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Banach gradient flows for various families of knot energies

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Abstract. We establish long-time existence of Banach gradient flows for generalised integral Menger curvatures and tangent-point energies, and for O'Hara's self-repulsive potentials $E^{\alpha,p}$. In order to do so, we employ the theory of curves of maximal slope in slightly smaller spaces compactly embedding into the respective energy spaces associated to these functionals and add a term involving the logarithmic strain, which controls the parametrisations of the flowing (knotted) loops. As a prerequisite, we prove in addition that O'Hara's knot energies $E^{\alpha,p}$ are continuously differentiable.

1. Introduction

It is an interesting and analytically challenging problem in geometric knot theory to evolve knots according to the gradient flow of a self-repelling interaction energy. Such energies are called *knot energies*, and one may categorise them into two types.

Firstly, there are singular *self-repulsive potentials* such as O'Hara's [32] two-parameter energy family $E^{\alpha,p}$ defined on closed regular curves $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ by

$$E^{\alpha,p}(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^2} \left(\frac{1}{|\gamma(x)-\gamma(y)|^\alpha} - \frac{1}{d_\gamma^\alpha(x,y)} \right)^p |\gamma'(x)||\gamma'(y)| \, \mathrm{d}x \, \mathrm{d}y \quad (1.1)$$

for $\alpha > 0$ and $p > 1$ satisfying $2 \le \alpha p < 2p+1$. Here, $d_\gamma(x,y)$ denotes the intrinsic distance between the points $\gamma(x)$ and $\gamma(y)$ along the curve. He [25] had shown short-time existence for the L^2-gradient flow of the *Möbius*[1] energy $E^{2,1}$ for smooth initial data, before Blatt investigated this flow systematically for the subfamily $E^{\alpha,1}$. He established long-time existence results and convergence to a critical point:

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A substantial part of the contents of Sects. 2 and 3 is contained in H. Matt's Ph.D. thesis [31]. Furthermore, Sect. 4 is a generalisation of [31, Chapter 4]. Some results of Sects. 3, 4, and Appendix A, as well as the content of Sect. 5 will be part of D. Steenebrügge's Ph.D. thesis [39]. The authors would like to thank the anonymous reviewer for their helpful comments and suggestions, especially for the corrections in the proof of Lemma 5.2.

[1]This name reflects the invariance of $E^{2,1}$ under Möbius transformations; see the seminal work on the Möbius energy by Freedman, He, and Wang in [21].

for $\alpha \in (2,3)$ in [7], and for $\alpha = 2$ and initial data sufficiently close to a local minimiser in [6]. Moreover, again for $E^{2,1}$ he proved an ε-regularity result together with a blow-up analysis in [8], resulting in the convergence to the round circle if one restricts the L^2-flow to planar loops. According to [11, p. 31] such strong results on the L^2-flow for the general energy family $E^{\alpha,p}$ may be out of reach because of a degenerate elliptic operator in the first variation formula for $E^{\alpha,p}$. As shown in [5] the underlying energy space of the Möbius energy $E^{2,1}$ is a fractional Sobolev space, the Sobolev-Slobodeckiĭ[2] space $W^{\frac{3}{2},2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, which is a Hilbert space. This fact was recently used by Reiter and Schumacher to establish short-time existence for a Hilbert gradient flow in that space for $E^{2,1}$ with a method, however, that seems to be restricted to $E^{2,1}$; see [34, Remark after Theorem 1.2].

The second type of knot energies are *geometric curvature energies* whose integrands are constructed from circles (as first suggested by Gonzalez and Maddocks in [23]), such as *integral Menger curvatures*, analysed in [41,42],

$$\mathscr{M}_p(\gamma) := \iiint_{(\mathbb{R}/\mathbb{Z})^3} \frac{1}{R^p(\gamma(x),\gamma(y),\gamma(z))} |\gamma'(x)||\gamma'(y)||\gamma'(z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.2}$$

for $p > 3$, where $R(a,b,c)$ denotes the circumcircle radius of the three points $a, b, c \in \mathbb{R}^n$, or the *tangent-point energies* [40]

$$\mathrm{TP}_q(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^2} \frac{1}{r_{\mathrm{tp}}^q[\gamma](\gamma(x),\gamma(y))} |\gamma'(x)||\gamma'(y)|\,\mathrm{d}x\,\mathrm{d}y \tag{1.3}$$

for $q > 2$, where $r_{\mathrm{tp}}[\gamma](\gamma(x),\gamma(y))$ is the radius of the unique circle through $\gamma(x)$ and $\gamma(y)$ that is tangent to the curve γ at the point $\gamma(x)$.

Replacing the p-th power of the circumcircle radius in (1.2) by the more general (but less geometric) expression $R^{(p,q)}(a,b,c) := \frac{(|b-c||b-a||c-a|)^p}{|(b-a)\wedge(c-a)|^q}$, or similarly, replacing the q-th power of the tangent-point radius in (1.3) by

$$r^{(p,q)}[\gamma](\gamma(x),\gamma(y)) := \frac{|\gamma(x)-\gamma(y)|^p}{\mathrm{dist}^q\big(\gamma(x)+\mathbb{R}\gamma'(x),\gamma(y)\big)}$$

one obtains *generalised integral Menger curvatures*

$$\mathrm{intM}^{(p,q)}(\gamma) := \iiint_{(\mathbb{R}/\mathbb{Z})^3} \frac{|\gamma'(x)||\gamma'(y)||\gamma'(z)|}{R^{(p,q)}(\gamma(x),\gamma(y),\gamma(z))}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1.4}$$

for $q \in (1,\infty)$, $p \in (\frac{2}{3}q+1, q+\frac{2}{3})$, and *generalised tangent-point energies*

$$\mathrm{TP}^{(p,q)} := \iint_{(\mathbb{R}/\mathbb{Z})^2} \frac{|\gamma'(x)||\gamma'(y)|}{r^{(p,q)}[\gamma](\gamma(x),\gamma(y))}\,\mathrm{d}x\,\mathrm{d}y \tag{1.5}$$

[2]For the definition and some facts concerning these spaces, see Appendix A; for a condensed selection of pertinent results on periodic Sobolev-Slobodeckiĭ spaces, see [27, Appendix A].

for $q \in (1, \infty)$, $p \in (q+2, 2q+1)$. These two-parameter energy families contain the original geometric curvature energies (1.2) and (1.3) up to a constant factor, i.e. $\mathscr{M}_p = 2^p \operatorname{intM}^{(p,p)}$ and $\mathrm{TP}_q = 2^q \mathrm{TP}^{(2q,q)}$. They were introduced by Blatt and Reiter to prove smoothness of critical points specifically of $\operatorname{intM}^{(p,2)}$ for $p \in (\frac{7}{3}, \frac{8}{3})$ [13, Theorem 4], and of $\mathrm{TP}^{(p,2)}$ for $p \in (4, 5)$ [12, Theorem 1.5]. There is no existence result on the L^2-flow for these energies yet, but the underlying Hilbert space structure in this specific range of parameters was recently used in [27] to prove long-time existence for a suitably projected Hilbert gradient flow for $\operatorname{intM}^{(p,2)}$ for $p \in (\frac{7}{3}, \frac{8}{3})$, where the projection was chosen to conserve the speed of the curves' parametrisations along the flow. A corresponding long-time existence result for $\mathrm{TP}^{(p,2)}$ for $p \in (4, 5)$ is also available [39]. Considering gradient flows as ordinary differential equations in the respective energy space has the additional potential of developing numerical procedures that are substantially more efficient and robust in comparison to the existing numerical methods for the L^2-flow; see [34] and [27, Sections 1.4 & 7].

