponding eigenspaces, are known. We study these properties of C_t, as well as their linear dynamics and mean ergodicity, when they act in certain non-normable sequence spaces contained in $\mathbb{C}^{\mathbb{N}_0}$. Besides $\mathbb{C}^{\mathbb{N}_0}$ itself, the Fréchet spaces considered are $\ell(p+)$, $ces(p+)$ and $d(p+)$, for $1 \leq p < \infty$, as well as the (LB)-spaces $\ell(p-)$, $ces(p-)$ and $d(p-)$, for $1 < p \leq \infty$.

Keywords Generalized Cesàro operator · Compactness · Spectra · Power boundedness · Uniform mean ergodicity · Sequence space · Fréchet space · (LB)-space

Mathematics Subject Classification Primary 46A45 · 47B37; Secondary 46A04 · 46A13 · 47A10 · 47A16 · 47A35

1 Introduction

The (discrete) generalized Cesàro operators C_t, for $t \in [0, 1]$, were first investigated by Rhaly, [52]. The action of C_t from $\omega := \mathbb{C}^{\mathbb{N}_0}$ into itself (with $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$) is given by

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$$C_t x := \left(\frac{t^n x_0 + t^{n-1}x_1 + \ldots + x_n}{n+1}\right)_{n\in\mathbb{N}_0}, \quad x = (x_n)_{n\in\mathbb{N}_0} \in \omega. \tag{1.1}$$

For $t = 0$ note that C_0 is the diagonal operator

$$D_\varphi x := \left(\frac{x_n}{n+1}\right)_{n\in\mathbb{N}_0}, \quad x = (x_n)_{n\in\mathbb{N}_0} \in \omega, \tag{1.2}$$

where $\varphi := \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}_0}$, and for $t = 1$ that C_1 is the classical Cesàro averaging operator

$$C_1 x := \left(\frac{x_0 + x_1 + \cdots + x_n}{n+1}\right), \quad x = (x_n)_{n\in\mathbb{N}_0} \in \omega. \tag{1.3}$$

The spectra of C_1 have been investigated in various Banach sequence spaces. For instance, we mention ℓ^p $(1 < p < \infty)$, [22, 23, 30, 40], c_0 [1, 40, 51], c [40], ℓ^∞ [50, 51], the Bachelis spaces N^p $(1 < p < \infty)$ [25], bv and bv_0 [47, 48], weighted ℓ^p spaces [7, 10], the discrete Cesàro spaces $ces(p)$ (for $p \in \{0\} \cup (1, \infty)$), [24], and their dual spaces d_s $(1 < s < \infty)$, [19]. For the class of generalized Cesàro operators C_t, for $t \in (0, 1)$, a study of their spectra and compactness properties (in ℓ^2) go back to Rhaly, [52, 53]. A similar investigation occurs for ℓ^p $(1 < p < \infty)$ in [58] and for c and c_0 in [55, 59]. The paper [55] also treats C_t when it acts on bv_0, bv, c, ℓ^1, ℓ^∞ and the Hahn sequence space h. In the recent paper [26] the setting for considering the operators C_t is a large class of Banach lattices in ω, which includes all rearrangement invariant sequence spaces (over $\mathbb{N}_0$ for counting measure), and many others.

Our aim is to study the compactness, the spectra and the dynamics of the generalized Cesàro operators C_t, for $t \in [0, 1)$, when they act in certain classical, *non-normable* sequence spaces $X \subseteq \omega$. Besides ω itself, the Fréchet spaces considered are $\ell(p+)$, $ces(p+)$ and $d(p+)$, for $1 \leq p < \infty$, as well as the (LB)-spaces $\ell(p-)$, $ces(p-)$ and $d(p-)$, for $1 < p \leq \infty$.

In Sect. 2 we formulate various preliminaries that will be needed in the sequel concerning particular properties of the spaces X that we consider, as well as linear operators between such spaces. We also collect some general results required to determine the spectra of operators T acting in the spaces X and the compactness of their dual operator T' acting in the strong dual space X'_β of X.

Section 3 is devoted to a detailed study of the operators C_t, for $t \in [0, 1)$, when they act in ω. These operators are *never* compact (c.f. Proposition 3.2) and their spectrum is completely described in Theorem 3.7 where, in particular, it is established that the set of all eigenvalues of C_t is independent of t and equals $\Lambda := \{\frac{1}{n+1} : n \in \mathbb{N}_0\}$. The 1-dimensional eigenspace corresponding to $\frac{1}{n+1}$, for each $n \in \mathbb{N}_0$, is identified in Lemma 3.4.

The situation for the other mentioned spaces $X \subseteq \omega$, which is rather different, is treated in Sects. 4 and 5. The operator C_t, for $t \in [0, 1)$, is *always* compact in these spaces; see Theorem 4.5(i) for the case of Fréchet spaces and Theorem 5.3(i) for the case of (LB)-spaces. The spectra of C_t are fully determined in Theorems 4.5(ii) and 5.3(ii), and the 1-dimensional eigenspace corresponding to each eigenvalue of C_t is identified in Theorems 4.5(iii) and 5.3(iii). We note, for all cases of X and $t \in [0, 1)$, that the set of all eigenvalues of C_t is again Λ. The main tool is a factorization result stating that $C_t = D_\varphi R_t$, where $D_\varphi : X \to X$ is a compact (diagonal) operator in X and $R_t : X \to X$ is a continuous linear operator; see Propositions 4.4(iii) and 5.2(iii).

For the definition of a mean ergodic operator and the notion of a supercyclic operator we refer to Sect. 6, where the relevant operators under consideration are C_t acting in the spaces X, for each $t \in [0, 1)$. It is necessary to determine some abstract results for linear operators in general lcHs' (c.f., Theorems 6.2 and 6.4), which are then applied to C_t to show that it is both power bounded and uniformly mean ergodic in all spaces $X \neq \omega$; see Theorem 6.6. The same is true for C_t acting in ω; see Theorem 6.1. In this section we also investigate the properties of the dual operators C_t' acting in X_β', which are given by (6.6) and (6.8). The operators C_t' are compact and their spectra are identified in Proposition 6.7, where it is also shown that the set of all eigenvalues of C_t' is Λ. Moreover, for each $n \in \mathbb{N}_0$, the eigenvector in X_β' spanning the 1-dimensional eigenspace corresponding to $\frac{1}{n+1} \in \Lambda$ is also determined. A consequence of C_t' having a rich supply of eigenvalues is that each operator $C_t\colon X \to X$, for $t \in [0, 1)$, fails to be supercyclic. Moreover, it is established in Proposition 6.8 that $C_t'\colon X_\beta' \to X_\beta'$ is power bounded, uniformly mean ergodic but, not supercyclic. It should be noted that the main results in this section are also new for C_t acting in the Banach spaces ℓ^p, $ces(p)$ and d_p.

2 Preliminaries

Given locally convex Haudorff spaces X, Y (briefly, lcHs) we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X into Y. If $X = Y$, then we simply write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. Equipped with the topology of pointwise convergence τ_s on X (i.e., the strong operator topology) the lcHs $\mathcal{L}(X)$ is denoted by $\mathcal{L}_s(X)$ and for the topology τ_b of uniform convergence on bounded sets the lcHs $\mathcal{L}(X)$ is denoted by $\mathcal{L}_b(X)$. Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X and by Γ_X a system of continuous seminorms determing the topology of X. The identity operator on X is denoted by I. The *dual operator* of $T \in \mathcal{L}(X)$ is denoted by T'; it acts in the topological dual space $X' := \mathcal{L}(X, \mathbb{C})$ of X. Denote by X_σ' (resp., by X_β') the space X' with the weak* topology $\sigma(X', X)$ (resp., with the strong topology $\beta(X', X)$); see [37, Sect. 21.2] for the definition. It is known that $T' \in \mathcal{L}(X_\sigma')$ and $T' \in \mathcal{L}(X_\beta')$, [38, p. 134]. For the general theory of functional analysis and operator theory relevant to this paper see, for example, [27, 33, 36, 44, 49, 56].

Lemma 2.1 *Let X be a lcHs and $T \in \mathcal{L}(X)$ be an isomorphism of X onto itself. Then T' is an isomorphism of X_β' onto itself. If, in addition, X is complete and barrelled, then T is an isomorphism of X onto itself if, and only if, T' is an isomorphism of X_β' onto itself.*

Proof If T is an isomorphism of X onto itself, then $T^{-1} \in \mathcal{L}(X)$ exists with $TT^{-1} = T^{-1}T = I$. It was already noted that $T', (T^{-1})' \in \mathcal{L}(X_\beta')$ and clearly $(T^{-1})'T' = T'(T^{-1})' = I$. Thus, $(T')^{-1}$ exists in $\mathcal{L}(X_\beta')$ and $(T')^{-1} = (T^{-1})'$; that is, T' is an isomorphism of X_β' onto itself.

Suppose that X is also complete and barrelled and that $T' \in \mathcal{L}(X_\beta')$ is an isomorphism of X_β' onto itself. As proved above, T'' is necessarily an isomorphism of X_β'' onto itself. By the proof of Lemma 3 in [6] it follows that T is an isomorphism of X onto itself. This completes the proof. □

Given a lcHs X and $T \in \mathcal{L}(X)$, the resolvent set $\rho(T; X)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The set $\sigma(T; X) := \mathbb{C}\backslash\rho(T; X)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{pt}(T; X)$ of T consists of all $\lambda \in \mathbb{C}$ (also called an eigenvalue of T) such that $(\lambda I - T)$ is not injective. An eigenvalue λ of T is called *simple* if $\dim\mathrm{Ker}(\lambda I - T) = 1$. Some authors (e.g. [56]) prefer the subset $\rho^*(T; X)$ of $\rho(T; X)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that the open disc $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\} \subseteq \rho(T; X)$ and $\{R(\mu, T) : \mu \in B(\lambda, \delta)\}$ is an equicontinuous subset of $\mathcal{L}(X)$. Define $\sigma^*(T; X) := \mathbb{C}\setminus\rho^*(T; X)$, which is a closed set with $\sigma(T; X) \subseteq \sigma^*(T; X)$. If X is a Banach space, then $\sigma(T; X) = \sigma^*(T; X)$. For the spectral theory of compact operators in lcHs' we refer to [27, 33], for example.

Corollary 2.2 *Let X be a complete, barrelled lcHs and $T \in \mathcal{L}(X)$. Then*

$$\rho(T; X) = \rho(T'; X'_\beta) \ \text{and} \ \sigma(T; X) = \sigma(T'; X'_\beta). \tag{2.1}$$

Moreover,

$$\sigma^*(T'; X'_\beta) \subseteq \sigma^*(T; X). \tag{2.2}$$

Proof The identities in (2.1) are an immediate consequence of Lemma 2.1.

Fix $\lambda \in \rho^*(T; X)$. Then there exists $\delta > 0$ such that $B(\lambda, \delta) \subseteq \rho(T; X)$ and $\{R(\mu; T) : \mu \in B(\lambda, \delta)\} \subseteq \mathcal{L}(X)$ is equicontinuous. For each $\mu \in B(\lambda, \delta)$ it follows from the proof of Lemma 2.1 that $R(\mu, T)' = ((\mu I - T)^{-1})' = (\mu I - T')^{-1} = R(\mu, T')$. Then [38, Sect. 39.3(6), p.138] implies that $\{R(\mu, T') : \mu \in B(\lambda, \delta)\} \subseteq \mathcal{L}(X'_\beta)$ is equicontinuous, that is, $\lambda \in \rho^*(T'; X'_\beta)$. So, we have established that $\rho^*(T; X) \subseteq \rho^*(T'; X'_\beta)$; taking complements yields (2.2). □

A linear map $T : X \to Y$, with X, Y lcHs', is called *compact* if there exists a neighbourhood $\mathcal{U}$ of 0 in X such that $T(\mathcal{U})$ is a relatively compact set in Y. It is routine to show that necessarily $T \in \mathcal{L}(X, Y)$. For the following result see [38, Sect. 42.1(1)] or [36, Proposition 17.1.1].

Lemma 2.3 *Let X be a lcHs. The compact operators are a 2-sided ideal in $\mathcal{L}(X)$.*

To establish the continuity of C_t, for $t \in [0, 1]$, in the Fréchet spaces considered in this paper we will need the following result, [14, Lemma 25].

Lemma 2.4 *Let $X = \cap_{n=1}^\infty X_n$ and $Y = \cap_{m=1}^\infty Y_m$ be two Fréchet spaces which resp. are the intersection of the sequence of Banach spaces $(X_n, \|\cdot\|_n)$, for $n \in \mathbb{N}$, and of the sequence of Banach spaces $(Y_m, |||\cdot|||_m)$, for $m \in \mathbb{N}$, satisfying $X_{n+1} \subset X_n$ with $\|x\|_n \le \|x\|_{n+1}$ for each $n \in \mathbb{N}$ and $x \in X_{n+1}$ and $Y_{m+1} \subset Y_m$ with $|||y|||_m \le |||y|||_{m+1}$ for each $m \in \mathbb{N}$ and $y \in Y_{m+1}$. Suppose that X is dense in X_n for each $n \in \mathbb{N}$. Then a linear operator $T : X \to Y$ is continuous if, and only if, for each $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the operator T has a unique continuous extension $T_{n,m} : X_n \to Y_m$.*

The following result, based on [8, Lemma 2.1], will be needed to determine the spectra of C_t, for $t \in [0, 1]$, in the Fréchet spaces considered in this paper.

Lemma 2.5 *Let $X = \cap_{n=1}^\infty X_n$ be a Fréchet space which is the intersection of a sequence of Banach spaces $(X_n, \|\cdot\|_n)$, for $n \in \mathbb{N}$, satisfying $X_{n+1} \subset X_n$ with $\|x\|_n \le \|x\|_{n+1}$ for each $n \in \mathbb{N}$ and $x \in X_{n+1}$. Let $T \in \mathcal{L}(X)$ satisfy the following condition:*

(A) For each $n \in \mathbb{N}$ there exists $T_n \in \mathcal{L}(X_n)$ such that the restriction of T_n to X (resp. of T_n to X_{n+1}) coincides with T (resp. with T_{n+1}).

Then the following properties are satisfied.

(i) *$\sigma(T; X) \subseteq \cup_{n=1}^\infty \sigma(T_n; X_n)$ and $\sigma_{pt}(T; X) \subseteq \cap_{n=1}^\infty \sigma_{pt}(T_n; X_n)$.*
(ii) *If $\cup_{n=1}^\infty \sigma(T_n; X_n) \subseteq \overline{\sigma(T; X)}$, then $\sigma^*(T; X) = \overline{\sigma(T; X)}$.*
(iii) *If $\dim\ker(\lambda I - T_m) = 1$ for each $\lambda \in \cap_{n=1}^\infty \sigma_{pt}(T_n; X_n)$ and $m \in \mathbb{N}$, then $\sigma_{pt}(T; X) = \cap_{n=1}^\infty \sigma_{pt}(T_n; X_n)$.*

Proof In view of [8, Lemma 2.1] it remains to show the validity of the inclusion $\sigma_{pt}(T; X) \subseteq \cap_{n=1}^{\infty}\sigma_{pt}(T_n, X_n)$ in the statement (i) and the identity in (iii).

The inclusion $\sigma_{pt}(T; X) \subseteq \cap_{n=1}^{\infty}\sigma_{pt}(T_n; X_n)$ is clear. Indeed, if $(\lambda I - T)x = 0$ for some $x \in X\setminus\{0\}$ and $\lambda \in \mathbb{C}$, then in view of $X \subseteq X_n$ and $T_n|_X = T$, for $n \in \mathbb{N}$, (see condition (A)), we have that $x \in X_n\setminus\{0\}$ and $(\lambda I - T_n)x = 0$ for every $n \in \mathbb{N}$. Hence, $\lambda \in \cap_{n=1}^{\infty}\sigma_{pt}(T_n; X_n)$.

To establish the validity of (iii), fix $\lambda \in \cap_{n=1}^{\infty}\sigma_{pt}(T_n; X_n)$. Then, for each $n \in \mathbb{N}$, there exists $x_n \in X_n\setminus\{0\}$ such that $(\lambda I - T_n)x_n = 0$. Since $x_{n+1} \in X_{n+1} \subseteq X_n$, for $n \in \mathbb{N}$, condition (A) implies that also $(\lambda I - T_n)x_{n+1} = 0$ in X_n for each $n \in \mathbb{N}$. So, for each $n \in \mathbb{N}$, we have that $x_{n+1} = \mu_n x_n$ for some $\mu_n \in \mathbb{C}\setminus\{0\}$. Therefore, $x_n = (\prod_{j=1}^{n-1}\mu_j)x_1$, with $\prod_{j=1}^{n-1}\mu_j \neq 0$. Accordingly, $x_1 \in X_n$ for each $n \in \mathbb{N}$ and hence, $x_1 \in X$. On the other hand, applying again condition (A), we can conclude that $(\lambda I - T)x_1 = (\lambda I - T_1)x_1 = 0$, i.e., $\lambda \in \sigma_{pt}(T; X)$. □

Fréchet spaces X which satisfy the assumptions of Lemma 2.5 are often called *countably normed Fréchet spaces*; for the general theory of such spaces see [29], for example.

A Hausdorff locally convex space (X, τ) is called an *(LB)-space* if there is a sequence $(X_k)_{k\in\mathbb{N}}$ of Banach spaces satisfying $X_k \subseteq X_{k+1}$ continuously for $k \in \mathbb{N}$, $X = \cup_{k=1}^{\infty}X_k$ and τ is the finest locally convex topology on X such that the natural inclusion $X_k \subset X$ is continuous for each $k \in \mathbb{N}$, [44, pp. 290–291]. In this case we write $X = \mathrm{ind}_k X_k$. If, in addition, X is a *regular* (LB)-space, [36, p. 83], then a set $B \subset X$ is bounded if and only if there exists $m \in \mathbb{N}$ such that $B \subset X_m$ and B is bounded in the Banach space X_m. Complete (LB)-spaces are regular, [37, Sect. 19.5(5)]. All of the (LB)-spaces of sequences considered in this note will be regular because of the following result, [44, Proposition 25.19(2)].

Lemma 2.6 *Let $X = \mathrm{ind}_k X_k$ be an (LB)-space with an increasing union of reflexive Banach spaces $X = \cup_{k=1}^{\infty}X_k$ such that each inclusion $X_k \subseteq X_{k+1}$, for $k \in \mathbb{N}$, is continuous. Then X is complete and hence, also regular.*

An (LB)-space $X = \mathrm{ind}_k X_k$ is said to be *boundedly retractive* if for every $B \in \mathcal{B}(X)$ there exists $k \in \mathbb{N}$ such that B is contained and bounded in X_k, and X and X_k induce the same topology on B. The (LB)-space X is said to be *sequentially retractive* if for every null sequence in X there exists $k \in \mathbb{N}$ such that the sequence is contained and converges to zero in X_k. Finally, the (LB)-space X is said to be *compactly regular* if for every compact subset C of X there exists $k \in \mathbb{N}$ such that C is compact in X_k. Each of these three notions implies the completeness of X, [57, Corollary 2.8]. Neus [46] proved that all these notions are equivalent even for inductive limits of normed spaces.

In the setting of boundedly retractive (LB)-spaces, the following general statement on the compactness of certain dual operators is valid.

Proposition 2.7 *Let X be a lcHs, $Y = \mathrm{ind}_k Y_k$ be a boundedly retractive (LB)-space and $T \in \mathcal{L}(X, Y)$ be compact. Then $T' \in \mathcal{L}(Y'_\beta, X'_\beta)$ is compact.*

Proof The compactness of T implies that there exists a closed, absolutely convex neighbourhood $\mathcal{U}$ of 0 in X such that $T(\mathcal{U})$ is a relatively compact set in Y. So, the closure $B := \overline{T(\mathcal{U})} \in \mathcal{B}(Y)$ of $T(\mathcal{U})$ is a compact set in Y. But, Y is a boundedly retractive (LB)-space. Accordingly, there exists $k \in \mathbb{N}$ such that B is contained and bounded in Y_k, and Y and Y_k induce the same topology on B. Therefore, B is also a compact set in Y_k and $T(X) \subseteq Y_k$. Accordingly, the operator T acts compactly from X into Y_k. Denote by T_1 the operator T when interpreted to be acting from X into Y_k and by i_k the continuous inclusion of Y_k into Y. So, $T_1 \in \mathcal{L}(X, Y_k)$ is compact and $T = i_k T_1$. Denote by p the continuous seminorm on X

corresponding to $\mathcal{U}$ and let X_p denote the normed quotient space $\left(\frac{X}{\operatorname{Ker} p}, p\right)$. Then there exists a unique continuous linear operator S from X_p into Y_k such that $SQ = T_1$, where Q denotes the canonical quotient map from X into X_p and hence, is an open map. Since $T_1 \in \mathcal{L}(X, Y_k)$ is compact and $Q \in \mathcal{L}(X, X_p)$ is open, the operator $S \in \mathcal{L}(X_p, Y_k)$ is necessarily compact. By Schauder's theorem, [38, Sect. 42(7), p. 202], it follows that $S' \in \mathcal{L}(Y_k', X_p')$ is compact. So, $T_1' = Q'S' \in \mathcal{L}(Y_k', X_\beta')$ is compact and hence, $T' = T_1' i_k' \in \mathcal{L}(Y_\beta', X_\beta')$ is compact (cf. Proposition 17.1.1 in [36]). This completes the proof. □

A Fréchet space X is said to be *quasinormable* if for every neighbourhhod $\mathcal{U}$ of 0 in X there exists a neighbourhhod $\mathcal{V}$ of 0 in X so that, for every $\varepsilon > 0$, there exists $B \in \mathcal{B}(X)$ satisfying $\mathcal{V} \subseteq B + \varepsilon\mathcal{U}$. Thus, every Fréchet-Schwartz space is quasinormable [44, Remark, p. 313]. The strong dual X_β' of a quasinormable Fréchet space X is necessarily a boundedly retractive (LB)-space [18, Theorem]. Thus, the strong dual of any Fréchet-Schwartz space (briefly, (DFS)-space) is a boundedly retractive (LB)-space.

Corollary 2.8 *Let X and Y be two Fréchet spaces and $T \in \mathcal{L}(X, Y)$. If $T'' \in \mathcal{L}(X_\beta'', Y_\beta'')$ is compact, then T is compact.*

If, in addition, X is quasinormable and T is compact, then $T'' \in \mathcal{L}(X_\beta'', Y_\beta'')$ is compact.

Proof Suppose that $T'' \in \mathcal{L}(X_\beta'', Y_\beta'')$ is compact. Since X, Y are Fréchet spaces, they are isomorphic to their respective natural image in X_β'', Y_β'' (in which they are closed subspaces). Moreover, the restriction of T'' to X coincides with T and takes its values in $Y \subseteq Y_\beta''$. Then the compactness of T follows from that of T''.

Suppose that X is quasinormable and that $T \in \mathcal{L}(X, Y)$ is compact. Since X is quasinormable, its strong dual X_β' is a boundedly retractive (LB)-space. Moreover, Y being a Fréchet space implies that $T' \colon Y_\beta' \to X_\beta'$ is compact, [27, Corollary 9.6.3]. It follows from Proposition 2.7, with Y_β' in place of X and X_β' in place of $Y = \operatorname{ind}_k Y_k$ and T' in place of T, that $T'' \in \mathcal{L}(X_\beta'', Y_\beta'')$ is compact. □

To identify the spectrum of C_t acting in the (LB)-spaces arising in this paper we will require the following two results; the first one, i.e. Lemma 2.9, is a direct consequence of Grothendieck's factorization theorem (see e.g. [44, Theorem 24.33]), and the second one, i.e. Lemma 2.10, is proved in [11, Lemma 5.2].

Lemma 2.9 *Let $X = \operatorname{ind}_n X_n$ and $Y = \operatorname{ind}_m Y_m$ be two (LB)-spaces with increasing unions of Banach spaces $X = \cup_{n=1}^\infty X_n$ and $Y = \cup_{m=1}^\infty Y_m$. Let $T \colon X \to Y$ be a linear map. Then T is continuous (i.e., $T \in \mathcal{L}(X, Y)$) if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T(X_n) \subseteq Y_m$ and the restriction $T \colon X_n \to Y_m$ is continuous.*

Lemma 2.10 *Let $X = \operatorname{ind}_k X_k$ be a Hausdorff inductive limit of a sequence of Banach spaces $(X_k, \|\cdot\|_k)$. Let $T \in \mathcal{L}(X)$ satisfy the following condition:*

(A′) For each $k \in \mathbb{N}$ the restriction T_k of T to X_k maps X_k into itself and $T_k \in \mathcal{L}(X_k)$.

Then the following properties are satisfied.

(i) $\sigma_{pt}(T; X) = \cup_{k=1}^\infty \sigma_{pt}(T_k; X_k)$.
(ii) *If* $\cup_{k=m}^\infty \sigma(T_k; X_k) \subseteq \overline{\sigma(T; X)}$ *for some* $m \in \mathbb{N}$, *then* $\sigma^*(T; X) = \overline{\sigma(T; X)}$.
(iii) $\sigma(T; X) \subseteq \cap_{m\in\mathbb{N}} \left(\cup_{n=m}^\infty \sigma(T_n; X_n)\right)$.

Another useful fact for our study is the following result.

Lemma 2.11 *Let $T \in \mathcal{L}(\omega)$. Let X be a Fréchet space or an (LB)-space continuously included in ω. If $T(X) \subseteq X$, then $T \in \mathcal{L}(X)$.*

Proof The result follows from the closed graph theorem, [44, Theorem 24.31], after recalling that X is ultrabornological, [44, Remark 24.15(c) & Proposition 24.16] and has a web, [44, Corollary 24.29 & Remark 24.36]. So, it is enough to show that the graph of T in X is closed. To do this, we assume that a net $(x_\alpha)_\alpha \subseteq X$ satisfies $x_\alpha \to x$ and $T(x_\alpha) \to y$ in X. Since the inclusion $X \subseteq \omega$ is continuous, $x_\alpha \to x$ in ω and hence, $T(x_\alpha) \to T(x)$ in ω. On the other hand, by the continuity of the inclusion $X \subseteq \omega$ also $T(x_\alpha) \to y$ in ω. Then $T(x) = y$. So, (x, y) belongs to the graph of T. This shows that the graph of T is closed. □

For X a barrelled lcHs, every bounded subset of $\mathcal{L}_s(X)$ is equicontinuous, [44, Proposition 23.27]. It is known that every Fréchet space is barrelled, [44, Remark, p. 296], and that every (LB)-space is barrelled, [44, Proposition 24.16].

The operator norm of a Banach space operator $T \in \mathcal{L}(X, Y)$ will be denoted by $\|T\|_{X\to Y}$. The Banach spaces $\ell^p = \ell^p(\mathbb{N}_0)$, for $1 \le p < \infty$, with their standard norm $\|\cdot\|_p$ are classical. For $1 < p < \infty$ these spaces are reflexive. The spectra of C_t acting in such spaces are given in the following result; see [58] for $1 < p < \infty$ and also [55, Sect. 8] for $1 \le p < \infty$. Recall from Sect. 1 that

$$\Lambda := \left\{ \frac{1}{n+1} : n \in \mathbb{N}_0 \right\}.$$

Proposition 2.12 *For each $t \in [0, 1)$ the operator $C_t \in \mathcal{L}(\ell^p)$, for $1 \le p < \infty$, is a compact operator satisfying*

$$\|C_t\|_{\ell^1\to\ell^1} = \frac{1}{t}\log\left(\frac{1}{1-t}\right), \quad t \in (0, 1),$$

and

$$\left(\sum_{n=0}^{\infty}\left(\frac{t^n}{n+1}\right)^p\right)^{1/p} \le \|C_t\|_{\ell^p\to\ell^p} \le \left(\frac{1}{t}\log\left(\frac{1}{1-t}\right)\right)^{1/p}, \ 1 < p < \infty, \ t \in (0, 1),$$

with $\|C_0\|_{\ell^p\to\ell^p} = 1$. Moreover,

$$\sigma_{pt}(C_t; \ell^p) = \Lambda \ \text{ and } \ \sigma(C_t; \ell^p) = \Lambda \cup \{0\}. \tag{2.3}$$

Concerning the classical Cesàro operator C_1 (c.f. (1.3)) in $\mathcal{L}(\ell^p)$ we have the following result.

Proposition 2.13 *Let $1 < p < \infty$.*

(i) *The operator $C_1 \in \mathcal{L}(\ell^p)$ with $\|C_1\|_{\ell^p\to\ell^p} = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*
(ii) *The spectra of C_1 are given by*

$$\sigma_{pt}(C_1; \ell^p) = \emptyset \ \text{ and } \ \sigma(C_1; \ell^p) = \left\{ z \in \mathbb{C} : \left| z - \frac{p'}{2} \right| \le \frac{p'}{2} \right\}.$$

Moreover, the range $(C_1 - zI)(\ell^p)$ is not dense in ℓ^p whenever $|z - \frac{p'}{2}| < \frac{p'}{2}$.

For part (i) we refer to [34, Theorem 326] and for part (ii) see [30, 40, 54] and the references therein. In particular, C_1 is a *not* a compact operator.

G. Bennett thoroughly investigated the discrete Cesàro spaces

$$ces(p) := \{x \in \omega : C_1|x| \in \ell^p\}, \quad 1 < p < \infty,$$

where $|x| := (|x_n|)_{n\in\mathbb{N}_0}$, which satisfy $\ell^p \subseteq ces(p)$ continuously and are reflexive Banach spaces relative to the norm

$$\|x\|_{ces(p)} := \|C_1|x|\|_p, \quad x \in ces(p); \tag{2.4}$$

see, for example, [17], as well as [15, 24, 31, 41] and the references therein. The following result, [26, Proposition 5.6] describes the spectra of C_t acting in $ces(p)$.

Proposition 2.14 *Let $t \in [0, 1)$ and $1 < p < \infty$. The operator $C_t \in \mathcal{L}(ces(p))$ is compact and satisfies*

$$\|C_t\|_{ces(p)\to ces(p)} \le \min\left\{\frac{1}{1-t}, \frac{p}{p-1}\right\}.$$

Moreover,

$$\sigma_{pt}(C_t; ces(p)) = \Lambda \ \textit{and}\ \sigma(C_t; ces(p)) = \Lambda \cup \{0\}. \tag{2.5}$$

The situation for $C_1 \in \mathcal{L}(ces(p))$ is quite different. Indeed, $\|C_1\|_{ces(p)\to ces(p)} = p'$ and the spectra are given by

$$\sigma_{pt}(C_1; ces(p)) = \emptyset \ \text{ and } \ \sigma(C_1; ces(p)) = \left\{z \in \mathbb{C} : \left|z - \frac{p'}{2}\right| \le \frac{p'}{2}\right\}$$

for each $1 < p < \infty$; see Theorem 5.1 and its proof in [24]. In particular, C_1 is *not* a compact operator.

The dual Banach spaces $(ces(p))'$, for $1 < p < \infty$, are rather complicated, [35]. A more transparent *isomorphic* identification of $(ces(p))'$ is given in Corollary 12.17 of [17]. It is shown there that

$$d_p := \left\{x \in \ell^\infty : \hat{x} := \left(\sup_{k\ge n} |x_k|\right)_{n\in\mathbb{N}_0} \in \ell^p\right\}, \quad 1 < p < \infty,$$

is a Banach space for the norm

$$\|x\|_{d_p} := \|\hat{x}\|_p, \quad x \in d_p, \tag{2.6}$$

which is isomorphic to $(ces(p'))'$, where p' is the conjugate exponent of p. The sequence $\hat{x}$ is called the *least decreasing majorant* of x. The duality is the natural one given by

$$\langle w, x\rangle := \sum_{n=0}^{\infty} w_n x_n, \quad w \in ces(p'),\ x \in d_p.$$

In particular, d_p is reflexive for each $1 < p < \infty$. Since $|x| \le |\hat{x}|$, it is clear that $\|x\|_p \le \|\hat{x}\|_p = \|x\|_{d_p}$, for $x \in d_p$, that is, $d_p \subseteq \ell^p$ continuously. So, for all $1 < p < \infty$, we have $d_p \subseteq \ell^p \subseteq ces(p)$ with continuous inclusions. The following result is Theorem 6.9 of [26].

Proposition 2.15 *Let $t \in [0, 1)$ and $1 < p < \infty$. The operator $C_t \in \mathcal{L}(d_p)$ is compact and satisfies*

$$\|C_t\|_{d_p\to d_p} \le (1-t)^{-1-(1/p)}.$$

Moreover,

$$\sigma_{pt}(C_t; d_p) = \Lambda \ \ \textit{and} \ \ \sigma(C_t; d_p) = \Lambda \cup \{0\}. \tag{2.7}$$

Concerning the operator $C_1 \in \mathcal{L}(d_p)$, $1 < p < \infty$, it is known that $\|C_1\|_{d_p \to d_p} = p'$ and that its spectra are given by

$$\sigma_{pt}(C_1; d_p) = \emptyset \ \text{ and } \ \sigma(C_1; d_p) = \left\{ z \in \mathbb{C} : \left| z - \frac{p'}{2} \right| \le \frac{p'}{2} \right\};$$

see Proposition 3.2 and Corollary 3.5 in [19].

3 The operators C_t acting in ω

Given an element $x = (x_n)_{n \in \mathbb{N}_0} \in \omega$ we write $x \ge 0$ if $x = |x| = (|x_n|)_{n \in \mathbb{N}_0}$. By $x \le z$ it is meant that $(z - x) \ge 0$. The sequence space ω is a non-normable Fréchet space for the Hausdorff locally convex topology of coordinatewise convergence, which is determined by the increasing sequence of seminorms

$$r_n(x) := \max_{0 \le j \le n} |x_j|, \quad x \in \omega, \tag{3.1}$$

for each $n \in \mathbb{N}_0$. Observe that $r_n(x) = r_n(|x|) \le r_n(|y|) = r_n(y)$ whenever $x, y \in \omega$ satisfy $|x| \le |y|$. Let $e_n := (\delta_{nj})_{j \in \mathbb{N}_0}$ for each $n \in \mathbb{N}_0$ and set $\mathcal{E} := \{e_n : n \in \mathbb{N}_0\}$. It is clear from (1.1) that each $C_t \colon \omega \to \omega$ is a linear map which is represented by a lower triangular matrix with respect to the unconditional basis $\mathcal{E}$ of ω. Namely,

$$C_t \simeq \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ t/2 & 1/2 & 0 & 0 & \cdots \\ t^2/3 & t/3 & 1/3 & 0 & \cdots \\ t^3/4 & t^2/4 & t/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.2}$$

with main diagonal the positive, decreasing sequence given by

$$\varphi := \left(\frac{1}{n+1} \right)_{n \in \mathbb{N}_0} \in c_0. \tag{3.3}$$

The following properties of C_t are recorded in [26, Lemma 2.1], except for part (iv).

Lemma 3.1 *Let* $t \in [0, 1)$.