It is the purpose of the present paper to prove long-time existence results for the gradient flows of the three two-parameter energy families $E^{\alpha,p}$, $\operatorname{intM}^{(p,q)}$, and $\mathrm{TP}^{(p,q)}$ for the respective complete range of parameters given in (1.1), (1.4), and (1.5). Since the underlying energy spaces are Sobolev-Slobodeckiĭ spaces that are in general only Banach spaces, we employ the general metric gradient approach of Ambrosio et al. [1] to construct curves of maximal slope, which turn out to be solutions of the doubly-nonlinear gradient flow equation. In order to do this, we need to control the curves' parametrisations along the flow, and for that we use the same constraint as in [27] but in a different manner. We add to the knot energy a suitable norm of the *logarithmic strain*

$$\Sigma(\gamma) := \log|\gamma'| \tag{1.6}$$

as a lower-order penalty term, instead of projecting onto the null space of its differential $D\Sigma[\gamma]$ as in [27]. As a second ingredient we restrict the total energy to a reflexive and uniformly convex Banach space $\mathcal{C}$ compactly embedded in the respective energy space $\mathcal{B}$ of closed curves, to account for the quite restrictive assumptions in the general existence theory for curves of maximal slope.

To summarise these ideas, given any knot energy $\mathcal{E} : \mathcal{B} \to (-\infty, \infty]$, and some number $\kappa > 1$, we consider the *total energy* $\phi : \mathcal{C} \to (-\infty, \infty]$ defined as

$$\phi(\gamma) := \begin{cases} \mathcal{E}(\gamma) + \|\Sigma(\gamma)\|_{\mathcal{A}}^{\kappa}, & \text{if } \gamma \in \mathcal{C} \text{ is regular and injective} \\ +\infty, & \text{else,} \end{cases} \tag{1.7}$$

where the Banach space $\mathcal{A}$ consists of real-valued functions with exactly one order lower in differentiability than the curves $\gamma \in \mathcal{B}$, since the scalar-valued logarithmic strain consumes one derivative.

By means of the θ*-duality mapping* $\mathfrak{J}_{\mathcal{C},\theta} : \mathcal{C} \to 2^{\mathcal{C}^*}$ *with* $\theta \in (1, \infty)$ defined by

$$\mathfrak{J}_{\mathcal{C},\theta}(x) := \left\{ \xi \in \mathcal{C}^* \colon \langle \xi, x \rangle_{\mathcal{C}^* \times \mathcal{C}} = \|x\|_{\mathcal{C}} \cdot \|\xi\|_{\mathcal{C}^*},\ \|x\|_{\mathcal{C}}^{\theta} = \|\xi\|_{\mathcal{C}^*}^{\beta} \right\},\ \frac{1}{\theta} + \frac{1}{\beta} = 1, \tag{1.8}$$

where $\mathcal{C}^*$ denotes the dual space of $\mathcal{C}$, we state our main result.

Theorem 1.1. (Long-time existence) *For any $\kappa, \theta \in (1, \infty)$, $\varepsilon \in (0, \infty)$, and any given regular and injective curve $\gamma_0 \in \mathcal{C}_\varepsilon$, there exists a mapping $\boldsymbol{u} \in C^1([0, \infty), \mathcal{C}_\varepsilon)$ with $\boldsymbol{u}(0) = \gamma_0$, such that the closed curves $\boldsymbol{u}(t)$ are injective and regular for all $t \geq 0$, satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t) = -\mathfrak{J}^{-1}_{\mathcal{C}_\varepsilon,\theta}\big(D\phi[\boldsymbol{u}(t)]\big) \quad \textit{for all} \quad t \geq 0, \tag{1.9}$$

for any one of the following choices:

(i) $\mathcal{E} := E^{\alpha,p}$ *for* $\alpha \in (0, \infty)$, $p \in [1, \infty)$ *satisfying* $2 < \alpha p < 2p + 1$, *and* $\mathcal{A} := W^{\frac{\alpha p - 1}{2p}, 2p}(\mathbb{R}/\mathbb{Z})$, $\mathcal{B} := W^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $\mathcal{C}_\varepsilon := W^{1+\frac{\alpha p-1}{2p}+\varepsilon,2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$;

(ii) $\mathcal{E} := \mathrm{intM}^{(p,q)}$ *for* $q \in (1, \infty)$, $p \in (\frac{2}{3}q + 1, q + \frac{2}{3})$, *and* $\mathcal{A} := W^{\frac{3p-2}{q}-2,q}(\mathbb{R}/\mathbb{Z})$, $\mathcal{B} := W^{\frac{3p-2}{q}-1,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $\mathcal{C}_\varepsilon := W^{\frac{3p-2}{q}-1+\varepsilon,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$;

(iii) $\mathcal{E} := \mathrm{TP}^{(p,q)}$ *for* $q \in (1, \infty)$, $p \in (q + 2, 2q + 1)$, *and* $\mathcal{A} := W^{\frac{p-1}{q}-1,q}(\mathbb{R}/\mathbb{Z})$, $\mathcal{B} := W^{\frac{p-1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $\mathcal{C}_\varepsilon := W^{\frac{p-1}{q}+\varepsilon,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

Since (1.9) is a gradient flow, the energy decreases along the flow. In addition, the Banach space $\mathcal{C}$ continuously embeds into $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ in all three alternatives. Therefore, the knot type $[\gamma_0]$ of the initial loop γ_0 is preserved along the flow, since knot classes are stable with respect to C^1-deformations [4,35].