(i) *Each* C_t *is a positive operator on* ω*,i.e.,* $C_t x \ge 0$ *whenever* $x \ge 0$.
(ii) *Let* $0 \le r \le s \le 1$. *Then*

$$0 \le |C_r x| \le C_r |x| \le C_s |x|, \quad x \in \omega.$$

(iii) *For each* $t \in [0, 1)$ *the identities*

$$C_t e_n = \sum_{k=0}^{\infty} \frac{t^k}{k+n+1} e_{k+n} \in \ell^1, \quad n \in \mathbb{N}_0,$$

and

$$C_t(e_n - t e_{n+1}) = \frac{1}{n+1} e_n, \quad n \in \mathbb{N}_0,$$

are valid.

(iv) *For each* $1 < q < \infty$ *we have* $d_q \subseteq \ell^q \subseteq ces(q) \subseteq \omega$ *with continuous inclusions.*

Proof (iv) In view of the discussion after (2.6) it remains to establish that $ces(q) \subseteq \omega$ continuously. Fix $x \in ces(q)$. Given $n \in \mathbb{N}_0$ observe that

$$|x_k| \leq (n+1)\frac{|x_0| + |x_1| + \cdots + |x_n|}{n+1} \leq (n+1)\|C_1|x|\|_q = (n+1)\|x\|_{ces(q)}, \ 0 \leq k \leq n.$$

It follows from (3.1) that $r_n(x) \leq (n+1)\|x\|_{ces(q)}$. Since $n \in \mathbb{N}_0$ is arbitrary, we can conclude that $ces(q) \subseteq \omega$ continuously. □

The classical Cesàro operator $C_1 \colon \omega \to \omega$ is a bicontinuous topological isomorphism (and hence, is *not* a compact operator) with spectra given by

$$\sigma(C_1; \omega) = \sigma_{pt}(C_1; \omega) = \Lambda \ \text{ and } \ \sigma^*(C_1; \omega) = \Lambda \cup \{0\};$$

see [8, p. 285 and Proposition 4.4]. So, we will only consider the case $t \in [0, 1)$.

Let $t \in [0, 1)$ and fix $n \in \mathbb{N}_0$. According to (1.1) and (3.1), for each $x \in \omega$, it is the case that

$$r_n(C_t x) = \max_{0 \leq k \leq n} \left| \frac{1}{k+1} \sum_{i=0}^{k-1} t^{k-i} x_i \right| \leq \max_{0 \leq k \leq n} \frac{1}{k+1} \sum_{i=0}^{k-1} |x_i| \leq r_n(x). \tag{3.4}$$

This implies that $C_t \in \mathcal{L}(\omega)$ and that the family of operators $\{C_t : t \in [0, 1)\}$ is an equicontinuous subset of $\mathcal{L}(\omega)$.

Proposition 3.2 *For each* $t \in [0, 1)$ *the operator* $C_t \in \mathcal{L}(\omega)$ *is a bicontinuous isomorphism of* ω *onto itself with inverse operator* $(C_t)^{-1} \colon \omega \to \omega$ *given by*

$$(C_t)^{-1} y = ((n+1)y_n - nty_{n-1})_{n \in \mathbb{N}_0}, \quad y \in \omega \ (\textit{with } y_{-1} := 0). \tag{3.5}$$

In particular, C_t *is not a compact operator.*

Proof Fix $t \in [0, 1)$. Let $x \in \omega$ satisfy $C_t x = 0$. Considering the coordinate 0 of $C_t x = 0$ yields $x_0 = 0$; see (1.1). The equation for coordinate 1 of $C_t x = 0$ is $\frac{tx_0 + x_1}{2} = 0$ (cf. (1.1)) which yields $x_1 = 0$. Proceed inductively for successive coordinates reveals that $x_n = 0$ for all $n \in \mathbb{N}_0$. Hence, C_t is injective.

Given $y \in \omega$ let $x \in \omega$ be the element on the right-side of (3.5). Direct calculation shows that $C_t x = y$. Accordingly, C_t is surjective.

By the open mapping theorem for Fréchet spaces (cf. Corollary 24.29 and Theorem 24.30 in [44]) the operator C_t is a bicontinuous isomorphism.

Since C_t is a bicontinuous isomorphism of ω, which is an infinite dimensional Fréchet space, C_t cannot be a compact operator. □

To determine the spectrum of $C_t \in \mathcal{L}(\omega)$ requires some preparation. Define

$$\mathcal{S} := \left\{ x \in \omega : \beta(x) := \lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|} < 1 \right\}, \tag{3.6}$$

with the understanding that there exists $N \in \mathbb{N}_0$ such that $x_n \neq 0$ for $n \geq N$ and the limit $\beta(x)$ exists. Analogously to d_p, for $1 < p < \infty$, define

$$d_1 := \left\{ x \in \ell^\infty : \hat{x} := \left(\sup_{k \geq n} |x_k| \right)_{n \in \mathbb{N}_0} \in \ell^1 \right\}; \tag{3.7}$$

see [17, 19, 24, 31] and the references therein. Then d_1 is a Banach lattice for the norm $\|x\|_{d_1} := \|\hat{x}\|_1$ and the coordinatewise order. Since $0 \le |x| \le \hat{x}$, for $x \in \ell^\infty$, it is clear that $\|x\|_1 \le \|x\|_{d_1}$ for $x \in d_1$, that is, $d_1 \subseteq \ell^1$ with a continuous inclusion. Clearly, $d_1 \subseteq d_p$, for all $1 < p < \infty$, and $d_1 \subseteq \ell^1$ implies that $d_1 \subseteq \ell^p$, for all $1 < p < \infty$. Moreover, $\ell^p \subseteq ces(p)$ (cf. Section 2) and so also $d_1 \subseteq ces(p)$, for $1 < p < \infty$. All inclusions are continuous. In view of Lemma 3.1(iv) it is clear that $d_1 \subseteq \omega$ and $\ell^1 \subseteq \omega$ continuously. It is known that $\mathcal{S} \subseteq d_1$, [26, Lemma 3.3].

Remark 3.3 Proposition 2.15 is also valid for $p = 1$; see [26, Theorem 6.9].

The following result, [26, Lemma 3.6], will be required.

Lemma 3.4 *Let $t \in [0, 1)$ and φ be as in (3.3). For each $m \in \mathbb{N}$ define $x^{[m]} \in \omega$ by*

$$x^{[m]} := \alpha_m \left(0, \dots, 0, 1, \frac{(m+1)!}{m!\,1!}t, \frac{(m+2)!}{m!\,2!}t^2, \frac{(m+3)!}{m!\,3!}t^3, \dots\right), \tag{3.8}$$

with $\alpha_m \in \mathbb{C}\backslash\{0\}$ arbitrary, where 1 is in position m. For $m = 0$ define $x^{[0]} := \alpha_0(t^n)_{n\in\mathbb{N}_0}$ with $\alpha_0 \in \mathbb{C}\backslash\{0\}$ arbitrary.

(i) *For each $m \in \mathbb{N}_0$, the vector $x^{[m]}$ is the unique solution in ω of the equation $C_t x = \varphi_m x = \frac{1}{m+1}x$ whose m-th coordinate is α_m.*
(ii) *The vector $x^{[m]} \in d_1 \subseteq \omega$, for each $m \in \mathbb{N}_0$.*

Remark 3.5 Let $t \in [0, 1)$ and X be any Banach space in $\{\ell^1, d_1\} \cup \{\ell^p, ces(p), d_p : 1 < p < \infty\}$. For each $\nu \in \sigma_{pt}(C_t; X) = \Lambda$, it is the case that $\dim \mathrm{Ker}(\nu I - C_t) = 1$. Indeed, $d_1 \subseteq X$; see the discussion prior to Remark 3.3. Given $\nu \in \Lambda$ there exists $m \in \mathbb{N}_0$ such that $\nu = \varphi_m$. According to Lemma 3.4 the 1-dimensional eigenspace corresponding to $\nu \in \sigma_{pt}(C_t; \omega)$ is spanned by $x^{[m]}$ with $x^{[m]} \in d_1$. The claim is thereby proved.

The next lemma places a restriction on where $\sigma(C_t; \omega)$ can be located in $\mathbb{C}$.

Lemma 3.6 *Let $t \in [0, 1)$. For each $\nu \in \mathbb{C}\backslash\Lambda$ the operator $C_t - \nu I$ is a bicontinuous isomorphism of ω onto itself. In particular, $\sigma(C_t; \omega) \subseteq \Lambda$.*

Proof Fix $\nu \notin \Lambda$. Let $(C_t - \nu I)x = 0$ for $x \in \omega$. It follows from (3.2), by equating the coordinate 0 of $C_t x = \nu x$, that $x_0 = \nu x_0$ and hence, as $\nu \ne 1$, that $x_0 = 0$. Equating the coordinate 1 of $C_t x = \nu x$ yields $\frac{tx_0 + x_1}{2} = \nu x_1$. Since $x_0 = 0$ and $\nu \ne \frac{1}{2}$, it follows that $x_1 = 0$. Considering coordinate 2 gives $\frac{t^2x_0 + tx_1 + x_2}{3} = \nu x_2$. Then $x_0 = x_1 = 0$ and $\nu \ne \frac{1}{3}$ imply $x_2 = 0$. Proceed inductively to conclude that $x = 0$, that is, $C_t - \nu I$ is injective.

To verify the surjectivity of $C_t - \nu I$ fix $y \in \omega$. It is required to show that there exists $x \in \omega$ satisfying $(C_t - \nu I)x = y$. Equating coordinate 0 gives $x_0 - \nu x_0 = y_0$, that is, $x_0 = y_0/(1-\nu)$. Considering coordinate 1 yields $\frac{tx_0}{2} + (\frac{1}{2} - \nu)x_1 = y_1$. Substituting for x_0 gives $(\frac{1}{2} - \nu)x_1 = y_1 - \frac{t}{2(1-\nu)}y_0$, that is,

$$x_1 = \frac{y_1}{\left(\frac{1}{2} - \nu\right)} - \frac{ty_0}{2\left(\frac{1}{2} - \nu\right)(1-\nu)}.$$

Next, an examination of coordinate 2 yields $\frac{t^2}{3}x_0 + \frac{t}{3}x_1 + (\frac{1}{3} - \nu)x_2 = y_2$. Substituting for x_0 and x_1 we can conclude that

$$x_2 = \frac{y_2}{\left(\frac{1}{3} - \nu\right)} - \frac{ty_1}{3\left(\frac{1}{3} - \nu\right)\left(\frac{1}{2} - \nu\right)} + \frac{\nu t^2 y_0}{3\left(\frac{1}{3} - \nu\right)\left(\frac{1}{2} - \nu\right)(1-\nu)}.$$

Continuing inductively yields

$$
\begin{aligned}
x_n = {} & \frac{y_n}{\left(\frac{1}{n+1}-\nu\right)} - \frac{t y_{n-1}}{(n+1)\left(\frac{1}{n+1}-\nu\right)\left(\frac{1}{n}-\nu\right)} + \\
& + \frac{\nu t^2 y_{n-2}}{(n+1)\left(\frac{1}{n+1}-\nu\right)\left(\frac{1}{n}-\nu\right)\left(\frac{1}{n-1}-\nu\right)} \\
& - \frac{\nu^2 t^3 y_{n-3}}{(n+1)\left(\frac{1}{n+1}-\nu\right)\left(\frac{1}{n}-\nu\right)\left(\frac{1}{n-1}-\nu\right)\left(\frac{1}{n-2}-\nu\right)} + \cdots \\
& + (-1)^n \frac{\nu^{n-1} t^n y_0}{(n+1)\left(\frac{1}{n+1}-\nu\right)\left(\frac{1}{n}-\nu\right)\ldots\left(\frac{1}{2}-\nu\right)(1-\nu)}.
\end{aligned} \tag{3.9}
$$

Then $x \in \omega$ satisfies $(C_t - \nu I)x = y$. Hence, $C_t - \nu I$ is surjective. □

Combining the previous results yields the main result of this section.

Theorem 3.7 *For each $t \in [0, 1)$ the spectra of $C_t \in \mathcal{L}(\omega)$ are given by*

$$\sigma(C_t; \omega) = \sigma_{pt}(C_t; \omega) = \Lambda,$$

with each eigenvalue being simple, and

$$\sigma^*(C_t; \omega) = \Lambda \cup \{0\}.$$

The 1-dimensional eigenspace corresponding to the eigenvalue $1/(m+1) \in \Lambda$ is spanned by $x^{[m]}$ (cf. (3.8)), for each $m \in \mathbb{N}_0$.

Proof It is clear from Lemma 3.4 that $\Lambda \subseteq \sigma_{pt}(C_t; \omega)$ and that each point $1/(m+1) \in \Lambda$ is a simple eigenvalue of C_t, whose corresponding eigenspace is spanned by $x^{[m]}$, for each $m \in \mathbb{N}_0$. Since $\sigma(C_t; \omega) \subseteq \Lambda$ (cf. Lemma 3.6) and $\sigma_{pt}(C_t; \omega) \subseteq \sigma(C_t; \omega)$, we can conclude that $\sigma(C_t; \omega) = \sigma_{pt}(C_t; \omega) = \Lambda$. The containment $\sigma(C_t; \omega) \subseteq \sigma^*(C_t; \omega)$ and the fact that $\sigma^*(C_t; \omega)$ is a closed set imply that $0 \in \sigma^*(C_t; \omega)$.

It remains to show that every $\nu \notin (\Lambda \cup \{0\})$ belongs to $\rho^*(C_t; \omega)$. So, fix $\nu \notin (\Lambda \cup \{0\})$. Select $\delta > 0$ such that the distance ϵ of $B(\nu, \delta)$ to the compact set $\Lambda \cup \{0\}$ is strictly positive. It follows from $0 \le t < 1$ and the identity (3.9) which is coordinate n of $(C_t - \nu I)^{-1} y$, for each $y \in \omega$, that for any given $k \in \mathbb{N}_0$ there exists $M_k > 0$ such that

$$r_k((C_t - \mu I)^{-1} y) \le \frac{M_k}{\epsilon^{k+1}} \left(\max_{0 \le j < k} |\nu|^j \right) r_k(y), \quad \mu \in B(\nu, \delta),$$

where r_k is the seminorm (3.1), with k in place of n. This implies that $\{(C_t - \mu I)^{-1} : \mu \in B(\nu, \delta)\}$ is a bounded set in $\mathcal{L}_s(\omega)$ and hence, by the barrelledness of ω, it is an equicontinuous subset of $\mathcal{L}(\omega)$. Accordingly, $\nu \in \rho^*(C_t; \omega)$. □

4 C_t acting in the Fréchet spaces $\ell(p+)$, $d(p+)$ and $ces(p+)$

Given $1 \le p < \infty$, consider any strictly decreasing sequence $\{p_k\}_{k \in \mathbb{N}} \subseteq (p, \infty)$ which satisfies $p_k \downarrow p$. Then $X_k := \ell^{p_k}$ satisfies $X_{k+1} \subseteq X_k$ with $\|x\|_{\ell^{p_k}} \le \|x\|_{\ell^{p_{k+1}}}$ for each $k \in \mathbb{N}$ and $x \in X_{k+1}$. Moreover, $X = \cap_{k=1}^{\infty} X_k$ (i.e., $\ell(p+) := \cap_{k=1}^{\infty} \ell^{p_k}$) is a Fréchet space of the type given in Lemma 2.5 whose topology is generated by the increasing sequence of *norms* u_k, for $k \in \mathbb{N}$, given by

$$u_k : x \mapsto \|x\|_{\ell^{p_k}}, \quad x \in \ell(p+). \tag{4.1}$$

That is, $u_k \le u_{k+1}$ for $k \in \mathbb{N}$. Moreover, $p_k > p$ implies that the natural inclusion map $\ell(p+) \hookrightarrow \ell^{p_k}$ is continuous for each $k \in \mathbb{N}$. Clearly the Banach space $\ell^p \subseteq \ell(p+)$ continuously and also $\ell(p+) \subseteq \omega$ continuously, as $\ell^q \subseteq \omega$ continuously, for every $1 \le q < \infty$ (cf. Lemma 3.1(iv)). The space $\ell(p+)$ is independent of the choice of $\{p_k\}_{k\in\mathbb{N}}$.

Changing the Banach spaces, now let $X_k := ces(p_k)$, in which case again $X_{k+1} \subseteq X_k$ with $\|x\|_{ces(p_k)} \le \|x\|_{ces(p_{k+1})}$ for each $k \in \mathbb{N}$ and $x \in X_{k+1}$; see [13, Proposition 3.2(iii)]. Then $X = \cap_{k=1}^{\infty} X_k$ (i.e., $ces(p+) := \cap_{k=1}^{\infty} ces(p_k)$) is a Fréchet space of the type given in Lemma 2.5 whose topology is generated by the increasing sequence of *norms* v_k, for $k \in \mathbb{N}$, given by

$$v_k : x \mapsto \|x\|_{ces(p_k)}, \quad x \in ces(p+). \tag{4.2}$$

That is, $v_k \le v_{k+1}$ for $k \in \mathbb{N}$. Again $ces(p) \subseteq ces(p+)$ (if $p > 1$) and $ces(p+) \subseteq \omega$ with both inclusions continuous, where we again use Lemma 3.1(iv). The Fréchet spaces $ces(p+)$, for $1 \le p < \infty$, have been intensively studied in [9, 14].