Corollary 1.2. *Any $\boldsymbol{u} \in \mathrm{AC}_{\mathrm{loc}}([0, \infty), \mathcal{C}_\varepsilon)$ starting at an injective and regular curve $\gamma_0 \in \mathcal{C}_\varepsilon$ and solving (1.9) for almost all $t > 0$ is a θ-curve of maximal slope with respect to the strong upper gradient[3] $\|D\phi[\cdot]\|_{\mathcal{C}^*_\varepsilon}$. In particular, the total energy ϕ decreases along $\boldsymbol{u}$, i.e. $\phi(\boldsymbol{u}(t)) \leq \phi(\boldsymbol{u}(s))$ for all $t > s \geq 0$. Additionally, $u \in C^1([0, \infty), \mathcal{C}_\varepsilon)$ and the knot type is preserved along the flow, that is, $[\boldsymbol{u}(t)] = [\gamma_0]$ for all $t \geq 0$.*

Remarks 1.3. 1. In Theorem 1.1 and Corollary 1.2 one can also choose the Möbius energy $E^{2,1}$ as knot energy $\mathcal{E}$. The flow equation (1.9) then reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t) = -\nabla_{\mathcal{C}_\varepsilon}\phi(\boldsymbol{u}(t)),$$

where $\nabla_{\mathcal{C}_\varepsilon}$ is the gradient in the Hilbert space $\mathcal{C}_\varepsilon := W^{\frac{3}{2}+\varepsilon,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, which compactly embeds into the energy space $W^{\frac{3}{2},2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ of $E^{2,1}$. The choice of the spaces $\mathcal{A}$ and $\mathcal{B}$, however, also needs to be adapted in this case to ensure that they embed into the spaces C^0 and C^1, respectively. So, we may choose $\mathcal{A} = \mathcal{A}_\varepsilon := W^{\frac{1}{2}+\frac{\varepsilon}{2},2}(\mathbb{R}/\mathbb{Z})$ and $\mathcal{B} = \mathcal{B}_\varepsilon := W^{\frac{3}{2}+\frac{\varepsilon}{2},2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ to maintain the compact embedding $\mathcal{C}_\varepsilon \hookrightarrow \mathcal{B}_\varepsilon$.

[3] See Sect. 2 where we briefly review the essentials for metric gradient flows.

Since all spaces considered in this situation are Hilbert spaces, other methods to prove existence of gradient flows may seem more appropriate.

2. Notice that in general the duality mapping $\mathfrak{J}_{\mathcal{C},\theta}$ defined in (1.8) is set-valued, so that one would expect to at most solve the differential inclusion

$$-D\phi[\boldsymbol{u}(\cdot)] \in \mathfrak{J}_{\mathcal{C},\theta}(\boldsymbol{u}'(\cdot))$$

on $[0,\infty)$ instead of the gradient flow equation (1.9). The specific properties of the Banach space $\mathcal{C} := \mathcal{C}_\varepsilon$ in Theorem 1.1, however, imply that $\mathfrak{J}_{\mathcal{C}_\varepsilon,\theta}$ is not only single-valued, but also a homeomorphism between $\mathcal{C}_\varepsilon$ and $\mathcal{C}_\varepsilon^*$. Its inverse $\mathfrak{J}_{\mathcal{C}_\varepsilon,\theta}^{-1}$ can be identified with $\mathfrak{J}_{\mathcal{C}_\varepsilon^*,\beta}$ for $\frac{1}{\theta}+\frac{1}{\beta}=1$; see the proof of Corollary 1.2 on page 17.

3. The choice of the Banach space $\mathcal{B}$ in the alternatives (i), (ii), and (iii) of Theorem 1.1 is maximal in the following sense: A regular injective curve $\gamma \in \mathcal{B}$ has finite energy $\mathcal{E}$, and, on the other hand, if a regular injective C^1-curve γ satisfies $\mathcal{E}(\gamma) < \infty$ then its suitably rescaled arc length parametrisation is contained in $\mathcal{B}$; see [5, Theorem 1.1][4], [13, Theorem 1], and [12, Theorem 1.1 & Remark 1.2].

4. There is complete freedom in the choice of the parameters $\kappa, \theta \in (1,\infty)$, and $\varepsilon > 0$ in Theorem 1.1. For $\kappa = 1$ we still obtain a curve of maximal slope for the total energy ϕ with respect to a strong upper gradient which, however, does not coincide with $\|D\phi\|_{\mathcal{C}^*}$ any more; see Proposition 3.3. But it is presently unclear if that curve of maximal slope also solves a differential inclusion.

The limiting process, $\varepsilon \searrow 0$, on the other hand, approximating the correct energy space $\mathcal{B}$ for the respective knot energy $\mathcal{E}$ in cases (i)–(iii) of Theorem 1.1, yields a subsequence of solutions of (1.9) that converges pointwise weakly to a limit mapping $\boldsymbol{u}^* \in AC^\theta([0,\infty),\mathcal{B})$, provided that the initial curves $\gamma_{0,\varepsilon}$ are well-prepared. For the precise statement see Corollary 4.3 and the more general result Proposition 2.2 for metric spaces, which is similar in spirit as [38, Theorem 2], where a more general limiting process is investigated. There, however, the existence of the limiting curve is assumed, see also [38, Remark 1]. While we know that the energy does not exceed its initial value $\phi(\boldsymbol{u}^*(0))$, it is unclear whether $\boldsymbol{u}^*$ is a curve of maximal slope in the limiting energy space $\mathcal{B}$. It would be if we had a weakly lower semicontinuous strong upper gradient for ϕ as shown in Lemma 2.3, a condition that is also used in [14, Theorem 2.5] and in [38, Theorem 2].

In addition, multiplying the term $\|\Sigma(\gamma)\|_{\mathcal{B}}^{\kappa}$ in (1.7) with a small prefactor $\vartheta > 0$ and sending ϑ to zero gives rise to another interesting limiting process, similarly as in [22], where such a procedure was used to study elastic knots. Letting ϑ tend to $+\infty$, on the other hand, would penalise the curves' deviation from arc length parametrisation enforcing an inextensibility constraint in the limit, which is comparable to the incompressible limit in material science; cf. [36,37].

5. We do not know at this point if the solution $\boldsymbol{u}(\cdot)$ of (1.9) is unique. Moreover, it is open whether $\boldsymbol{u}(t)$ subconverges or even converges to a critical point of the total

[4]Unfortunately, there is a typographical error in said theorem, for the correct constant cf. its proof or later papers by the same author, for example, [11].

energy ϕ as $t \to \infty$, not to speak of any information about convergence rates. These convergence issues would possibly require to study the second variation of the total energy ϕ and a suitable Łojasiewicz-Simon inequality as carried out for the L^2-flow of the Möbius energy $E^{2,1}$ in [6, Sections 4 & 5]; see also the initial analysis of the kernel of the second variation of $E^{2,1}$ in [3, Section 3].

6. Let us finally mention why we preferred the metric gradient approach to the study of ordinary differential equations in Banach spaces. First of all, the Picard-Lindelöf theory requires Lipschitz continuity of the right-hand side of the equation. This seems out of reach in the present context, where—at least in the case $\theta = 2$—the inverse duality mapping $\mathfrak{J}_{\mathcal{C},\theta}^{-1}$ fails to be Lipschitz unless the underlying Banach space is Lindenstrauß convex [46]. Indeed, here we deal with Sobolev-Slobodeckiĭ spaces, and the Lindenstrauß convexity requires an integrability that is at most quadratic; see [16, Theorem 7] in combination with [17, Proposition 3.6]. However, more general existence results with a compact operator on the right-hand side such as [30, Ch. VI, Thm. 3.1] could probably be used to obtain at least short-time existence. To extend this to long-time existence would then require further estimates which do not seem to provide a short cut. An interesting alternative approach could be a vanishing viscosity method as performed recently for the p-curvature integral by Blatt and Vorderobermeier, and this might lead to stronger results; see [9] in comparison to [10].