Finally, consider the family of Banach spaces $X_k := d_{p_k}$, in which case $X_{k+1} \subseteq X_k$ with $\|x\|_{d_{p_k}} \le \|x\|_{d_{p_{k+1}}}$ for each $k \in \mathbb{N}$ and $x \in X_{k+1}$; see [19, Proposition 5.1(iii)]. So, $X = \cap_{k=1}^{\infty} X_k$ (i.e., $d(p+) := \cap_{k=1}^{\infty} d_{p_k}$) is a Fréchet space of the type given in Lemma 2.5 whose topology is generated by the increasing sequence of *norms* w_k, for $k \in \mathbb{N}$, given by

$$w_k : x \mapsto \|x\|_{d_{p_k}}, \quad x \in d(p+). \tag{4.3}$$

That is, $w_k \le w_{k+1}$ for $k \in \mathbb{N}$. With continuous inclusions we have $d_p \subseteq d(p+) \subseteq \omega$; see [20, Sect. 4] or, argue as for ℓ^p and $\ell(p+)$.

It is known that the canonical vectors $\mathcal{E}$ belong to $\ell(p+)$, $d(p+)$ and $ces(p+)$, for $1 \le p < \infty$, and form an *unconditional basis* in each of these spaces; see [20, Proposition 3.1], [20, Lemma 4.1] and [9, Proposition 3.5(i)], respectively.

In this section we consider the compactness and determine the spectra of C_t when they act in the Fréchet spaces $\ell(p+)$, $d(p+)$ and $ces(p+)$, for $1 \le p < \infty$. The decreasing sequence $\{p_k\}_{k\in\mathbb{N}}$ always has the properties listed above. Crucial for the proofs is the existence of a particular factorization available for C_t (cf. Proposition 4.4).

The decreasing sequence φ given in (3.3) satisfies $\|\varphi\|_\infty = 1$. Define the linear map $D_\varphi : \omega \to \omega$ by

$$D_\varphi x := (\varphi_0 x_0, \varphi_1 x_1, \varphi_2 x_2, \ldots) = \left(\frac{x_n}{n+1} \right)_{n\in\mathbb{N}_0}, \quad x \in \omega. \tag{4.4}$$

The diagonal (multiplication) operator $D_\varphi \in \mathcal{L}(\omega)$ since, for each $n \in \mathbb{N}_0$,

$$r_n(D_\varphi x) \le r_n(x), \quad x \in \omega,$$

where r_n is the seminorm (3.1). Define the *right-shift operator* $S : \omega \to \omega$ by

$$Sx := (0, x_0, x_1, \ldots), \quad x \in \omega. \tag{4.5}$$

For each $n \in \mathbb{N}$ note that $r_n(Sx) = \max_{0\le k<n} |x_k| \le r_n(x)$ and for $n = 0$ that $r_0(Sx) = 0 \le r_0(x)$ for each $x \in \omega$. So, for every $n \in \mathbb{N}_0$, the operator S satisfies

$$r_n(Sx) \le r_n(x), \quad x \in \omega, \tag{4.6}$$

which implies that $S \in \mathcal{L}(\omega)$. The following result is Lemma 2.2 in [26].

Lemma 4.1 *For each $t \in [0, 1)$ we have the representation*

$$C_t = \sum_{n=0}^{\infty} t^n D_\varphi S^n$$

with the series being convergent in $\mathcal{L}_s(\omega)$. Equivalently,

$$C_t x = \sum_{n=0}^{\infty} t^n D_\varphi S^n x, \quad x \in \omega,$$

with the series being convergent in ω.

Fix $t \in [0, 1)$ and $x \in \omega$. For each $n \in \mathbb{N}_0$ it follows from (4.6) that

$$r_n \left(\sum_{k=0}^{\infty} t^k S^k x \right) \leq \sum_{k=0}^{\infty} r_n(t^k S^k x) \leq \frac{1}{1-t} r_n(x).$$

Accordingly, the series

$$R_t := \sum_{n=0}^{\infty} t^n S^n, \quad t \in [0, 1), \tag{4.7}$$

is absolutely convergent in the quasicomplete lcHs $\mathcal{L}_s(\omega)$. In particular, $R_t \in \mathcal{L}(\omega)$. Combining this with Lemma 4.1 and the fact that $D_\varphi \in \mathcal{L}(\omega)$ yields the following factorization of C_t.

Proposition 4.2 *For each $t \in [0, 1)$ the operators D_φ, R_t, C_t belong to $\mathcal{L}(\omega)$ and*

$$C_t = D_\varphi R_t = \sum_{n=0}^{\infty} t^n D_\varphi S^n, \tag{4.8}$$

with the series being absolutely convergent in $\mathcal{L}_s(\omega)$.

Our aim is to to extend Proposition 4.2 to $\mathcal{L}(X)$ with $X \in \{\ell(p+), ces(p+), d(p+) : 1 \leq p < \infty\}$, to show that $D_\varphi \in \mathcal{L}(X)$ is compact and then to apply Lemma 2.3 to conclude that $C_t \in \mathcal{L}(X)$ is compact.

Proposition 4.3 *Let X be any Fréchet space in $\{\ell(p+), ces(p+), d(p+) : 1 \leq p < \infty\}$. Then D_φ maps X into X and $D_\varphi \in \mathcal{L}(X)$ is compact.*

Proof Recall that $\varphi \in c_0$ with $\|\varphi\|_\infty = 1$. We consider each of the three possible cases for X. It was shown above that $D_\varphi \in \mathcal{L}(\omega)$ and that $X \subseteq \omega$ continuously.

(a) Suppose that $X = \ell(p+)$ for some $1 \leq p < \infty$. Clearly, $D_\varphi(X_k) \subseteq X_k$ for each $k \in \mathbb{N}$ and so $D_\varphi \in \mathcal{L}(X)$; see Lemma 2.11. In the notation of [14] it is clear from (4.4) that D_φ is precisely the multiplication operator M_φ defined there. Such a multiplication operator is compact if and only if $\varphi \in \ell(\infty-) = \cup_{s>1} \ell^s$, [14, Proposition 17], which is surely the case as $\varphi \in \ell^2$, for example. So, $D_\varphi \in \mathcal{L}(\ell(p+))$ is a compact operator.

(b) Suppose that $X = ces(p+)$ for some $1 \leq p < \infty$. It follows from (2.4) that $D_\varphi(X_k) \subseteq X_k$ for each $k \in \mathbb{N}$ and so $D_\varphi \colon X \to X$. Lemma 2.11 yields that $M_\varphi = D_\varphi \in \mathcal{L}(ces(p+))$. Moreover, if $\varphi \in d(\infty-) = \cup_{s>1} d_s$, then M_φ is also compact, [14, Proposition 10]. But, φ

is a positive decreasing sequence and so $\varphi = \hat{\varphi}$. Accordingly, by choosing $s = 2$ say, we see that

$$\|\varphi\|_{d_2} := \|\hat{\varphi}\|_2 = \|\varphi\|_2 < \infty.$$

Hence, $\varphi \in d_2 \subseteq d(\infty-)$ and so $D_\varphi = M_\varphi \in \mathcal{L}(ces(p+))$ is indeed compact.

(c) Suppose $X = d(p+)$ for some $1 \leq p < \infty$. Since $|D_\varphi x| = D_\varphi |x| \leq |x|$, for $x \in \ell^\infty$, it is clear that $\widehat{D_\varphi x} \leq \hat{x}$. Then (2.6) implies that $D_\varphi(X_k) \subseteq X_k$ for all $k \in \mathbb{N}$ and so $D_\varphi \colon X \to X$. Again Lemma 2.11 yields that $D_\varphi \in \mathcal{L}(d(p+))$. Note that the operator $M^{\varphi}_{d(p+)}$ in [21] is precisely $D_\varphi \colon d(p+) \to d(p+)$. It was verified in (b) above that $\varphi \in d(\infty-)$ which, together with $D_\varphi \in \mathcal{L}(d(p+))$, implies that D_φ is compact, [21, Theorem 4.13(i)]. □

Proposition 4.4 *Let $t \in [0, 1)$, and X be any Fréchet space in $\{\ell(p+), ces(p+), d(p+) : 1 \leq p < \infty\}$.*

(i) *The generalized Cesàro operator C_t maps X into itself and $C_t \in \mathcal{L}(X)$.*
(ii) *The right-shift operator S given by (4.5) maps X into itself and belongs to $\mathcal{L}(X)$.*
(iii) *The operator R_t given by (4.7) maps X into itself and belongs to $\mathcal{L}(X)$, with the series $\sum_{n=0}^{\infty} t^n S^n$ being absolutely convergent in $\mathcal{L}_s(X)$. Moreover,*

$$C_t = D_\varphi R_t = \sum_{n=0}^{\infty} t^n D_\varphi S^n.$$

Proof (i) Again we consider the three possible cases for X. Fix $t \in [0, 1)$. According to Proposition 3.2 the operator $C_t \in \mathcal{L}(\omega)$.

(a) Suppose that $X = \ell(p+)$ for some $1 \leq p < \infty$. Proposition 2.12 implies that $C_t(X_k) \subseteq X_k$ for all $k \in \mathbb{N}$, with $X_k = \ell^{p_k}$, and so $C_t(X) \subseteq X$. In view of Lemma 2.11, with $T := C_t$, it follows that $C_t \in \mathcal{L}(\ell(p+))$.

(b) Suppose that $X = ces(p+)$ for some $1 \leq p < \infty$. Proposition 2.14 shows that $C_t(X_k) \subseteq X_k$ for all $k \in \mathbb{N}$, with $X_k = ces(p_k)$, and so $C_t(X) \subseteq X$. Again, for $T := C_t$, Lemma 2.11 implies that $C_t \in \mathcal{L}(ces(p+))$.

(c) Suppose that $X = d(p+)$ for some $1 \leq p < \infty$. Proposition 2.15 shows that $C_t(X_k) \subseteq X_k$ for all $k \in \mathbb{N}$, with $X_k = d_{p_k}$, and so $C_t(X) \subseteq X$. Yet again, for $T := C_t$, Lemma 2.11 implies that $C_t \in \mathcal{L}(d(p+))$.

(ii) Again we check the three separate cases for X. Prior to Lemma 4.1 it was shown that $S \in \mathcal{L}(\omega)$.

(a) Suppose that $X = \ell(p+)$ for some $1 \leq p < \infty$. Using the fact that the Banach space right-shift operator $S \colon \ell^{p_k} \to \ell^{p_k}$ is an isometry, for every $k \in \mathbb{N}$, we see that $S(X) \subseteq X$. It follows that $S \in \mathcal{L}(\ell(p+))$; see Lemma 2.11 for $T := S \in \mathcal{L}(\omega)$.

(b) Suppose that $X = ces(p+)$ for some $1 \leq p < \infty$. It is known, for each $k \in \mathbb{N}$, that $S \in \mathcal{L}(ces(p_k))$ and $\|S\|_{ces(p_k) \to ces(p_k)} \leq 1$, [26, Lemma 5.4]. Accordingly, $S(X) \subseteq X$ and so Lemma 2.11, for $T := S \in \mathcal{L}(\omega)$, implies that $S \in \mathcal{L}(ces(p+))$.

(c) Suppose that $X = d(p+)$ for some $1 \leq p < \infty$. Fix $k \in \mathbb{N}$. It is known that $S \in \mathcal{L}(d_{p_k})$ and

$$\|S^m\|_{d_{p_k} \to d_{p_k}} = (m+1)^{1/p_k}, \quad m \in \mathbb{N}_0, \tag{4.9}$$

[26, Lemma 6.2]. For $m = 1$ we can conclude that $S(d_{p_k}) \subseteq d_{p_k}$ for $k \in \mathbb{N}$, that is, $S(X) \subseteq X$. So, in view of Lemma 2.11, for $T := S \in \mathcal{L}(\omega)$, it follows that $S \in \mathcal{L}(d(p+))$.

(iii) (a) Suppose that $X = \ell(p+)$ for some $1 \leq p < \infty$. Fix $k \in \mathbb{N}$ and $x \in \ell(p+) \subseteq \ell^{p_k}$. It follows from S being an isometry in ℓ^{p_k} that $u_k(S^n x) = u_k(x)$ for all $n \in \mathbb{N}_0$ and hence, that

$$\sum_{n=0}^{\infty} u_k(t^n S^n x) = \sum_{n=0}^{\infty} t^n u_k(S^n x) \leq \frac{1}{1-t} u_k(x) < \infty.$$

Accordingly, the series $\sum_{n=0}^{\infty} t^n S^n x$ is absolutely convergent in the Fréchet space $\ell(p+)$ for each $x \in \ell(p+)$. By part (ii) the sequence $\{\sum_{n=0}^{m} t^n S^n\}_{m \in \mathbb{N}_0} \subseteq \mathcal{L}(\ell(p+))$ and so, by the Banach-Steinhaus theorem (as $\ell(p+)$ is barrelled), the series $\sum_{n=0}^{\infty} t^n S^n$ is absolutely convergent in $\mathcal{L}_s(\ell(p+))$; its sum is denoted by $R_t \in \mathcal{L}(\ell(p+))$.

It has been established that each of the operators C_t, D_φ, R_t belongs to $\mathcal{L}(\ell(p+))$. The identities $C_t = D_\varphi R_t = \sum_{n=0}^{\infty} t^n D_\varphi S^n$ are valid in $\mathcal{L}(\ell(p+))$ because they are valid in $\mathcal{L}(\omega)$; see Lemma 4.1 and both (4.7) and (4.8).

(b) Suppose $X = ces(p+)$ for some $1 \leq p < \infty$. Fix $k \in \mathbb{N}$ and $x \in ces(p+) \subseteq ces(p_k)$. Using $\|S^n\|_{ces(p_k) \to ces(p_k)} \leq 1$, for all $n \in \mathbb{N}_0$ (see the proof of part (ii)(b)), we can argue as in (a) to conclude that

$$\sum_{n=0}^{\infty} v_k(t^n S^n x) \leq \frac{1}{1-t} v_k(x) < \infty.$$

Hence, the series $\sum_{n=0}^{\infty} t^n S^n x$ is absolutely convergent in $ces(p+)$ for each $x \in ces(p+)$. Then argue as in (a) to deduce that the series $R_t := \sum_{n=0}^{\infty} t^n S^n$ is absolutely convergent in $\mathcal{L}_s(ces(p+))$, with $R_t \in \mathcal{L}(ces(p+))$, and that the identities $C_t = D_\varphi R_t = \sum_{n=0}^{\infty} t^n D_\varphi S^n$ are valid in $\mathcal{L}(ces(p+))$.

(c) Let $X = d(p+)$ for some $1 \leq p < \infty$. Fix $k \in \mathbb{N}$ and $x \in d(p+) \subseteq d_{p_k}$. It follows from (4.9) that

$$w_k(S^m x) = \|S^m x\|_{d_{p_k}} \leq \|S^m\|_{d_{p_k} \to d_{p_k}} \|x\|_{d_{p_k}} = (m+1)^{1/p_k} w_k(x), \quad m \in \mathbb{N}_0,$$

and hence, since $0 \leq t < 1$, that

$$\sum_{n=0}^{\infty} w_k(t^n S^n x) \leq \left(\sum_{n=0}^{\infty} t^n (n+1)^{1/p_k} \right) w_k(x) < \infty.$$

Now argue as in (a) to conclude that the series $R_t := \sum_{n=0}^{\infty} t^n S^n$ is absolutely convergent in $\mathcal{L}_s(d(p+))$, with $R_t \in \mathcal{L}(d(p+))$, and that the identities $C_t = D_\varphi R_t = \sum_{n=0}^{\infty} t^n D_\varphi S^n$ are valid in $\mathcal{L}(d(p+))$. □

We come to the main result of this section, which should be compared with Proposition 3.2 and Theorem 3.7.

Theorem 4.5 *Let $t \in [0,1)$ and X be any Fréchet space in $\{\ell(p+), ces(p+), d(p+) : 1 \leq p < \infty\}$.*

(i) *The generalized Cesàro operator $C_t \in \mathcal{L}(X)$ is compact.*
(ii) *The spectra of C_t are given by*

$$\sigma_{pt}(C_t; X) = \Lambda \tag{4.10}$$

and

$$\sigma^*(C_t; X) = \sigma(C_t; X) = \Lambda \cup \{0\}. \tag{4.11}$$

(iii) *For each* $\lambda \in \sigma_{pt}(C_t; X)$ *the subspace* $(\lambda I - C_t)(X)$ *is closed in* X *with* codim $(\lambda I - C_t)(X) = 1$. *Moreover, the* 1*-dimensional eigenspace* $\mathrm{Ker}(\frac{1}{m+1} I - C_t) = \mathrm{span}(x^{[m]})$, *for each* $m \in \mathbb{N}_0$, *with* $x^{[m]} \in d_1 \subseteq X$ *given by* (3.8).

Proof (i) Since $D_\varphi \in \mathcal{L}(X)$ is compact (cf. Proposition 4.3) and $R_t \in \mathcal{L}(X)$ (cf. Proposition 4.4(iii)), the compactness of C_t follows from the factorization $C_t = D_\varphi R_t$ (cf. Proposition 4.4(iii)) and Lemma 2.3.

(ii) Since $X \subseteq \omega$, we can conclude from Theorem 3.7 that

$$\sigma_{pt}(C_t; X) \subseteq \sigma_{pt}(C_t; \omega) = \Lambda. \tag{4.12}$$

Fix $1 \leq p < \infty$. Then $d_1 \subseteq \ell^1 \subseteq \ell^p \subseteq \ell(p+)$. Since $\ell^p \subseteq ces(p) \subseteq ces(p+)$ (cf. (1) on p. 2 of [24]), it follows that also $d_1 \subseteq ces(p+)$. Moreover, $d_1 \subseteq d_p \subseteq d(p+)$. So, $d_1 \subseteq X$. Given $\nu \in \Lambda$ there exists $m \in \mathbb{N}_0$ such that $\nu = \varphi_m$. According to Lemma 3.4 the 1-dimensional eigenspace corresponding to $\nu \in \sigma_{pt}(C_t; \omega)$ is spanned by $x^{[m]}$ with $x^{[m]} \in d_1$. Since $d_1 \subseteq X$, it follows that $\nu \in \sigma_{pt}(C_t; X)$. So, it has been established that $\Lambda \subseteq \sigma_{pt}(C_t; X)$. Combined with (4.12) we can conclude that (4.10) is valid.

The spectrum of a compact operator in a lcHs is necessarily a compact subset of $\mathbb{C}$ (see [27, Theorem 9.10.2], [33, Theorem 4 & Proposition 6]) and it is either a finite set or a countable sequence of non-zero eigenvalues with limit point 0. It follows from part (i) and (4.10) that

$$\sigma(C_t; X) = \Lambda \cup \{0\}. \tag{4.13}$$

The discussion in the first three paragraphs of this section, with the notation from there, shows that $X = \cap_{k=1}^\infty X_k$ is a Fréchet space of the type given in Lemma 2.5. Setting there $T := C_t \in \mathcal{L}(X)$ and $T_n := C_t \in \mathcal{L}(X_n)$ for $n \in \mathbb{N}$ (see Propositions 2.12, 2.14 and 2.15), it is clear that condition (A) is satisfied. Moreover, $\sigma(T_n; X_n) = \Lambda \cup \{0\}$ for every $n \in \mathbb{N}$ (cf. (2.3), (2.5) and (2.7) with p_n in place of p) and so, via (4.13), we have that

$$\cup_{n=1}^\infty \sigma(T_n; X_n) = \Lambda \cup \{0\} = \sigma(T; X) = \sigma(C_t; X).$$

In particular, $\cup_{n=1}^\infty \sigma(T_n; X_n) \subseteq \overline{\sigma(T; X)}$ and so we can conclude from Lemma 2.5 that (4.11) is valid.

(iii) First observe that $(\nu I - C_t) = \nu(I - \nu^{-1} C_t)$, for $\nu \in \mathbb{C}\backslash\{0\}$, with $\nu^{-1} C_t$ being a compact operator by part (i). So, by [27, Theorem 9.10.1(i)], the subspace $(\nu I - C_t)(X)$ is closed in X with codim $(\nu I - C_t)(X) = \dim \mathrm{Ker}(\nu I - C_t)$ for every $\nu \in \sigma_{pt}(C_t; X)$. But, $\dim \mathrm{Ker}(\nu I - C_t) = 1$ for $\nu \in \sigma_{pt}(C_t; X)$, as observed in the proof of part (ii), where it was also established that $\mathrm{Ker}(\frac{1}{m+1} I - C_t) = \mathrm{span}(x^{[m]})$, for each $m \in \mathbb{N}_0$. □

Remark 4.6 (i) The identity (4.10), established in the proof of part (ii) of Theorem 4.5, can also be deduced from Lemma 2.5(ii).

(ii) Let $t \in [0, 1)$ and X be any Fréchet space in $\{\ell(p+), ces(p+), d(p+) : 1 \leq p < \infty\}$. Since $X \subseteq \omega$ and $C_t \in \mathcal{L}(\omega)$ is injective (cf. Lemma 3.6), also $C_t \in \mathcal{L}(X)$ is injective. Moreover, as $C_t \in \mathcal{L}(X)$ is compact (cf. Theorem 4.5(i)) it cannot be surjective, otherwise it would be an isomorphism thereby implying that $0 \in \rho(C_t; X)$, which is *not* the case (see (4.11)). Recall that $\mathcal{E}$ is a basis for X and, by Lemma 3.1(iii), that the range $C_t(X)$ is a proper, dense subspace of X. Hence, 0 belongs to the *continuous spectrum* of C_t. This is in contrast to the situation of ω, where $0 \in \rho(C_t; \omega)$; see Theorem 3.7.

(iii) Concerning the case when $t = 1$, it is known that $\sigma_{pt}(C_1; \ell(p+)) = \emptyset$ and

$$\sigma(C_1; \ell(p+)) = \left\{ z \in \mathbb{C} \,:\, \left| z - \frac{p'}{2} \right| < \frac{p'}{2} \right\} \cup \{0\} \text{ and } \sigma^*(C_1; \ell(p+)) = \overline{\sigma(C_1; \ell(p+))}, \tag{4.14}$$

for every $1 < p < \infty$, [8, Theorem 2.2]. For $p = 1$, again $\sigma_{pt}(C_1; \ell(1+)) = \emptyset$ whereas

$$\sigma(C_1; \ell(1+)) = \{z \in \mathbb{C} \,:\, \operatorname{Re} z > 0\} \cup \{0\} \text{ and } \sigma^*(C_1; \ell(1+)) = \overline{\sigma(C_1; \ell(1+))}, \tag{4.15}$$

[8, Theorem 2.4]. For the Fréchet space $ces(p+)$, both (4.14) and (4.15) are also valid (with $ces(p+)$, resp. with $ces(1+)$, in place of $\ell(p+)$, resp. in place of $\ell(1+)$), as well as $\sigma_{pt}(C_1; ces(p+)) = \emptyset$ for all $1 \le p < \infty$, [14, Theorem 3]. For the Fréchet space $d(p+)$, both (4.14) and (4.15) are again valid with $d(p+)$ (resp. with $d(1+)$), in place of $\ell(p+)$ (resp. of $\ell(1+)$), as well as $\sigma_{pt}(C_1; d(p+)) = \emptyset$ for all $1 \le p < \infty$, [21, Theorem 3.2].

5 C_t acting in the (LB)-spaces $\ell(p-)$, $d(p-)$ and $ces(p-)$

Given $1 < p \le \infty$, consider any strictly increasing sequence $\{p_k\}_{k\in\mathbb{N}} \subseteq (1, p)$ which satisfies $p_k \uparrow p$. The Banach spaces $X_k := \ell^{p_k}$ satisfy $X_k \subset X_{k+1}$ with a continuous inclusion, for each $k \in \mathbb{N}$, and $X = \cup_{k=1}^{\infty} X_k$ is an (LB)-space, necessarily *regular* by Lemma 2.6. The (LB)-space X is denoted by $\ell(p-) = \operatorname{ind}_k \ell^{p_k}$. If we set $X_k := ces(p_k)$, then again $X_k \subset X_{k+1}$ for $k \in \mathbb{N}$ (see the discussion prior to Proposition 3.3 in [13]) with a continuous inclusion. The (LB)-space $X := \cup_{k=1}^{\infty} X_k$, necessarily *regular* by Lemma 2.6, is denoted by $ces(p-) := \operatorname{ind}_k ces(p_k)$. Finally, the Banach spaces $X_k := d_{p_k}$ satisfy $X_k \subset X_{k+1}$ with a continuous inclusion, for $k \in \mathbb{N}$ (see Propositions 2.7(ii) and 5.1(iii) in [19]). The (LB)-space $X := \cup_{k=1}^{\infty} X_k$, necessarily *regular* by Lemma 2.6, is denoted by $d(p-) := \operatorname{ind}_k d_{p_k}$. The discussion after (3.7) shows that d_1 is continuously included in each space in $\{\ell^p, ces(p), d_p \,:\, 1 < p < \infty\}$, from which it follows that $d_1 \subseteq X$ continuously, for each $X \in \{\ell(p-), ces(p-), d(p-) \,:\, 1 < p \le \infty\}$. Indeed, by the definition of the inductive limit topology, $\ell^p \subseteq \ell(p-)$ and $d_p \subseteq d(p-)$ and $ces(p) \subseteq ces(p-)$ with all inclusions continuous. In all of these (LB)-spaces the canonical vectors $\mathcal{E}$ form a Schauder basis. Indeed, concerning $\ell(p-)$ recall that $\mathcal{E}$ is a basis for each Banach space ℓ^{p_k} and the natural inclusion $\ell^{p_k} \subseteq \ell(p-)$ is continuous for each $k \in \mathbb{N}$. It follows that $\mathcal{E}$ is a Schauder basis for $\ell(p-)$. For the (LB)-spaces $ces(p-)$, resp. $d(p-)$, see [12, Proposition 2.1], resp. [20, Theorem 4.6]. It follows from [44, Proposition 24.7] together with Lemma 3.1(iv) that $X \subseteq \omega$ continuously. For further properties of the (LB)-spaces $\ell(p-)$, $ces(p-)$ and $d(p-)$, and operators acting in them, we refer to [12, 20, 21], for example, and the references therein.

For each of the three cases above it is clear that the diagonal (multiplication) operator $D_\varphi \in \mathcal{L}(\omega)$ as defined in (4.4) satisfies $D_\varphi(X_k) \subseteq X_k$ for all $k \in \mathbb{N}$ (cf. proof of Proposition 4.3) and so $D_\varphi(X) \subseteq X$. By Lemma 2.11 it follows that $D_\varphi \in \mathcal{L}(X)$. Actually, $D_\varphi \in \mathcal{L}(X)$ is a compact operator. For the case $X = \ell(p-)$, since $\varphi \in \ell^2 \subseteq \ell(\infty-)$, Proposition 4.5 of [12] implies that $D_\varphi \in \mathcal{L}(\ell(p-))$ is compact. Suppose now that $X := ces(p-)$. By Proposition 4.2 of [12] it follows that $D_\varphi \in \mathcal{L}(ces(p-))$ is compact provided that $\hat{\varphi} \in \ell^t$ for some $t > q$ (with $\frac{1}{p} + \frac{1}{q} = 1$). But, it is clear from (3.3) that $\hat{\varphi} = \varphi \in \cap_{s>1} \ell^s$ and so D_φ is a compact operator in $ces(p-)$. Consider now when $X := d(p-)$. Since $\hat{\varphi} \in \ell^2$ and $\hat{\varphi} = \varphi$, it follows that $\varphi \in d_2 \subseteq d(\infty-)$ and so Proposition 4.13(ii) of [21] implies that D_φ is a compact operator in $d(p-)$. So, we have established the following result.

Proposition 5.1 *Let X be any (LB)-space in $\{\ell(p-), ces(p-), d(p-) : 1 < p \leq \infty\}$. Then D_φ maps X into itself and $D_\varphi \in \mathcal{L}(X)$ is a compact operator.*

The following result will also be required.

Proposition 5.2 *Let $t \in [0, 1)$ and X be any (LB)-space in $\{\ell(p-), ces(p-), d(p-) : 1 < p \leq \infty\}$.*

(i) The right-shift operator S given by (4.5) maps X into X and belongs to $\mathcal{L}(X)$.
(ii) The generalized Cesàro operator C_t maps X into X and satisfies $C_t \in \mathcal{L}(X)$.
(iii) The operator R_t given by (4.7) maps X into X and belongs to $\mathcal{L}(X)$, with the series $\sum_{n=0}^{\infty} t^n S^n$ being convergent in $\mathcal{L}_s(X)$. Moreover,

$$C_t = D_\varphi R_t = \sum_{n=0}^{\infty} t^n D_\varphi S^n. \tag{5.1}$$

Proof (i) It was observed in the proof of Proposition 4.4(ii) that $S \in \mathcal{L}(\omega)$ as well as $S(\ell^{p_k}) \subseteq \ell^{p_k}$ and $S(ces(p_k)) \subseteq ces(p_k)$ and $S(d_{p_k}) \subseteq d_{p_k}$, for each $k \in \mathbb{N}$, from which it is clear that $S(X) \subseteq X$. By Lemma 2.11 it follows that $S \in \mathcal{L}(X)$.

(ii) In each of the three cases $\ell(p-)$, $ces(p-)$, $d(p-)$ for X it is clear that $C_t : \omega \to \omega$ (cf. (1.1)) satisfies $C_t(X_k) \subseteq X_k$ for all $k \in \mathbb{N}$ (see the proof of Proposition 4.4(i)) and hence, $C_t(X) \subseteq X$. Since $C_t \in \mathcal{L}(\omega)$, via Proposition 3.2, again by Lemma 2.11 we can conclude that $C_t \in \mathcal{L}(X)$.

(iii) According to part (i) the sequence $\{\sum_{n=0}^{k} t^n S^n\}_{k\in\mathbb{N}_0} \subseteq \mathcal{L}(X)$.

Claim. $\{\sum_{n=0}^{k} t^n S^n : k \in \mathbb{N}_0\}$ *is an equicontinuous subset of* $\mathcal{L}(X)$.
Suppose first that $X = \ell(p-)$ or $X = ces(p-)$. Since X is barrelled, to establish the *Claim* it suffices to show, for each $x \in X$, that

$$B(x) := \left\{ \sum_{n=0}^{k} t^n S^n x \ : \ k \in \mathbb{N}_0 \right\}$$

is a bounded subset of $X = \operatorname{ind}_r X_r$. Since X is a regular (LB)-space, the set $B(x)$ will be bounded if there exists $m \in \mathbb{N}$ such that $B(x) \subseteq X_m$ and $B(x)$ is bounded in the Banach space X_m. But, $x \in X = \cup_{r=1}^{\infty} X_r$ and so there exists $m \in \mathbb{N}$ such that $x \in X_m$. Since $S^n \in \mathcal{L}(X_m)$ for all $n \in \mathbb{N}_0$, it is clear that $B(x) \subseteq X_m$. Moreover, in the proof of Proposition 4.4(ii) it was noted that $\|S\|_{X_m \to X_m} \leq 1$ and hence, $\|S^n\|_{X_m \to X_m} \leq 1$ for all $n \in \mathbb{N}_0$. Accordingly,

$$\|\sum_{n=0}^{k} t^n S^n x\|_{X_m} \leq \sum_{n=0}^{\infty} t^n \|S^n x\|_{X_m} \leq \sum_{n=0}^{\infty} t^n \|S^n\|_{X_m \to X_m} \|x\|_{X_m} \leq \frac{\|x\|_{X_m}}{(1-t)}, \quad k \in \mathbb{N}_0,$$

which implies that $B(x)$ is a bounded set in X_m. In the event that $X = d(p-)$, an analogous argument applies except that now $X_m = d_{p_m}$ and so $\|S^n\|_{d_{p_m} \to d_{p_m}} = (n+1)^{1/p_m}$ for $n \in \mathbb{N}_0$; see (4.9). In this case the previous inequality becomes

$$\|\sum_{n=0}^{k} t^n S^n x\|_{d_{p_m}} \leq \left(\sum_{n=0}^{\infty} t^n (n+1)^{1/p_m} \right) \|x\|_{d_{p_m}}, \quad k \in \mathbb{N}_0,$$

which implies that $B(x)$ is a bounded set in d_{p_m} as $\sum_{n=0}^{\infty} t^n (n+1)^{1/p_m} < \infty$. The proof of the *Claim* is thereby complete.

In view of the *Claim*, to show that the series $\sum_{n=0}^{\infty} t^n S^n$ converges in $\mathcal{L}_s(X)$ it suffices to show that the limit

$$R_t x := \lim_{k\to\infty} \sum_{n=0}^{k} t^n S^n x = \sum_{n=0}^{\infty} t^n S^n x \tag{5.2}$$

exists in X for all $x \in X$ in some dense subset of X. Since $\mathcal{E}$ is a Schauder basis for X, its linear span span $\mathcal{E}$ is a dense subspace of X and so it suffices to show that the limit in (5.2) exists for each $x \in \mathcal{E}$. Let $x := e_r = (0, \ldots, 0, 1, 0, \ldots)$, for any fixed $r \in \mathbb{N}_0$, where 1 is in position r. Then $S^n e_r = e_{r+n}$ for all $n \in \mathbb{N}_0$. Fix $k \in \mathbb{N}_0$. It follows that

$$\sum_{n=0}^{k} t^n S^n e_r = \sum_{n=0}^{k} t^n e_{r+n} = (0, \ldots, 1, t, t^2, \ldots, t^k, 0, 0, \ldots), \tag{5.3}$$

where 1 is in position r and t^k is in position $r + k$. Observe that $\|e_j\|_{\ell^{p_1}} = 1$ for $j \in \mathbb{N}_0$. Direct calculation via (2.6) shows that $\|e_j\|_{d_{p_1}} = (j+1)^{1/p_1}$, for $j \in \mathbb{N}_0$, and by Lemma 4.7 in [17], there exists $K > 0$ such that $\|e_j\|_{ces(p_1)} \leq K$ for all $j \in \mathbb{N}_0$. It follows that $\sum_{j=r}^{\infty} t^j \|e_j\|_{\ell^{p_1}} = \frac{t^r}{(1-t)} \leq \frac{1}{(1-t)}$, that $\sum_{j=r}^{\infty} t^j \|e_j\|_{ces(p_1)} \leq \frac{Kt^r}{(1-t)} \leq \frac{K}{(1-t)}$ and that $\sum_{j=r}^{\infty} t^j \|e_j\|_{d_{p_1}} \leq \sum_{j=r}^{\infty} t^j (j+1)^{1/p_1} < \infty$. Accordingly, the series

$$y^{[r]} := \sum_{j=r}^{\infty} t^j e_j = (0, \ldots, 0, 1, t, t^2, \ldots), \tag{5.4}$$

with 1 in position r, is absolutely convergent in the Banach space X_1 belonging to $\{\ell^{p_1}, ces(p_1), d_{p_1}\}$ and defines an element of X_1, that is, $y^{[r]} \in X_1$. Since the inclusion $X_1 \subseteq X$ is continuous, the series (5.4) is also convergent to $y^{[r]}$ in X. For any $k > r$ we have

$$\|y^{[r]} - \sum_{n=0}^{k} t^n S^n e_r\|_{X_1} = \|\sum_{j=r+k+1}^{\infty} t^j e_j\|_{X_1} \to 0, \quad k \to \infty,$$

being the tail of the absolutely convergent series (5.2). So, the sequence in (5.3) converges to $y^{[r]}$ in X_1 for $k \to \infty$ and hence, also to $y^{[r]}$ in X. Since $r \in \mathbb{N}_0$ is arbitrary, we have proved that the limit in (5.2) exists in X for each $x \in$ span $\mathcal{E}$ and hence, by the *Claim*, it exists for *every* $x \in X$. Accordingly, the limit operator $R_t = \lim_{k\to\infty} \sum_{n=0}^{k} t^n S^n$ exists in $\mathcal{L}_s(X)$. Since $D_\varphi, R_t, C_t \in \mathcal{L}(X)$ and $X \subseteq \omega$ continuously, the equality $C_t = D_\varphi R_t = \sum_{n=0}^{\infty} t^n D_\varphi S^n$ follows from Proposition 4.2. □

The main result of this section is as follows.