The paper is structured as follows. In Sect. 2 we recall the basic notions of the metric gradient flow approach following [1] but slightly adapted to our context. There, we also revisit how these notions manifest in Banach spaces. In Sect. 3 we prove an abstract existence theorem for curves of maximal slope for the total energy ϕ in (1.7) under certain assumptions on an otherwise arbitrary knot energy $\mathcal{E}$; see Theorem 3.2. We also treat in Proposition 3.3 the limiting case $\kappa = 1$. In Sect. 4 we verify in detail the assumptions of Theorem 3.2 for the three energy families $E^{\alpha,p}$, $\mathrm{intM}^{(p,q)}$, and $\mathrm{TP}^{(p,q)}$, thus proving Theorem 1.1. One of these assumptions is the continuous differentiability of the knot energy, which is known for the generalised integral Menger curvature and tangent-point energies, but—to the best of our knowledge—for O'Hara's energies $E^{\alpha,p}$ so far only for $p = 1$; see [11, Theorem 1.1]. Kawakami and Nagasawa [26] established L^1-bounds for the integrands of the first and second variation of a variant of O'Hara's energy that coincides with $E^{\alpha,p}$ only on curves parametrised by arc length. Since we need bounds for more general parametrisations and their ansatz seems non-transferable, we have included a full proof of continuous differentiability of $E^{\alpha,p}$ in Sect. 5; see Theorem 5.1, which may be of independent interest. As mentioned in Remark 1.3.4, the limiting process $\varepsilon \to 0$ for solutions of (1.9) is treated in Proposition 2.2 in the general metric setting, and specified in Corollary 4.3 to the situation described in Theorem 1.1. The appendix contains some technical material on the geometry of Sobolev-Slobodeckiĭ spaces and some differentiation rules.

2. Preliminaries

2.1. Curves of maximal slope in metric spaces

Given a complete metric space $(\mathscr{S}, d)$, an interval $I \subset \mathbb{R}$, and a number $\theta \geq 1$, a *θ-absolutely continuous curve* is a curve $\boldsymbol{u} : I \to \mathscr{S}$ such that there exists a map $m \in L^\theta(I)$ with the property $d\big(\boldsymbol{u}(s), \boldsymbol{u}(t)\big) \leq \int_s^t m(r)\,\mathrm{d}r$ for all $s, t \in I$ with $s < t$. The set of all such curves is denoted by $AC^\theta(I, \mathscr{S})$ (writing $\mathrm{AC}^\theta_{\mathrm{loc}}(I, \mathscr{S})$ for the local variant of this space if m is only in $L^\theta_{\mathrm{loc}}(I)$, and using the abbreviation $AC(I, \mathscr{S}) := AC^1(I, \mathscr{S})$). According to [1, Theorem 1.1.2] every $\boldsymbol{u} \in AC^\theta_{\mathrm{loc}}(I, \mathscr{S})$ is *metrically differentiable* almost everywhere in the following sense: its *metric derivative* $|\boldsymbol{u}'|(t) := \lim_{s\to t} \frac{d(\boldsymbol{u}(t),\boldsymbol{u}(s))}{|t-s|}$ exists for almost every $t \in I$.

Any functional $\phi\colon \mathscr{S} \to (-\infty, \infty]$ with non-empty *effective domain*

$$\mathscr{D}(\phi) := \{u \in \mathscr{S} \colon \phi(u) < \infty\}$$

admits a so-called *local slope* $|\partial\phi|(u)$ of ϕ at $u \in \mathscr{D}(\phi)$ given by the expression $|\partial\phi|(u) := \limsup_{d(v,u)\to 0} \frac{(\phi(u)-\phi(v))^+}{d(u,v)} \in [0, \infty]$, where $a^+ := \max\{0, a\}$ for $a \in \mathbb{R}$ [1, Definition 1.2.4]. A map $g : \mathscr{S} \to [0, \infty]$ is a *strong upper gradient* for ϕ on $\mathscr{D}(\phi)$, if for every curve $\boldsymbol{u} \in AC(I, \mathscr{D}(\phi))$ the composition $g \circ \boldsymbol{u}$ is Borel-measurable and

$$\big|\phi \circ \boldsymbol{u}(t) - \phi \circ \boldsymbol{u}(s)\big| \leq \int_s^t g \circ \boldsymbol{u}(r)|\boldsymbol{u}'|(r)\,\mathrm{d}r \quad \text{for all } s, t \in I \text{ with } s < t. \tag{2.1}$$

Notice that this notion of a strong upper gradient modifies slightly that of [1, Definition 1.2.1] in that inequality (2.1) is only required for absolutely continuous curves whose image is contained in the effective domain $\mathscr{D}(\phi)$. This restriction does not affect Theorem 2.1 below upon which our existence results build. In the situations encountered in this work, the local slope turns out to be a strong upper gradient.

With these notions at hand, one can define a *θ-curve of maximal slope for ϕ with respect to the strong upper gradient g starting at $u_0 \in \mathscr{D}(\phi)$*. It is a curve $\boldsymbol{u} \in AC^\theta_{\mathrm{loc}}\big([0, \infty), \mathscr{S}\big)$ that satisfies $\boldsymbol{u}(0) = u_0$ and the *energy dissipation equality*

$$\phi(\boldsymbol{u}(t)) - \phi(\boldsymbol{u}(s)) = -\frac{1}{\beta}\int_s^t g^\beta(\boldsymbol{u}(r))\,\mathrm{d}r - \frac{1}{\theta}\int_s^t |\boldsymbol{u}'|^\theta(r)\,\mathrm{d}r \tag{2.2}$$

for all $0 \leq s < t < \infty$, where $\theta^{-1} + \beta^{-1} = 1$. At first sight, this definition seems to differ from that given in [1, Definition 1.3.2]. Notice, however, that in the present more specific situation, where $u_0 \in \mathscr{D}(\phi)$ and g is a strong upper gradient, our definition is actually equivalent to that in [1]. Indeed, since $\phi(u_0) < \infty$ and $\phi(u) > -\infty$ for all $u \in \mathscr{S}$, equality (2.2) with $s = 0$ and Young's inequality imply that $g \circ \boldsymbol{u}|\boldsymbol{u}'|$ is integrable on every compact interval $[0, t]$. Hence, by the defining inequality (2.1) for strong upper gradients, $\phi \circ \boldsymbol{u}$ is locally absolutely continuous and we deduce the defining inequality (1.3.13) of [1, Definition 1.3.2] with $\varphi = \phi \circ \boldsymbol{u}$. Conversely, [1, Remark 1.3.3] implies that (2.2) holds. In that case, we also have the identities

$$g^{\beta}(\boldsymbol{u}(r)) = |\boldsymbol{u}'|^{\theta}(r) = |\boldsymbol{u}'|(r) g \circ \boldsymbol{u}(r) \quad \text{for a.e. } r \in (0, \infty). \tag{2.3}$$

In fact, one can also require (2.2) only to hold with "$\leq$" instead of equality, since the converse inequality is always satisfied as a consequence of Young's inequality and (2.1).