Theorem 5.3 *Let $t \in [0, 1)$ and X be any (LB)-space in $\{\ell(p-), ces(p-), d(p-) : 1 < p \leq \infty\}$.*

(i) *The generalized Cesàro operator $C_t \in \mathcal{L}(X)$ is compact.*
(ii) *The spectra of C_t are given by*

$$\sigma_{pt}(C_t; X) = \Lambda \tag{5.5}$$

and

$$\sigma^*(C_t; X) = \sigma(C_t; X) = \Lambda \cup \{0\}. \tag{5.6}$$

(iii) *For each* $\lambda \in \sigma_{pt}(C_t; X)$ *the subspace* $(\lambda I - C_t)(X)$ *is closed in* X *with* codim $(\lambda I - C_t)(X) = 1$. *Moreover, the* 1*-dimensional eigenspace* $\mathrm{Ker}(\frac{1}{m+1}I - C_t) = \mathrm{span}(x^{[m]})$, *for each* $m \in \mathbb{N}_0$, *with* $x^{[m]} \in d_1 \subseteq X$ *given by* (3.8).

Proof (i) Since $D_\varphi \in \mathcal{L}(X)$ is compact (cf. Proposition 5.1) and $R_t \in \mathcal{L}(X)$ (cf. Proposition 5.2(iii)), the compactness of $C_t \in \mathcal{L}(X)$ follows from the factorization in (5.1) and Lemma 2.3.

(ii) The (LB)-space $X = \mathrm{ind}_k X_k$ is an inductive limit of the type in Lemma 2.10. Moreover, $T := C_t \in \mathcal{L}(X)$ has the property, for each $k \in \mathbb{N}$, that the restriction T_k of T to the Banach space X_k maps X_k into itself and satisfies $T_k \in \mathcal{L}(X_k)$. That is, T satisfies condition (A′) of Lemma 2.10. Then, by Lemma 2.10(i) it follows that $\sigma_{pt}(C_t; X) = \cup_{k=1}^{\infty}\sigma_{pt}(T_k; X_k) = \Lambda$ (cf. Propositions 2.12, 2.14 and 2.15). Since $C_t \in \mathcal{L}(X)$ is compact by part (i), the analogous argument used to prove (4.13), now with (4.10) replaced by (5.5), can be used to show that

$$\sigma(C_t; X) = \Lambda \cup \{0\}. \tag{5.7}$$

Moreover, $\sigma(T_k; X_k) = \sigma(C_t; X_k) = \Lambda \cup \{0\}$ for every $k \in \mathbb{N}$ and so, for $m = 1$ say, we note (via (5.7)) that

$$\cup_{k=m}^{\infty}\sigma(T_k; X_k) = \Lambda \cup \{0\} \subseteq \overline{\sigma(T; X)}.$$

We can conclude again from Lemma 2.10(ii) that $\sigma^*(C_t; X) = \overline{\sigma(C_t; X)}$. Combined with (5.7) this yields (5.6).

(iii) The analogous argument used to prove part (iii) of Theorem 4.5 also applies to establish the given statement. Again, since $d_1 \subseteq X$ (see the introduction to Sect. 5), it follows that $\mathrm{Ker}(\frac{1}{m+1}I - C_t) = \mathrm{span}(x^{[m]})$, for each $m \in \mathbb{N}_0$. □

Remark 5.4 (i) An examination of the arguments given in Remark 4.6 shows that, when suitably adapted, they also apply here to conclude that $C_t(X)$ is a proper, dense subspace of X. That is, 0 belongs to the *continuous spectrum* of C_t.

(ii) Concerning $t = 1$, it is known that $\sigma_{pt}(C_1; ces(p-)) = \emptyset$, [12, Proposition 3.1] with

$$\{0\} \cup \left\{z \in \mathbb{C} : \left|z - \frac{p'}{2}\right| < \frac{p'}{2}\right\} \subseteq \sigma(C_1; ces(p-)) \subseteq \left\{z \in \mathbb{C} : \left|z - \frac{p'}{2}\right| \le \frac{p'}{2}\right\} \tag{5.8}$$

and

$$\sigma^*(C_1; ces(p-)) = \left\{z \in \mathbb{C} : \left|z - \frac{p'}{2}\right| \le \frac{p'}{2}\right\} = \overline{\sigma(C_1; ces(p-))},\ 1 < p \le \infty, \tag{5.9}$$

[12, Propositions 3.2 and 3.3].

For the (LB)-space $d(p-)$, both (5.8) and (5.9) are also valid (with $d(p-)$ in place of $ces(p-)$) as well as $\sigma_{pt}(C_1; d(p-)) = \emptyset$, for all $1 < p \le \infty$; see Theorem 3.6 in [21].

The spectrum of C_1 acting in $\ell(p-)$ is covered by the next result.

Recall that the space $\ell(p'+)$ is the strong dual of $\ell(p-)$, [20, Proposition 3.4(i)], and that the dual operator $C_1' \in \mathcal{L}(\ell(p'+))$ of $C_1 \in \mathcal{L}(\ell(p-))$ is given by

$$C_1'x = \left(\sum_{i=n}^{\infty}\frac{x_i}{i+1}\right)_{n\in\mathbb{N}_0}, \quad x = (x_n)_{n\in\mathbb{N}_0} \in \ell(p'+),$$

see, for instance, [40, p. 123].

Proposition 5.5 *Let* $p \in (1, \infty]$ *and let* $p' \in [1, \infty)$ *satisfy* $\frac{1}{p} + \frac{1}{p'} = 1$.

(i) $\sigma_{pt}(C_1; \ell(p-)) = \emptyset$ *and* $\{z \in \mathbb{C} : |z - \frac{p'}{2}| < \frac{p'}{2}\} \subseteq \sigma_{pt}(C_1'; \ell(p'+))$.

(ii) $\{0\} \cup \{z \in \mathbb{C} : |z - \frac{p'}{2}| < \frac{p'}{2}\} \subseteq \sigma(C_1; \ell(p-)) \subseteq \{z \in \mathbb{C} : |z - \frac{p'}{2}| \leq \frac{p'}{2}\}$.

(iii) $\sigma^*(C_1; \ell(p-)) = \{z \in \mathbb{C} : |z - \frac{p'}{2}| \leq \frac{p'}{2}\} = \overline{\sigma(C_1; \ell(p-))}$.

Proof (i) The first part of (i) follows from Lemma 2.10(i), the definition $\ell(p-) = \cup_{k=1}^{\infty} \ell^{p_k}$ with $1 < p_k \uparrow p$, and the fact that $\sigma_{pt}(C_1; \ell^q) = \emptyset$ for every $1 < q < \infty$; see Proposition 2.13(ii).

To establish the second part, fix $z \in \mathbb{C}$ with $|z - \frac{p'}{2}| < \frac{p'}{2}$. Since $1 < p_k \uparrow p$, it follows that $p_k' \downarrow p'$ and hence, the open disk $B(\frac{p'}{2}, \frac{p'}{2}) \subseteq B(\frac{p_k'}{2}, \frac{p_k'}{2})$ for every $k \in \mathbb{N}$. Accordingly, $|z - \frac{p_k'}{2}| < \frac{p_k'}{2}$ for all $k \in \mathbb{N}$. So, by [40, Theorem 1(b)], for each $k \geq 1$ there exists $x_k \in \ell^{p_k'} \setminus \{0\}$ such that $C_1' x_k = z x_k$ with $x_k = (x_{k,i})_{i \in \mathbb{N}_0}$ satisfying $x_{k,i+1} = x_{k,0} \prod_{h=0}^{i} (1 - \frac{1}{z(h+1)})$ for all $i \in \mathbb{N}_0$ (see (1) on p. 125 of [40]) for some $x_{k,0} \in \mathbb{C} \setminus \{0\}$. Setting $x_{k,0} := 1$ for each $k \in \mathbb{N}$, it follows that $x_k = x_1 =: x$ for all $k \in \mathbb{N}$ and hence, $x \in \cap_{k \in \mathbb{N}} \ell^{p_k'} = (\ell(p-))' = \ell(p'+)$. On the other hand, it is clear that $C_1' x = z x$. This shows the second part of (i).

(ii) To establish the second containment in (ii) we note that an analogous proof as that given for Proposition 3.2 in [12] also applies here. The use of Theorem 3.1 and Lemma 3.1(ii) there needs to be replaced, respectively, with the fact that $\sigma(C_1; \ell^q) = \{z \in \mathbb{C} : |z - \frac{q'}{2}| \leq \frac{q'}{2}\}$ for $1 < q < \infty$ (cf. Proposition 2.13(ii)) and Lemma 2.10(iii).

Concerning the first containment in (ii), observe that C_1 is *not surjective* on $\ell(p-)$. Indeed, the element $y := (\frac{1-(-1)^{n+1}}{2(n+1)})_{n \in \mathbb{N}_0}$ belongs to ℓ^{p_1} with $\ell^{p_1} \subseteq \ell(p-)$ and so $y \in \ell(p-)$. On the other hand, $x := C_1^{-1} y = ((-1)^n)_{n \in \mathbb{N}_0}$ belongs to ω but, $x \notin \ell^{p_k}$ for every $k \in \mathbb{N}$ implies that $x \notin \ell(p-) = \cup_{k=1}^{\infty} \ell^{p_k}$. Since x is the unique element in ω satisfying $y = C_1 x$ (as $C_1 \in \mathcal{L}(\omega)$ is a bicontinuous isomorphism), it follows that y is *not* in the range of $C_1 \in \mathcal{L}(\ell(p-))$ for every $1 < p \leq \infty$. In particular, $0 \in \sigma(C_1; \ell(p-))$.

Fix $\lambda \in \mathbb{C} \setminus \{0\}$. If $\lambda \in \rho(C_1; \ell(p-))$, then $(\lambda I - C_1)(\ell(p-)) = \ell(p-)$. Since $\ell(p-)$ is dense in ℓ^p, it follows (with the bar denoting the closure in ℓ^p) that

$$\ell^p = \overline{\ell^p} = \overline{(\lambda I - C_1)(\ell(p-))} \subseteq \overline{(\lambda I - C_1)(\ell^p)} \subseteq \ell^p.$$

By Proposition 2.13 we can conclude that $|\lambda - \frac{p'}{2}| \geq \frac{p'}{2}$. Accordingly, $|\lambda - \frac{p'}{2}| < \frac{p'}{2}$ implies that $\lambda \in \sigma(C_1; \ell(p-))$.

(iii) An analogous argument used for the proof of Propostion 3.3 in [12] also applies here. One only needs to replace the use of Proposition 3.2 and Theorem 3.1 there by part (ii) above and Proposition 2.13, respectively. □

6 Dynamics of the generalized Cesàro operators C_t

The aim of this section is to investigate the mean ergodicity and linear dynamics of the operator C_t, for $t \in [0, 1]$, in ω, in the Fréchet spaces $\{\ell(p+), ces(p+), d(p+) : 1 \leq p < \infty\}$ and in the (LB)-spaces $\{\ell(p-), ces(p-), d(p-) : 1 < p \leq \infty\}$. For the Banach spaces ℓ^1, d_1 and ℓ^p, $ces(p)$, d_p, for $1 < p < \infty$, these results are also new. We also study the compactness, spectra and linear dynamics of the dual operators C_t'.

An operator $T \in \mathcal{L}(X)$, with X a lcHs, is called *power bounded* if $\{T^n : n \in \mathbb{N}\}$ is an equicontinuous subset of $\mathcal{L}(X)$. Here $T^n := T \circ ... \circ T$ is the composition of T with itself

n times. For a Banach space X, this means precisely that $\sup_{n\in\mathbb{N}} \|T^n\|_{X\to X} < \infty$. Given $T \in \mathcal{L}(X)$, its sequence of averages

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}, \tag{6.1}$$

is called the *Cesàro means* of T. The operator T is said to be *mean ergodic* (resp., *uniformly mean ergodic*) if $(T_{[n]})_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{L}_s(X)$ (resp., in $\mathcal{L}_b(X)$). It follows from (6.1) that

$$\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n} T_{[n-1]},$$

for $n \geq 2$. Hence, necessarily $\frac{T^n}{n} \to 0$ in $\mathcal{L}_s(X)$ (resp., in $\mathcal{L}_b(X)$) as $n \to \infty$, whenever T is mean ergodic (resp., uniformly mean ergodic). A relevant text is [39].

Concerning the dynamics of a continuous linear operator T defined on a separable lcHs X, recall that T is said to be *hypercyclic* if there exists $x \in X$ whose orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X. If, for some $x \in X$, the projective orbit $\{\lambda T^n x : \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Clearly, any hypercyclic operator is also supercyclic. As general references, we refer to [16, 32].

We begin with a study of the dynamics of generalized Cesàro operators acting in ω. For this, we will require, for each fixed $n \in \mathbb{N}_0$, the combinatorial identity

$$\sum_{k=n-i}^{n} (-1)^{(n-i)-k}\binom{n+1}{k+1} = \binom{n}{i}, \quad i = 0, \ldots, n. \tag{6.2}$$

For the proof we proceed by induction on $i = 0, \ldots, n$. For $i = 0$ observe that

$$\sum_{k=n}^{n}(-1)^{n-k}\binom{n+1}{k+1} = (-1)^0\binom{n+1}{n+1} = 1 = \binom{n}{0}.$$

Assume that (6.2) is valid for some $0 \leq i < n$. For $i+1$ it follows that

$$\begin{aligned}\sum_{k=n-(i+1)}^{n} (-1)^{(n-i-1)-k}\binom{n+1}{k+1} &= (-1)^0\binom{n+1}{n-i} + (-1)^{-1}\sum_{k=n-i}^{n}(-1)^{(n-i)-k}\binom{n+1}{k+1}\\ &= \binom{n+1}{n-i} - \binom{n}{i} = \frac{(n+1)!}{(n-i)!(i+1)!} - \frac{n!}{i!(n-i)!}\\ &= \frac{n!}{i!(n-i)!}\left[\frac{n+1}{i+1} - 1\right] = \frac{n!}{(i+1)!(n-i-1)!} = \binom{n}{i+1}.\end{aligned}$$

Since this is identity (6.2) for $i+1$, the proof is complete.

Theorem 6.1 *Let $t \in [0,1)$ and $x^{[0]} := \alpha_0(t^n)_{n\in\mathbb{N}_0}$ with $\alpha_0 \in \mathbb{C}\setminus\{0\}$; see (3.8).*

(i) *The generalized Cesàro operator $C_t \in \mathcal{L}(\omega)$ is power bounded and uniformly mean ergodic.*

(ii) $\mathrm{Ker}(I - C_t) = \mathrm{span}\,\{x^{[0]}\}$ *and the range*

$$(I - C_t)(\omega) = \{x \in \omega : x_0 = 0\} = \overline{\mathrm{span}\,\{e_r : r \in \mathbb{N}\}} \tag{6.3}$$

of $(I - C_t)$ is closed in ω.

(iii) *The operator C_t is not supercyclic in ω.*

Proof (i) That C_t is power bounded follows from the barrelledness of ω and $r_n(C_t x) \leq r_n(x)$, for $x \in \omega$ and $n \in \mathbb{N}_0$ (cf. (3.4)), which implies, for every $x \in \omega$, that

$$r_n(C_t^m x) \leq r_n(x), \quad m, n \in \mathbb{N}_0.$$

Since ω is Montel, C_t is uniformly mean ergodic, [3, Proposition 2.8].

(ii) By part (i) and [5, Theorem 3.5] we can conclude that $(I - C_t)(\omega)$ is closed in ω and that

$$\omega = \mathrm{Ker}(I - C_t) \oplus (I - C_t)(\omega). \tag{6.4}$$

Moreover, Lemma 3.4(i) yields that $\mathrm{Ker}(I - C_t) = \mathrm{span}\{x^{[0]}\}$. Since $(C_t x)_0 = x_0$ for each $x \in \omega$ (cf. (1.1)), we have $(I - C_t)(\omega) \subseteq \{x \in \omega : x_0 = 0\} = \overline{\mathrm{span}\,\{e_r : r \in \mathbb{N}\}}$. In order to establish (6.3), it remains to show that $e_r \in (I - C_t)(\omega)$ for each $r \geq 1$. Observe, via Lemma 3.1(iii), that

$$(I - C_t)(e_n - te_{n+1}) = (e_n - te_{n+1}) - \frac{1}{n+1} e_n = \frac{n}{n+1} e_n - te_{n+1}, \quad n \in \mathbb{N}_0. \tag{6.5}$$

Arguing by induction and using (6.5) we can conclude that $e_r \in (I - C_t)(\omega)$ for each $r \geq 1$. Indeed, if $n = 0$, then (6.5) yields $(I - C_t)(e_0 - te_1) = -te_1$ and hence, $e_1 \in (I - C_t)(\omega)$. Suppose that $e_n \in (I - C_t)(\omega)$. Then (6.5) implies that $\frac{n}{n+1} e_n - te_{n+1} = (I - C_t)(e_n - te_{n+1}) \in (I - C_t)(\omega)$. Since $e_n \in (I - C_t)(\omega)$, by the induction hypothesis, it follows that $e_{n+1} \in (I - C_t)(\omega)$. This completes the proof of (6.3).