A common way to obtain a curve of maximal slope is by carrying out three steps. Firstly, for a given step size, a discretised version of the energy dissipation equation is solved by iteratively solving minimisation problems associated with the step size. Secondly, using an interpolation method, each of the thus obtained piecewise constant solutions is transformed into a continuous curve. These curves form a relatively compact set and a limit curve is extracted. This limit curve is commonly referred to as a *(generalised) minimising movement* and satisfies a weaker form of the energy dissipation equality. Thirdly, it remains to check that the minimising movement in fact also satisfies the energy dissipation equality. General assumptions on the functional have been formulated that ensure that each of the previous steps can be executed. We state these assumptions collected from [1, Section 2.1 & Remark 2.3.4] and the corresponding existence theorem.

Assumption (M) on the metric space. *The complete metric space $(\mathscr{S}, d)$ is endowed with an additional weak topology σ which is Hausdorff, weaker than the topology induced by the metric d, and such that d is sequentially weakly lower semi-continuous*[5], *i.e. for all sequences $u_k \overset{\sigma}{\rightharpoonup} u$ and $v_k \overset{\sigma}{\rightharpoonup} v$ as $k \to \infty$ it holds true that*

$$d(u, v) \leq \liminf_{k \to \infty} d(u_k, v_k).$$

Assumptions on the functional $\boldsymbol{\phi} : \mathscr{S} \to (-\infty, \infty]$ with $\mathscr{D}(\boldsymbol{\phi}) \neq \emptyset$ and fixed $\boldsymbol{\theta} \in (1, \infty)$.

(Φ1) The functional ϕ is sequentially weakly lower semi-continuous on d-bounded sets, i.e. if $\sup_{k,l \in \mathbb{N}} d(u_k, u_l) < \infty$ for a sequence $u_k \overset{\sigma}{\rightharpoonup} u$ as $k \to \infty$, then $\phi(u) \leq \liminf_{k \to \infty} \phi(u_k)$.

(Φ2) The functional ϕ is coercive in the sense that there exists $u_* \in \mathscr{S}$, $B \in \mathbb{R}$, and $C > 0$ such that $\phi(u) \geq B - C d(u, u_*)^{\theta}$ for all $u \in \mathscr{S}$.

(Φ3) Every d-bounded subset of a sublevel set of ϕ is relatively weakly sequentially compact, i.e. for a sequence $(u_k)_k \subset \mathscr{S}$ with $\sup_{k \in \mathbb{N}} \phi(u_k) < \infty$ and $\sup_{k,l \in \mathbb{N}} d(u_k, u_l) < \infty$ there is $u \in \mathscr{S}$ and a subsequence $(u_{k_m})_m \subset (u_k)_k$ such that $u_{k_m} \overset{\sigma}{\rightharpoonup} u$ as $m \to \infty$.

(Φ4) The local slope $|\partial \phi|$ of ϕ is weakly sequentially lower semi-continuous on d-bounded subsets of sublevel sets of ϕ, that is, for every sequence $u_k \overset{\sigma}{\rightharpoonup} u$ as

[5] In our application, σ will be the weak topology on a Banach space, so there is no risk of confusion regarding the expression "weak".

$k \to \infty$ which satisfies $\sup_{k\in\mathbb{N}}\{d(u_k, u), \phi(u_k)\} < \infty$ one has

$$|\partial\phi|(u) \leq \liminf_{k\to\infty} |\partial\phi|(u_n).$$

(Φ5) The local slope of ϕ is a strong upper gradient on[6] $\mathscr{D}(\phi)$.

Theorem 2.1 (Curves of maximal slope exist). *Let* $(\mathscr{S}, d)$ *be a metric space satisfying Assumption* (M)*, and suppose* $\phi : \mathscr{S} \to (-\infty, \infty]$ *satisfies for given* $\theta \in (1, \infty)$ *Assumptions* (Φ1)–(Φ5)*, and let* $u_0 \in \mathscr{D}(\phi)$*. Then there exists a* θ*-curve of maximal slope for* ϕ *with respect to the strong upper gradient* $|\partial\phi|$ *starting at* u_0.

Proof. Notice that Assumption (Φ2) is equivalent to

$$\inf_{v\in\mathscr{S}} [Cd(v, u_*)^\theta + \phi(v)] > -\infty,$$

which ensures that $\tau_* := (\theta C)^{-\frac{1}{\theta-1}}$ satisfies

$$\inf_{v\in\mathscr{S}} [\frac{1}{\theta}\tau_*^{1-\theta} d(u_*, v)^\theta + \phi(v)] > -\infty.$$

Therefore, for every uniform partition $\{0 < \frac{1}{m} < \frac{2}{m} < \cdots < \frac{k}{m} < \ldots\}$ of $[0, \infty)$ with $m \in \mathbb{N}$ such that $\frac{1}{m} < \tau_*$, Assumptions (M) and (Φ1)–(Φ3) enable us to employ [1, Corollary 2.2.2] to find a solution of the recursive minimisation problem [1, (2.0.4)] starting at u_0. Then, [1, Proposition 2.2.3] implies that the family of these discrete solutions admit, up to a subsequence, an absolutely continuous limit curve $\boldsymbol{u}$ (called a *generalised minimising movement*). Now, Assumption (Φ4) implies that the relaxed slope $|\partial^-\phi|$, [1, (2.3.1)], is equal to the local slope on $\mathscr{D}(\phi)$. Moreover, by Assumption (Φ5), it is also a strong upper gradient on $\mathscr{D}(\phi)$. Therefore, Theorem [1, Theorem 2.3.3] is applicable and we deduce that $\boldsymbol{u}$ is a curve of maximal slope with respect to $|\partial^-\phi| = |\partial\phi|$. Notice that our additional restrictions in the definition of upper gradients and Assumption (Φ5) are non-essential, as both only play a role at [1, (3.4.2)]. There, the curve $\boldsymbol{u}$ has finite energy because of [1, (3.4.1)]. □

The following proposition shows how curves of maximal slope in smaller subspaces of a given metric space give rise to a limiting curve.

Proposition 2.2. *Let* $\theta \in (1, \infty)$*. Let* $(\mathscr{S}_0, d_0, \sigma)$ *be a metric space that satisfies Assumption* (M) *and let* $\big((\mathscr{S}_\varepsilon, d_\varepsilon)\big)_{\varepsilon>0}$ *be metric spaces such that* $\mathscr{S}_\varepsilon \subset \mathscr{S}_0$ *and such that there exists a constant* $c_0 > 0$ *such that* $d_0(u, v) \leq c_0 d_\varepsilon(u, v)$ *holds for all* $u, v \in \mathscr{S}_\varepsilon$ *and all* $\varepsilon > 0$*. Let* $\phi : \mathscr{S} \to (-\infty, \infty]$ *be a functional that satisfies Assumptions* (Φ1)–(Φ3) *with* $\mathscr{S} = \mathscr{S}_0$ *and let* $u_0 \in \mathscr{D}(\phi)$*. Assume that for every* $\varepsilon > 0$ *there exists* $u_{0,\varepsilon} \in \mathscr{S}_\varepsilon$ *such that*

[6] This is a slightly restricted version of [1, Remark 2.3.4 (i)], but this does not affect the validity of Theorem 2.1 below.

$$\sup_{\varepsilon>0} d_0(u_{0,\varepsilon}, u_0) < \infty, \quad u_{0,\varepsilon} \xrightarrow[(\varepsilon\to 0)]{\sigma} u_0, \quad \textit{and} \quad \phi(u_{0,\varepsilon}) \xrightarrow[(\varepsilon\to 0)]{} \phi(u_0) \tag{2.4}$$

and let $\boldsymbol{u}_\varepsilon \in AC^\theta_{\mathrm{loc}}([0,\infty),(\mathscr{S}_\varepsilon, d_\varepsilon))$ *be a* θ*-curve of maximal slope for* ϕ *with respect to the strong upper gradient* g_ε *starting at* $u_{0,\varepsilon}$*. Then there exists a subsequence* $\varepsilon_k \to 0$ *and a curve* $\boldsymbol{u}^* \in AC^\theta_{\mathrm{loc}}([0,\infty),(\mathscr{S}_0, d_0))$ *such that* $\boldsymbol{u}^*(0) = u_0$ *and* $\boldsymbol{u}_{\varepsilon_k}(t) \xrightarrow[(k\to\infty)]{\sigma} \boldsymbol{u}^*(t)$ *holds for every* $t \geq 0$*. Moreover,* $\phi(\boldsymbol{u}^*(t)) \leq \phi(u_0)$ *for all* $t \geq 0$.

Proof. Let us first fix some notation. For a given metric space $\mathscr{S} \in \{\mathscr{S}_0, \mathscr{S}_\varepsilon\}$, a locally absolutely continuous curve $\boldsymbol{v} \in AC_{\mathrm{loc}}([0,\infty),\mathscr{S})$ and a point $v \in \mathscr{S}$, we denote by $|\boldsymbol{v}'|_{\mathscr{S}}(r)$ the metric derivative of $\boldsymbol{v}$ at $r \geq 0$ taken with respect to the metric on $\mathscr{S}$. Moreover, since we are only interested in small ε, we may assume that $\sup_{\varepsilon>0} \phi(u_{0,\varepsilon}) < \infty$.

The arguments presented here are continuous analogues of those used in [1, Lemma 3.2.2] and [1, Corollary 3.3.4]. The existence of a converging subsequence relies on the theorem of Arzelà-Ascoli [1, Proposition 3.3.1]. We will check its prerequisites, which are consequences of the energy dissipation equalities

$$\phi\big(\boldsymbol{u}_\varepsilon(t)\big) + \frac{1}{\theta}\int_s^t |\boldsymbol{u}'_\varepsilon|^\theta_{\mathscr{S}_\varepsilon}(r)\,\mathrm{d}r + \frac{1}{\beta}\int_s^t g^\beta_\varepsilon\big(\boldsymbol{u}_\varepsilon(r)\big)\,\mathrm{d}r = \phi\big(\boldsymbol{u}_\varepsilon(s)\big),\ 0 \leq s < t < \infty. \tag{2.5}$$

From the assumptions on the metrics d_0, d_ε, it follows that

$$d_0(\boldsymbol{u}_\varepsilon(t), \boldsymbol{u}_\varepsilon(s)) \leq c_0 d_\varepsilon(\boldsymbol{u}_\varepsilon(t), \boldsymbol{u}_\varepsilon(s)) \leq c_0 \int_s^t |\boldsymbol{u}'_\varepsilon|_{\mathscr{S}_\varepsilon}(r)\,\mathrm{d}r$$

for all $0 \leq s < t < \infty$. Therefore, $\boldsymbol{u}_\varepsilon \in AC^\theta_{\mathrm{loc}}([0,\infty),(\mathscr{S}_0, d_0))$ with $|\boldsymbol{u}'_\varepsilon|_{\mathscr{S}_0}(r) \leq c_0|\boldsymbol{u}'_\varepsilon|_{\mathscr{S}_\varepsilon}(r)$ for almost every $r \geq 0$. Notice that Assumption ($\Phi 2$) also holds for $u_* = u_0$, possibly with different constants. We deduce from Assumption ($\Phi 2$) and (2.5) with $s = 0$ that

$$\begin{aligned}\int_0^t |\boldsymbol{u}'_\varepsilon|^\theta_{\mathscr{S}_0}(r)\,\mathrm{d}r &\leq c_0^\theta \int_0^t |\boldsymbol{u}'_\varepsilon|^\theta_{\mathscr{S}_\varepsilon}(r)\,\mathrm{d}r \leq \theta c_0^\theta\big(\phi(u_{0,\varepsilon}) - \phi(\boldsymbol{u}_\varepsilon(t))\big)\\ &\leq \theta c_0^\theta\big(\phi(u_{0,\varepsilon}) - B\big) + C\theta c_0^\theta\, d_0^\theta(\boldsymbol{u}_\varepsilon(t), u_0).\end{aligned} \tag{2.6}$$

On the other hand, since $\boldsymbol{u}_\varepsilon$ and $d_0(\boldsymbol{u}_\varepsilon(\cdot), u_0)$ are locally absolutely continuous, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r} d_0(\boldsymbol{u}_\varepsilon(r), u_0) &= \lim_{s\to r} \frac{d_0(\boldsymbol{u}_\varepsilon(s), u_0) - d_0(\boldsymbol{u}_\varepsilon(r), u_0)}{s-r}\\ &\leq \lim_{s\to r} \frac{d_0(\boldsymbol{u}_\varepsilon(s), \boldsymbol{u}_\varepsilon(r))}{|s-r|} = |\boldsymbol{u}'_\varepsilon|_{\mathscr{S}_0}(r)\end{aligned}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}r} d_0^\theta(\boldsymbol{u}_\varepsilon(r), u_0) \ \le \theta d_0^{\theta-1}(\boldsymbol{u}_\varepsilon(r), u_0)|\boldsymbol{u}'_\varepsilon|_{\mathscr{S}_0}(r) \quad \text{for almost every } r \ge 0.$$

Using Young's inequality with $\delta > 0$, $a = \sqrt[\theta]{\delta}|\boldsymbol{u}'_\varepsilon|_{\mathscr{S}_0}$, $b = \theta \sqrt[\theta]{\delta^{-1}} d_0^{\theta-1}(\boldsymbol{u}_\varepsilon(r), u_0)$, and recalling that $\beta = \frac{\theta}{\theta-1}$, we obtain for almost every $r \ge 0$

$$\frac{\mathrm{d}}{\mathrm{d}r} d_0^\theta(\boldsymbol{u}_\varepsilon(r), u_0) \le \frac{\delta}{\theta}|\boldsymbol{u}'_\varepsilon|^\theta_{\mathscr{S}_0}(r) + \frac{\theta^\beta}{\beta\delta^{\beta-1}} d_0^\theta(\boldsymbol{u}_\varepsilon(r), u_0). \tag{2.7}$$