(iii) To verify that $C_t \in \mathcal{L}(\omega)$ is not supercyclic we proceed as follows. It follows from (6.4), by a duality argument, that $(\omega)'_\beta = \mathrm{Ker}(I - C'_t) \oplus (I - C'_t)((\omega)'_\beta)$ and that $\dim \mathrm{Ker}(I - C'_t) = \mathrm{codim}\,(I - C_t)(\omega) = 1$, where $C'_t \in \mathcal{L}((\omega)'_\beta)$ is the dual operator of C_t. Accordingly, $1 \in \sigma_{pt}(C'_t; (\omega)'_\beta)$. On the other hand, a direct calculation shows that the dual operator $C'_t \in \mathcal{L}((\omega)'_\beta)$ is given by the transpose matrix of (3.2), that is,

$$C'_t z = \left(\sum_{k=i}^{\infty} \frac{t^{k-i}}{k+1} z_k \right)_{i \in \mathbb{N}_0}, \quad z = (z_k)_{k \in \mathbb{N}_0} \in (\omega)'_\beta. \tag{6.6}$$

Recall that $(\omega)'_\beta$ consists of vectors $z = (z_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$ with only finitely many non-zero coordinates. Define

$$z^{[n]} := \sum_{i=0}^{n} (-1)^i \binom{n}{i} t^i e_{n-i} \in (\omega)'_\beta \setminus \{0\}, \quad n \in \mathbb{N}_0.$$

It is shown below that

$$C'_t z^{[n]} = \frac{1}{n+1} z^{[n]}, \quad n \in \mathbb{N}_0. \tag{6.7}$$

This reveals that $\Lambda = \{\frac{1}{n+1} : n \in \mathbb{N}_0\} \subseteq \sigma_{pt}(C'_t; (\omega)'_\beta)$. Since $\sigma(C_t; \omega) = \sigma_{pt}(C_t; \omega) = \Lambda$ (cf. Theorem 3.7), it follows via (2.1) in Corollary 2.2 that also $\sigma_{pt}(C'_t; (\omega)'_\beta) \subseteq \sigma(C'_t; (\omega)'_\beta) = \Lambda$. So,

$$\sigma_{pt}(C'_t; (\omega)'_\beta) = \sigma(C'_t; (\omega)'_\beta) = \Lambda.$$

In particular, C'_t has a plenty of eigenvalues which implies that C_t cannot be supercyclic, [16, Proposition 1.26].

It remains to establish (6.7). Note, for $n \in \mathbb{N}_0$ fixed, that $(z^{[n]})_i = 0$ if $i > n$ and $(z^{[n]})_{n-i} = (-1)^i \binom{n}{i} t^i$ for $i = 0, \ldots, n$. In particular, $z^{[n]} \in (\omega)'_\beta \setminus \{0\}$. For $i > n$ it is clear that

$$(C'_t z^{[n]})_i = \sum_{k=i}^{\infty} \frac{t^{k-i}}{k+1} (z^{[n]})_k = 0 = \frac{1}{n+1} \cdot 0 = \frac{1}{n+1} (z^{[n]})_i.$$

To verify that $(C'_t z^{[n]})_{n-i} = \frac{1}{n+1} (z^{[n]})_{n-i}$ for $i = 0, \ldots, n$ observe that

$$\begin{aligned}
(C'_t z^{[n]})_{n-i} &= \sum_{k=n-i}^{\infty} \frac{t^{k-(n-i)}}{k+1} (z^{[n]})_k = \sum_{k=n-i}^{n} \frac{t^{k-(n-i)}}{k+1} (z^{[n]})_k \\
&= \sum_{k=n-i}^{n} \frac{t^{k-(n-i)}}{k+1} (z^{[n]})_{n-(n-k)} = \sum_{k=n-i}^{n} \frac{t^{k-(n-i)}}{k+1} (-1)^{n-k} \binom{n}{n-k} t^{n-k} \\
&= \sum_{k=n-i}^{n} \frac{t^i}{k+1} (-1)^{n-k} \frac{n!}{(n-k)!\,k!} \cdot \frac{n+1}{n+1} \\
&= \frac{t^i (-1)^i}{n+1} \sum_{k=n-i}^{n} (-1)^{(n-i)-k} \binom{n+1}{k+1} = \frac{(-1)^i}{n+1} t^i \binom{n}{i},
\end{aligned}$$

where the last equality follows from (6.2). But, as noted above, $(-1)^i \binom{n}{i} t^i = (z^{[n]})_{n-i}$ and so $(C'_t z^{[n]})_{n-i} = \frac{1}{n+1} (z^{[n]})_{n-i}$ for $i = 0, \ldots, n$. The identity (6.7) is thereby established and the proof is complete. □

We now turn to the dynamics of generalized Cesàro operators C_t acting in the other sequence spaces considered in this paper, for which we first need to establish some general results on bounded linear operators acting in lcHs'. Recall that a linear operator $T\colon X \to Y$, with X, Y lcHs', is said to be *bounded* if there exists a neighbourhood $\mathcal{U}$ of $0 \in X$ such that $T(\mathcal{U})$ is a bounded subset of Y. It is routine to verify that necessarily $T \in \mathcal{L}(X, Y)$. A lcHs X is called *locally complete* if, for each closed, absolutely convex subset $B \in \mathcal{B}(X)$, the space $X_B := \operatorname{span}(B)$ equipped with the Minkowski functional $\|\cdot\|_B$, [44, p. 47], is a Banach space, whose closed unit ball is B. Such a set B is also called a Banach disc, [36, Sect. 8.3].

Theorem 6.2 *Let X be a locally complete lcHs and $T \in \mathcal{L}(X)$ be a bounded operator satisfying $\sigma(T; X) \subseteq \overline{B(0, \delta)}$ for some $\delta \in (0, 1)$. Then $T^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. In particular, T is both power bounded and uniformly mean ergodic.*

Proof Since T is a bounded operator, there exists a closed, absolutely convex neighbourhood $\mathcal{U}$ of $0 \in X$ such that $T(\mathcal{U}) \in \mathcal{B}(X)$. So, we can select a closed, absolutely convex subset $B \in \mathcal{B}(X)$ such that $T(\mathcal{U}) \subseteq B$. By the assumptions, $(X_B, \|\cdot\|_B)$ is a Banach space. Since $T(\mathcal{U}) \subseteq B$, the map $S\colon X \to X_B$ defined by $Sx := Tx$ for $x \in X$, is well defined and it is clearly continuous. Let $j\colon X_B \to X$ denote the canonical inclusion of X_B into X, i.e., $j(x) := x$ for $x \in X_B$. Then $j \in \mathcal{L}(X_B, X)$ and $T = jS \in \mathcal{L}(X)$. On the other hand $Sj \in \mathcal{L}(X_B)$. So, by [33, Proposition 5, p. 199] we have that

$$\sigma(jS; X)\setminus\{0\} = \sigma(Sj; X_B)\setminus\{0\}.$$

Accordingly, $\sigma(Sj; X_B) = \sigma(T; X) \subseteq \overline{B(0, \delta)}$. This implies that the spectral radius $r(Sj)$ of Sj satisfies $r(Sj) \le \delta < 1$. Since $r(Sj) = \lim_{n\to\infty} \left(\|(Sj)^n\|_{X_B \to X_B}\right)^{1/n}$, it follows via standard arguments that $(Sj)^n \to 0$ in $\mathcal{L}_b(X_B)$ as $n \to \infty$. The claim is that this implies

$T^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. To establish the claim, fix any $C \in \mathcal{B}(X)$ and any absolutely convex neighbourhood $\mathcal{V}$ of $0 \in X$. Then there exist $\lambda > 0$ such that $C \subseteq \lambda\mathcal{U}$ and $\mu > 0$ such that $B \subseteq \mu\mathcal{V}$. Since B is the unit closed ball of X_B and $(Sj)^n \to 0$ in $\mathcal{L}_b(X_B)$, there exists $n_0 \in \mathbb{N}$ such that $(Sj)^n(B) \subseteq \frac{1}{\lambda\mu}B$ for all $n \geq n_0$. So, for each $n > n_0$, it follows that

$$\begin{aligned} T^n(C) &\subseteq \lambda T^n(\mathcal{U}) = \lambda T^{n-1}T(\mathcal{U}) \subseteq \lambda T^{n-1}(B) = \lambda T^{n-1}(j(B)) = \lambda (jS)^{n-1}(j(B)) \\ &= \lambda j(Sj)^{n-2}S(j(B)) = \lambda j[(Sj)^{n-1}(B)] \subseteq \lambda j\left(\left(\frac{1}{\lambda\mu}\right)B\right) = \left(\frac{1}{\mu}\right)j(B) \\ &= \left(\frac{1}{\mu}\right)B \subseteq \mathcal{V}. \end{aligned}$$

This means, with $W(C, \mathcal{V}) := \{R \in \mathcal{L}(X) : R(C) \subseteq \mathcal{V}\}$, that $T^n \in W(C, \mathcal{V})$ for each $n > n_0$. Since $C \in \mathcal{B}(X)$ and $\mathcal{V}$ are arbitrary and the sets $W(C, \mathcal{V})$ form a basis of neighbourhoods for 0 in $\mathcal{L}_b(X)$, the claim is proved, i.e., $T^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. It follows that T is power bounded (clearly) and that $T_{[n]} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ (i.e., T is uniformly mean ergodic). Indeed, let q be any τ_b-continuous seminorm. Then (6.1) implies that $q(T_{[n]}) \leq \frac{1}{n}\sum_{m=1}^{n} q(T^m)$ for $n \in \mathbb{N}$. Since $q(T^n) \to 0$ in $[0, \infty)$, also its arithmetic means $\frac{1}{n}\sum_{m=1}^{n} q(T^m) \to 0$ for $n \to \infty$, that is, $\lim_{n\to\infty} q(T_{[n]}) = 0$. So, we can conclude that $T_{[n]} \to 0$ in $\mathcal{L}_b(X)$ for $n \to \infty$. □

Theorem 6.2 permits us to formulate and prove the following general criterion for power boundedness and uniform mean ergodicity. To state it, recall that a lcHs X is said to be *ultrabornological* if it is an inductive limit of Banach spaces, [36, Sect. 13.1], [44, p. 283]. For instance, Fréchet spaces, [36, Corollary 13.1.4], and (LB)-spaces are ultrabornological. A lcHs X is called *a webbed space* if a *web* can be defined on X. For the definition of a web and the properties of webbed spaces we refer to [36, Sect. 5.2] and [38, Ch. 2.4]. Recall from Sect. 2 that Fréchet spaces and (LB)-spaces are webbed spaces. Moreover, sequentially closed subspaces and quotients of webbed spaces are webbed spaces, [36, Theorem 5.3.1].

For what follows we require the next result concerning algebraic sums in ultrabornological lcHs' which can be found in [38, Sect. 35.5(4), p. 66].

Proposition 6.3 *Let X be an ultrabornological lcHs such that $X = X_1 \oplus X_2$ algebraically with both $X_1, X_2 \subseteq X$ webbed spaces for the topology induced by X. Then X_1 and X_2 are closed subspaces of X and $X = X_1 \oplus X_2$ topologically, i.e., the canonical projections $P_i : X \to X_i$ are continuous for $i = 1, 2$.*

In general compact operators need not be mean ergodic. Just consider $T = \alpha I$ with $|\alpha| > 1$ in a finite dimensional space.

Theorem 6.4 *Let X be a locally complete, webbed and ultrabornological lcHs. Let $T \in \mathcal{L}(X)$ be a compact operator such that $1 \in \sigma(T; X)$ with $\sigma(T; X)\backslash\{1\} \subseteq \overline{B(0, \delta)}$ for some $\delta \in (0, 1)$ and satisfying* $\mathrm{Ker}(I - T) \cap (I - T)(X) = \{0\}$. *Then T is both power bounded and uniformly mean ergodic.*

Proof Since $T \in \mathcal{L}(X)$ is a compact operator, the following properties hold true: (a) $(I - T)(X)$ is closed in X, (b) $\dim\mathrm{Ker}(I - T) < \infty$ (1 is necessarily an eigenvalue of T as it is an isolated point of $\sigma(T; X)$ and T is compact), and (c) $\mathrm{codim}\,(I - T)(X) = \dim \mathrm{Ker}(I - T) < \infty$, see, e.g., [27, Theorem 9.10.1]. Since $\mathrm{Ker}(I - T) \cap (I - T)(X) = \{0\}$ by assumption, it follows that $X = \mathrm{Ker}(I - T) \oplus (I - T)(X)$ algebraically. Moreover, $(I - T)(X)$ and $\mathrm{Ker}(I - T)$ are closed complemented subspaces of X and hence, are webbed spaces, [36, Theorem 5.3.1]. So, we can apply Proposition 6.3 to conclude that $X = \mathrm{Ker}(I - T) \oplus (I - T)(X)$ holds topologically.

Set $Y := (I - T)(X)$ and $S := T|_Y$. It is routine to verify that $S(Y) \subseteq Y$ and $S\colon Y \to Y$ is a compact operator. So, $\sigma(S; Y)\backslash\{0\} = \sigma_{pt}(S; Y) \subseteq \sigma_{pt}(T; X) \subseteq \sigma(T; X)$. But, $1 \notin \sigma(S; Y)$. Otherwise, there exists $y \in Y\backslash\{0\}$ such that $Sy = y$, i.e., $Ty = y$ or, equivalently, $(I - T)y = 0$. Thus, $y \in Y \cap \mathrm{Ker}(I - T) = (I - T)(X) \cap \mathrm{Ker}(I - T) = \{0\}$ and hence, $y = 0$; a contradiction. Hence, $\sigma(S; Y) \subseteq \sigma(T; X)\backslash\{1\} \subseteq \overline{B(0, \delta)}$ with $\delta \in (0, 1)$. Since S is compact, it is also bounded and hence, we can apply Theorem 6.2 to conclude that $S^n \to 0$ in $\mathcal{L}_b(Y)$ as $n \to \infty$, after noting that the closed subspace Y of X is locally complete.

Denote by $P\colon X \to X$ the continuous projection onto $\mathrm{Ker}(I-T)$ along $(I-T)(X) = Y$, i.e., for each $z \in X$ there exist unique elements $x \in \mathrm{Ker}(I - T)$ and $y \in Y$ such that $z = x + y$ and so $Pz := x$. The claim is that $T^n \to P$ in $\mathcal{L}_b(X)$ as $n \to \infty$. To establish this fix $B \in \mathcal{B}(X)$ and a neighbourhood $\mathcal{U}$ of $0 \in X$. As $(I - P) \in \mathcal{L}(X)$, we have that $(I - P)(B) \in \mathcal{B}(Y)$. Taking into account that $S^n \to 0$ in $\mathcal{L}_b(Y)$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $S^n((I - P)(B)) \subseteq \mathcal{U} \cap Y$ for every $n \geq n_0$. On the other hand, for each $z \in X$ we have that $Pz \in \mathrm{Ker}(I - T)$, i.e., $TPz = Pz$, and hence, $T^n(Pz) = Pz$ for each $n \in \mathbb{N}$. Accordingly, as $S = T$ on $(I - P)(X) = (I - T)(X) = Y$ we get, for each $z \in B$ and $n \geq n_0$, that

$$\begin{aligned}T^n z - Pz &= T^n(Pz + (z - Pz)) - Pz = T^n(z - Pz) = T^n((I - P)z)\\ &= S^n((I - P)z) \in S^n((I - P)(B)) \subseteq \mathcal{U} \cap Y,\end{aligned}$$

where we used the fact that $(I - P)z \in Y$. Since $z \in B$ is arbitrary, this implies that $T^n - P \in W(B, \mathcal{U}) := \{R \in \mathcal{L}(X) : R(B) \subseteq \mathcal{U}\}$ for each $n \geq n_0$. So, by the arbitrariness of B and $\mathcal{U}$, the claim is proved. □

Remark 6.5 (i) Let X be a sequentially complete lcHs and $T \in \mathcal{L}(X)$. If $\frac{T^n}{n} \to 0$ in $\mathcal{L}_s(X)$ as $n \to \infty$, then $\sigma(T; X) \subseteq \overline{B(0, 1)}$, [2, Proposition 5.1 & Remark 5.3]; see also [28, Proposition 4.4]. In particular, if T is power bounded, then $\sigma(T; X) \subseteq \overline{B(0, 1)}$. In view of this fact, Theorem 6.4 can be seen as a sort of converse result (observe that every sequentially complete lcHs is locally complete, [44, Corollary 23.14]).

(ii) Theorem 6.2 should also be compared with [6, Theorem 10] in which it is proved, for $T \in \mathcal{L}(X)$ with X a prequojection Fréchet space, that $T^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ if, and only if, $\sigma(T; X) \subseteq B(0, 1)$ and $\frac{T^n}{n} \to 0$ in $\mathcal{L}_b(X)$. Since $\sigma(C_t; \omega) \nsubseteq B(0, 1)$ (as $1 \in \sigma(C_t; \omega)$ but $1 \notin B(0, 1)$) and ω is a prequojection Fréchet space, for each $t \in [0, 1)$, it follows that $(C_t)^n \not\to 0$ in $\mathcal{L}_b(\omega)$ for $n \to \infty$.

Combining Theorem 6.4 with the results in the preceding sections we get the following result.