Therefore,

$$\begin{aligned} d_0^\theta(\boldsymbol{u}_\varepsilon(t), u_0) \ &= \ d_0^\theta(u_{0,\varepsilon}, u_0) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}r} d_0^\theta(\boldsymbol{u}_\varepsilon(r), u_0) \,\mathrm{d}r \\ &\overset{(2.7)}{\le} d_0^\theta(u_{0,\varepsilon}, u_0) + \frac{\delta}{\theta}\int_0^t |\boldsymbol{u}'_\varepsilon|^\theta_{\mathscr{S}_0}(r)\,\mathrm{d}r + \frac{\theta^\beta}{\beta\delta^{\beta-1}}\int_0^t d_0^\theta(\boldsymbol{u}_\varepsilon(r), u_0)\,\mathrm{d}r \\ &\overset{(2.6)}{\le} d_0^\theta(u_{0,\varepsilon}, u_0) + \delta c_0^\theta\big(\phi(u_{0,\varepsilon}) - B\big) + \delta C c_0^\theta\, d_0^\theta(\boldsymbol{u}_\varepsilon(t), u_0) \\ &\quad + \frac{\theta^\beta}{\beta\delta^{\beta-1}}\int_0^t d_0^\theta(\boldsymbol{u}_\varepsilon(r), u_0)\,\mathrm{d}r. \end{aligned}$$

Choosing, for example, $\delta = (2Cc_0^\theta)^{-1}$, and using (2.4), we deduce for some constants $c_1, c_2 > 0$ independent of ε and t the inequality

$$d_0^\theta(\boldsymbol{u}_\varepsilon(t), u_0) \le c_1 + c_2\int_0^t d_0^\theta(\boldsymbol{u}_\varepsilon(r), u_0)\,\mathrm{d}r.$$

It follows from Gronwall's inequality [24, Corollary 6.6] that

$$d_0^\theta(\boldsymbol{u}_\varepsilon(t), u_0) \le c_1 \exp(c_2 t). \tag{2.8}$$

Now fix $T > 0$. It follows immediately from (2.5) and from (2.8) that

$$\sup_{\varepsilon>0}\sup_{0\le t\le T} \phi(\boldsymbol{u}_\varepsilon(t)) < \infty, \quad \text{and} \quad \sup_{\varepsilon>0}\sup_{0\le t\le T} d_0(\boldsymbol{u}_\varepsilon(t), u_0) < \infty. \tag{2.9}$$

Therefore, the maps $\boldsymbol{u}_\varepsilon\big|_{[0,T]}$ take their values in a d_0-bounded sublevel set of ϕ, which, by Assumption $(\Phi 3)$ is relatively weakly sequentially compact. Moreover, a consequence of (2.6) and (2.8) with $t = T$ is that

$$\sup_{\varepsilon>0}\big\||\boldsymbol{u}'_\varepsilon|_{\mathscr{S}_0}\big\|^\theta_{L^\theta([0,T])} = \sup_{\varepsilon>0}\int_0^T |\boldsymbol{u}'_\varepsilon|^\theta_{\mathscr{S}_0}(t)\,\mathrm{d}t < \infty. \tag{2.10}$$

Since $L^\theta([0,T])$ is reflexive, there exists a null-sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that, with the notation $\boldsymbol{u}_k := \boldsymbol{u}_{\varepsilon_k}$, the metric derivatives $|\boldsymbol{u}'_k|_{\mathscr{S}_0}$ converge weakly to some $A_T \in L^\theta([0,T])$. For all $0 \le s < t \le T$ the characteristic function $\chi_{[s,t]}$ is of class $L^\beta([0,T])$ (recall that $\theta^{-1} + \beta^{-1} = 1$) and therefore

$$\limsup_{k\to\infty} d_0(\boldsymbol{u}_k(t), \boldsymbol{u}_k(s)) \le \limsup_{k\to\infty}\int_s^t |\boldsymbol{u}'_k|_{\mathscr{S}_0}(r)\,\mathrm{d}r = \int_s^t A_T(r)\,\mathrm{d}r. \tag{2.11}$$

Using [1, Proposition 3.3.1] with $\omega(s,t) := |\int_s^t A_T(r)\,\mathrm{d}r|$, we deduce from (2.11) and (2.9) that there exists a map $\boldsymbol{u}_T^* : [0,T] \to \mathscr{S}_0$, such that, up to a subsequence,

$$\boldsymbol{u}_k(t) \xrightarrow[(k\to\infty)]{\sigma} \boldsymbol{u}_T^*(t) \quad \text{for all} \quad t \in [0,T]. \tag{2.12}$$

By a diagonal argument, we find $\boldsymbol{u}^* : [0,\infty) \to \mathscr{S}_0$ and $A \in L^\theta_{\mathrm{loc}}([0,\infty))$ such that, up to another subsequence, (2.12) holds with $\boldsymbol{u}^*$ for every $t \geq 0$ and such that for every $T > 0$ there holds $\boldsymbol{u}^* = \boldsymbol{u}_T^*$ and $A = A_T$ almost everywhere on $[0,T]$. Since the metric d_0 is sequentially weakly lower semicontinuous and because of (2.11) and (2.12), we deduce that for all $0 \leq s < t < \infty$

$$d_0(\boldsymbol{u}^*(t), \boldsymbol{u}^*(s)) \leq \liminf_{k\to\infty} d_0(\boldsymbol{u}_k(t), \boldsymbol{u}_k(s)) \leq \int_s^t A(r)\,\mathrm{d}r \tag{2.13}$$

whence $\boldsymbol{u}^* \in AC^\theta_{\mathrm{loc}}([0,\infty), (\mathscr{S}_0, d_0))$ and $|(\boldsymbol{u}^*)'|_{\mathscr{S}_0}(r) \leq A(r)$ for almost every $r \geq 0$.

Because of (2.12), Assumption (Φ1), (2.5) and (2.4), the energy remains bounded by its initial value:

$$\phi(\boldsymbol{u}^*(t)) \leq \liminf_{k\to\infty} \phi(\boldsymbol{u}_k(t)) \leq \liminf_{k\to\infty} \phi(\boldsymbol{u}_k(0)) = \phi(u_0).$$

□

The following lemma shows that under additional assumptions the limiting curve is actually a curve of maximal slope. We will not need this result in the remainder of this paper.