Theorem 6.6 *Let $t \in [0, 1)$. Let X belong to any one of the sets: $\{d_p, \ell^p : 1 \leq p < \infty\} \cup \{ces(p) : 1 < p < \infty\}$ or $\{\ell(p+), ces(p+), d(p+) : 1 \leq p < \infty\}$ or $\{\ell(p-), ces(p-), d(p-) : 1 < p \leq \infty\}$. Then $C_t \in \mathcal{L}(X)$ is power bounded and uniformly mean ergodic, but not supercyclic.*

Proof From the results of the preceding sections recall that $C_t \in \mathcal{L}(X)$ is a compact operator on X and $\sigma(C_t; X) = \Lambda \cup \{0\}$. Hence, $\sigma(C_t; X)\backslash\{1\} \subseteq \overline{B(0, 1/2)}$. Moreover, $(I - C_t)(X)$ is also closed in X. Since $x^{[0]} \in d_1 \subseteq X$, we can adapt the arguments in the proof of Theorem 6.1 to argue that $(I - C_t)(X) = \{x \in X : x_0 = 0\} = \overline{\mathrm{span}\{e_r : r \in \mathbb{N}\}}$ and $\mathrm{Ker}(I - C_t) = \mathrm{span}\,\{x^{[0]}\}$. Hence, $\mathrm{Ker}(I - C_t) \cap (I - C_t)(X) = \{0\}$. So, all the assumptions of Theorem 6.4 (for $\delta = \frac{1}{2}$ and $T := C_t$) are satisfied. Then we can conclude that C_t is power bounded and uniformly mean ergodic.

To show that $C_t\colon X \to X$ is not supercyclic we proceed as follows. Since $C_t \in \mathcal{L}(X)$ is compact, the operators $C_t\colon X \to X$ and $C_t'\colon X_\beta' \to X_\beta'$ have the same non-zero eigenvalues, [27, Theorem 9.10.2(2)]. Hence, $\sigma_{pt}(C_t'; X_\beta') = \sigma_{pt}(C_t; X) = \Lambda$. According to [16, Proposition 1.26] it follows that the operator $C_t\colon X \to X$ cannot be supercyclic. □

A first consequence of the results collected above is the following one concerning the dual operators C_t'. First we recall the relevant dual spaces involved. Namely, for p, p' satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ we have (see Proposition 3.4(i), Proposition 4.3 and Remark 4.4 in [20], respectively):

$\ell(p-) \simeq (\ell(p'+))_\beta'$ and $(\ell(p-))_\beta' \simeq \ell(p'+)$, for $1 < p \leq \infty$;
$d(p-) \simeq (ces(p'+))_\beta'$ and $(ces(p-))_\beta' \simeq d(p'+)$, for $1 < p \leq \infty$;
$ces(p-) \simeq (d(p'+))_\beta'$ and $ces(p'+) \simeq (d(p-))_\beta'$, for $1 < p \leq \infty$.

Proposition 6.7 *Let $t \in [0, 1)$ and X belong to any one of the sets: $\{d_p, \ell^p : 1 \leq p < \infty\} \cup \{ces(p) : 1 < p < \infty\}$ or $\{\ell(p+), ces(p+), d(p+) : 1 \leq p < \infty\}$ or $\{\ell(p-), ces(p-), d(p-) : 1 < p \leq \infty\}$.*

(i) The dual operator $C_t' \in \mathcal{L}(X_\beta')$ of $C_t \in \mathcal{L}(X)$ is compact and is given by

$$C_t' y = \left(\sum_{k=i}^{\infty} \frac{t^{k-i}}{k+1} y_k\right)_{i\in\mathbb{N}_0}, \quad y = (y_k)_{k\in\mathbb{N}_0} \in X_\beta'. \tag{6.8}$$

(ii) The point spectrum of $C_t' \in \mathcal{L}(X_\beta')$ is given by

$$\sigma_{pt}(C_t'; X_\beta') = \sigma_{pt}(C_t; X) = \Lambda. \tag{6.9}$$

Each eigenvalue $\frac{1}{n+1}$, for $n \in \mathbb{N}_0$, is simple and its corresponding eigenspace is spanned by

$$y^{[n]} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} t^i e_{n-i} \in X_\beta' \setminus \{0\}, \; n \in \mathbb{N}_0.$$

Moreover,

$$\sigma^*(C_t'; X_\beta') = \sigma(C_t'; X_\beta') = \Lambda \cup \{0\}.$$

Proof (i) Recall that $\mathcal{E}$ is an unconditional basis in $\ell(p+), ces(p+), d(p+)$, for $1 \leq p < \infty$ (cf. Section 4) and an unconditional basis in $\ell(p-), ces(p-), d(p-)$, for $1 < p \leq \infty$ (cf. Section 5). Moreover, $\mathcal{E}$ is also an unconditional basis in the dual Banach spaces $(\ell^p)' = \ell^{p'}$ for $1 < p < \infty$, in the dual Banach spaces $(ces(p))' \simeq d_{p'}$ for $1 < p < \infty$, [19], and in the dual Banach spaces $(d_p)' \simeq ces(p')$ for $1 < p < \infty$ (cf. [17, 24]), as well as in $(d_1)' \simeq ces(0)$, [25, Sect. 6]. In view of the description of X_β' (for X non-normable) given prior to this Proposition it follows, for all $X \neq \ell^1$, that the linear space span$(\mathcal{E}) = (\omega)'$ is dense in X_β'. The continuity of $C_t'\colon X_\beta' \to X_\beta'$ then implies that (6.6) can be extended to an inequality for every $y \in X_\beta'$, that is, (6.8) is valid.

For $X = \ell^1$, the linear space span$(\mathcal{E}) = (\omega)'$ is not dense in $X_\beta' = \ell^\infty$. So, in this case we argue as follows. Define $Ty := \left(\sum_{k=i}^{\infty} \frac{t^{k-i}}{k+1} y_k\right)_{i\in\mathbb{N}_0}$ for $y \in \ell^\infty$, in which case $T \in \mathcal{L}(\ell^\infty)$.

Indeed, for $y \in \ell^\infty$, note that

$$\|Ty\|_\infty = \sup_{i\in\mathbb{N}_0}\left|\sum_{k=i}^{\infty}\frac{t^{k-i}}{k+1}y_k\right| \leq \sup_{i\in\mathbb{N}_0}\sum_{k=i}^{\infty}\frac{t^{k-i}}{k+1}|y_k| \leq \|y\|_\infty \sup_{i\in\mathbb{N}_0}\sum_{k=i}^{\infty}\frac{t^{k-i}}{k+1}$$
$$\leq \|y\|_\infty \sup_{i\in\mathbb{N}_0}\sum_{k=i}^{\infty}t^{k-i} = \|y\|_\infty\sum_{j=0}^{\infty}t^j = \frac{1}{1-t}\|y\|_\infty \ \text{ (as } 0 \leq t < 1).$$

Accordingly, $\|T\|_{\ell^\infty\to\ell^\infty} \leq \frac{1}{1-t}$, that is, $T \in \mathcal{L}(\ell^\infty)$. For each $x \in \ell^1$ and $y \in \ell^\infty$, a direct calculation yields

$$\langle C_t x, y\rangle = \langle x, Ty\rangle,$$

which implies that $T = C_t'$.

For any Fréchet space $X \in \{\ell(p+), ces(p+), d(p+) \ : \ 1 \leq p < \infty\}$ and any Banach space $X \in \{\ell^1, d_1\} \cup \{\ell^p, ces(p), d_p \ : \ 1 < p < \infty\}$ the operator $C_t \in \mathcal{L}(X)$ is compact (cf. Propositions 2.12, 2.14, 2.15 and Remark 3.3 and Theorem 4.5(i)). Accordingly, the dual operator $C_t' \in \mathcal{L}(X_\beta')$ of $C_t \in \mathcal{L}(X)$ is compact, [27, Corollary 9.6.3].

For any (LB)-space $X \in \{\ell(p-), ces(p-), d(p-) \ : \ 1 < p \leq \infty\}$ the operator $C_t \in \mathcal{L}(X)$ is also compact (cf. Theorem 5.3(i)). So, the compactness of $C_t' \in \mathcal{L}(X_\beta')$ follows from Proposition 2.7, after observing that X is a boundedly retractive (LB)-space. Indeed, $X = \ell(p-)$, for $1 < p \leq \infty$, is a boundedly retractive (LB)-space, as it is the strong dual of the quasinormable Fréchet space $\ell(p'+)$, [45, p. 12]. On the other hand, $X \in \{ces(p-), d(p-) : 1 < p \leq \infty\}$ is a boundedly retractive (LB)-space, as it is a (DFS)-space, [20, Proposition 2.5(ii) & Lemma 4.2(i)].

(ii) It was shown in the proof of Theorem 6.1 that each vector $z^{[n]} \in (\omega)_\beta'\setminus\{0\} \subseteq X_\beta'$ satisfies $C_t' z^{[n]} = \frac{1}{n+1}z^{[n]}$, for every $n \in \mathbb{N}_0$. Accordingly.

$$\Lambda \subseteq \sigma_{pt}(C_t'; X_\beta'). \tag{6.10}$$

Moreover, $0 \notin \sigma_{pt}(C_t'; X_\beta')$ as C_t' is injective. To verify this let $z \in X_\beta'$ satisfy $C_t' z = 0$. By considering the individual coordinates in (6.8) it follows that

$$\frac{1}{i+1}z_i = (C_t'z)_i - t(C_t'z)_{i+1}, \quad i \in \mathbb{N}_0,$$

that is, $z = 0$ and so indeed $0 \notin \sigma_{pt}(C_t'; X_\beta')$. The compactness of $C_t' \in \mathcal{L}(X_\beta')$ then implies that

$$\sigma(C_t'; X_\beta') = \{0\} \cup \sigma_{pt}(C_t'; X_\beta') \ \text{ and } \ 0 \notin \sigma_{pt}(C_t'; X_\beta'). \tag{6.11}$$

It follows from (2.1) in Corollary 2.2 (with $T := C_t$), from (6.11) and from the fact that $\sigma_{pt}(C_t; X) = \Lambda$, that (6.9) is valid.

Parts (1) and (2) of [27, Proposition 9.10.2] imply that each eigenvalue of C_t' is simple, as this is the case for C_t; see Propositions 2.12, 2.14, 2.15 and Remark 3.3 and Theorems 4.5, 5.3, which also include the identities

$$\sigma^*(C_t; X) = \sigma(C_t; X) = \Lambda \cup \{0\}. \tag{6.12}$$

Setting $T := C_t$ it follows from (2.2) in Corollary 2.2, together with (6.12), that

$$\sigma^*(C_t'; X_\beta') \subseteq \sigma^*(C_t; X) = \Lambda \cup \{0\}.$$

From general theory (cf. Section 2) we also have that

$$\sigma(C_t'; X_\beta') \subseteq \sigma^*(C_t'; X_\beta').$$

Since (6.9) and (6.11) imply that $\sigma(C_t'; X_\beta') = \Lambda \cup \{0\}$, we can conclude that

$$\Lambda \cup \{0\} = \sigma(C_t'; X_\beta') \subseteq \sigma^*(C_t'; X_\beta') \subseteq \Lambda \cup \{0\}.$$

This, together with (6.12), yields $\sigma^*(C_t'; X_\beta') = \sigma^*(C_t; X) = \Lambda \cup \{0\}$. □

A consequence of Theorem 6.6 is the next result.

Proposition 6.8 *Let* $t \in [0, 1)$. *Let* X *belong to any one of the sets:* $\{d_p, \ell^p : 1 \leq p < \infty\} \cup \{ces(p) : 1 < p < \infty\}$ *or* $\{\ell(p+), ces(p+), d(p+) : 1 \leq p < \infty\}$ *or* $\{\ell(p-), ces(p-), d(p-) : 1 < p \leq \infty\}$. *Then* $C_t' \in \mathcal{L}(X_\beta')$ *is power bounded and uniformly mean ergodic, but not supercyclic.*

Proof By Theorem 6.6 the operator $C_t \in \mathcal{L}(X)$ is power bounded. Since $(C_t')^n = (C_t^n)'$, for every $n \in \mathbb{N}_0$, it follows from [38, Sect. 39.3(6)] that also $C_t' \in \mathcal{L}(X_\beta')$ is power bounded. The operator $C_t \in \mathcal{L}(X)$ is also uniformly mean ergodic in X, again by Theorem 6.6. Since X is barrelled (hence, quasi-barrelled), Lemma 2.1 in [4] implies that C_t' is uniformly mean ergodic in X_β'. If $X \notin \{\ell^1, d_1\}$, then X_β' is reflexive with $(X_\beta')_\beta' = X$ (cf. the proof of Proposition 6.7) and hence, $(C_t')' = C_t$. It follows from (6.9) that $C_t'' = C_t$ has plenty of eigenvalues so that $C_t' \in \mathcal{L}(X_\beta')$ cannot be supercyclic [16, Proposition 1.26]. Finally, suppose that $X \in \{\ell^1, d_1\}$. Since C_t is compact with $\sigma_{pt}(C_t; X) = \Lambda$ (cf. Proposition 2.12 and Remark 3.3), it follows that $\sigma_{pt}(C_t'; X_\beta') = \sigma_{pt}(C_t; X) = \Lambda$; see [27, Proposition 9.10.2(2)]. Schauder's theorem implies that $C_t' \in \mathcal{L}(X_\beta')$ is also compact and hence, again by Proposition 9.10.2(2) of [27], now applied to C_t', we can conclude that $\sigma_{pt}(C_t''; X_\beta'') = \sigma_{pt}(C_t'; X_\beta') = \Lambda$. So, $C_t'' \in \mathcal{L}(X_\beta'')$ has plenty of eigenvalues which implies that C_t' is not supercyclic. □

Remark 6.9 The dynamics of $C_1 \in \mathcal{L}(X)$, with $X \notin \{\ell^1, d_1\}$ belonging to one of the sets in Theorem 6.6, is quite different. Consider first the Banach space case. For $1 < p < \infty$, the operator $C_1 \in \mathcal{L}(\ell^p)$ is neither power bounded nor mean ergodic, [5, Proposition 4.2]. Since $\{z \in \mathbb{C} : |z - \frac{p'}{2}| < \frac{p'}{2}\} \subseteq \sigma_{pt}(C_1'; \ell^{p'})$ with $\frac{1}{p} + \frac{1}{p'} = 1$, [40, Theorem 1(b)], $C_1 \in \mathcal{L}(\ell^p)$ cannot be supercyclic, [16, Proposition 1.26]. Similarly, $C_1 \in \mathcal{L}(ces(p))$, for $1 < p < \infty$, is not mean ergodic, not power bounded and not supercyclic, [13, Proposition 3.7(ii)]. Also, $C_1 \in \mathcal{L}(d_p)$ is not mean ergodic and not supercyclic, [19, Propositions 3.10 & 3.11]. Since power bounded operators in reflexive Banach spaces are necessarily mean ergodic, [43], C_1 cannot be power bounded in d_p. Turning to Fréchet spaces, for $1 \leq p < \infty$ the operator $C_1 \in \mathcal{L}(\ell(p+))$ is not mean ergodic, not power bounded and not supercyclic, [8, Theorems 2.3 & 2.5], as is the case for $C_1 \in \mathcal{L}(ces(p+))$, [14, Proposition 5], and for $C_1 \in \mathcal{L}(d(p+))$, [21, Proposition 3.5]. For (LB)-spaces, with $1 < p \leq \infty$, the operator $C_1 \in \mathcal{L}(ces(p-))$ is not mean ergodic, not power bounded and not supercyclic, [12, Propositions 3.4 & 3.5], as is the case for $C_1 \in \mathcal{L}(d(p-))$, [21, Proposition 3.8]. Finally, the dynamics of $C_1 \in \mathcal{L}(\omega)$ is the same as for $C_t \in \mathcal{L}(\omega)$, with $t \in [0, 1)$; see Theorem 6.1 above and [8, Proposition 4.3].

The dynamics of C_1 acting in $\ell(p-)$ is covered by our final result.

Proposition 6.10 *Let* $p \in (1, \infty]$. *The Cesàro operator* $C_1 \in \mathcal{L}(\ell(p-))$ *is not mean ergodic, not power bounded and not supercyclic.*

Proof In view of Proposition 5.5(i) the proof follows in a similar way to that of [8, Theorem 2.3]. For the sake of completeness, we indicate the details.

By the discussion prior to Proposition 6.7 we know that $(\ell(p-))'_\beta \simeq \ell(p'+)$. Proposition 5.5(i) implies that $\frac{1+p'}{2} > 1$ belongs to $\sigma_{pt}(C_1'; \ell(p'+))$, where $\frac{1}{p} + \frac{1}{p'} = 1$. So, there exists a non-zero vector $u \in \ell(p'+)$ satisfying $C_1'(u) = \frac{1+p'}{2}u$. Choose any $x \in \ell(p-)$ such that $\langle x, u\rangle \neq 0$. Then

$$\left\langle \frac{1}{n}(C_1)^n(x), u \right\rangle = \left\langle x, \frac{1}{n}(C_1')^n(u) \right\rangle = \frac{1}{n}\left(\frac{1+p'}{2}\right)^n \langle x, u\rangle, \quad n \in \mathbb{N}.$$

This means that the sequence $\{\frac{1}{n}(C_1)^n(x)\}_{n\in\mathbb{N}} \subseteq \ell(p-)$ cannot be bounded in $\ell(p-)$. Accordingly, C_1 is not mean ergodic and not power bounded.

Applying again Proposition 5.5(i), we see that C_1' has a plenty of eigenvalues. So, C_1 cannot be supercyclic, [16, Proposition 1.26]. □

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