Lemma 2.3. *If in addition to the assumptions of Proposition 2.2 with $c_0 = 1$, we also have for a strong upper gradient g of ϕ with respect to $\mathscr{S}_0$ and for every $u_\varepsilon \xrightarrow[(\varepsilon\to 0)]{\sigma} u$ in $\mathscr{S}_0$ that $\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) \geq g(u)$, then the limiting curve $\boldsymbol{u}^*$ is a θ-curve of maximal slope for ϕ starting at u_0.*

Proof. Let $\boldsymbol{u}^*$ and $(\boldsymbol{u}_k)_k$ be as in the proof of Proposition 2.2. Since $c_0 = 1$, we have $|\boldsymbol{u}_\varepsilon'|_{\mathscr{S}_0}(r) \leq |\boldsymbol{u}_\varepsilon'|_{\mathscr{S}_\varepsilon}(r)$ for a.e. $r \geq 0$. Using this inequality, (Φ1), the weak convergence of $|\boldsymbol{u}_k'|_{\mathscr{S}_0}$ to A together with the inequality (2.13), the newly introduced assumption on the upper gradients, and (2.4), we may pass to the limit inferior in the energy dissipation equality (2.5) with $s = 0$ to deduce that the limit curve $\boldsymbol{u}^*$ satisfies

$$\phi(\boldsymbol{u}^*(t)) + \frac{1}{\theta}\int_0^t |(\boldsymbol{u}^*)'|^\theta_{\mathscr{S}_0}(r)\,\mathrm{d}r + \frac{1}{\beta}\int_0^t g^\beta(\boldsymbol{u}^*(r))\,\mathrm{d}r \leq \phi(u_0). \tag{2.14}$$

Since g is a strong upper gradient, we may use (2.1) in combination with Young's inequality to find that (2.14) actually holds with equality. Subtracting equality (2.14) with s instead of t from equality (2.14) yields the energy dissipation equality (2.2) for $\boldsymbol{u}^*$. □

2.2. Curves of maximal slope in Banach spaces

If the metric space is actually a real Banach space $(\mathscr{S}, d) = (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ with dual space $(\mathcal{B}^*, \|\cdot\|_{\mathcal{B}^*})$, then the objects of the previous section can be characterised in terms of classical derivatives; see, for example, [31, Section 2.3] for a proof of the following result regarding the local slope and the metric derivative.

Proposition 2.4. *Let* $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ *be a Banach space,* $\phi : \mathcal{B} \to (-\infty, \infty]$ *a mapping, and suppose* $\boldsymbol{u} : I \to \mathcal{B}$ *is a curve defined on an interval* $I \subset \mathbb{R}$*. If* ϕ *is Fréchet differentiable at some point* $v \in \mathcal{B}$ *with derivative* $D\phi[v]$*, then* $|\partial\phi|(v) = \|D\phi[v]\|_{\mathcal{B}^*}$*, and if* $\boldsymbol{u}$ *is differentiable at* $t \in I$ *with derivative* $\boldsymbol{u}'(t)$*, then* $|\boldsymbol{u}'|(t) = \left\|\boldsymbol{u}'(t)\right\|_{\mathcal{B}}$*.*

If one has a more specific Banach space $\mathcal{C}$ with additional properties, then θ-curves of maximal slope satisfy differential inclusions. The following result is a simplified and slightly modified version of [1, Proposition 1.4.1] and is stated in terms of the duality mapping $\mathfrak{J}_{\mathcal{C},\theta} : \mathcal{C} \to 2^{\mathcal{C}^*}$ with weight $\theta \in (1, \infty)$ defined in (1.8) of the introduction.

Proposition 2.5. *Suppose* $\mathcal{C}$ *is a Banach space,* $\phi : \mathcal{C} \to \mathbb{R} \cup \{\infty\}$ *a mapping that is Fréchet-differentiable*[7] *on an open subset* $\Omega \subset \mathcal{C}$*, and let* $\boldsymbol{u} \in AC^{\theta}_{\mathrm{loc}}\big([0, \infty), \Omega\big)$ *be a* θ*-curve of maximal slope for* ϕ *with respect to its local slope* $|\partial\phi|$*.*

(i) If $\mathcal{C}$ *has the Radon-Nikodym property then*

$$-D\phi[\boldsymbol{u}(t)] \in \mathfrak{J}_{\mathcal{C},\theta}\big(\boldsymbol{u}'(t)\big) \quad \text{for a.e. } t > 0. \tag{2.15}$$

(ii) If $\mathcal{C}$ *is reflexive, strictly convex, and has a Gâteaux-differentiable norm on* $\mathcal{C}\setminus\{0\}$*, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t) = -\mathfrak{J}^{-1}_{\mathcal{C},\theta}\big(D\phi[\boldsymbol{u}(t)]\big) \quad \text{for a.e. } t > 0. \tag{2.16}$$

Proof. (i) The Radon-Nikodym property implies that absolutely continuous curves are differentiable almost everywhere, see, for example, [2, Definition 1.2.5]. Using the energy dissipation equality (2.2) together with (2.3) and Proposition 2.4, we infer that

$$\begin{aligned}\phi \circ \boldsymbol{u}(t) - \phi \circ \boldsymbol{u}(s) &= -\int_s^t |\boldsymbol{u}'|(r)|\partial\phi| \circ \boldsymbol{u}(r)\,\mathrm{d}r \\ &= -\int_s^t \left\|\boldsymbol{u}'(r)\right\|_{\mathcal{C}} \|D\phi[\boldsymbol{u}(r)]\|_{\mathcal{C}^*}\,\mathrm{d}r\end{aligned}$$

holds for all $0 \le s < t < \infty$. By the chain rule we obtain

$$-\left\|\boldsymbol{u}'(t)\right\|_{\mathcal{C}} \|D\phi[\boldsymbol{u}(t)]\|_{\mathcal{C}^*} = \frac{\mathrm{d}}{\mathrm{d}t}\phi \circ \boldsymbol{u}(t) = \langle D\phi[\boldsymbol{u}(t)], \boldsymbol{u}'(t)\rangle_{\mathcal{C}^*\times\mathcal{C}} \quad \text{for a.e. } t \ge 0.$$

[7] One can easily show that under this strong assumption on ϕ its local slope $|\partial\phi|$ is a strong upper gradient; see, for example, [31, Lemma 2.3.7].

Together with identity (2.3) for $g := |\partial\phi|$ in combination with the definition of the duality mapping (1.8) and Proposition 2.4, this implies (2.15).

(ii) If $\mathcal{C}$ is reflexive, it has the Radon-Nikodym property [2, Corollary 1.2.7]. The differentiability of the norm $\|\cdot\|_{\mathcal{C}}$ on $\mathcal{C} \setminus \{0\}$ ensures that the duality mapping $\mathfrak{J}_{\mathcal{C},\theta}$ is single-valued [18, Corollary I.4.5]. The reflexivity of $\mathcal{C}$ guarantees that $\mathfrak{J}_{\mathcal{C},\theta}$ is surjective [18, Theorem II.3.4], and the strict convexity of $\mathcal{C}$ implies that it is injective [18, Theorem II.1.8][8]. Thus, the inclusion (2.15) is actually an identity and we deduce (2.16). □

3. Curves of maximal slope for abstract knot energies regularised by the logarithmic strain

For most of the known knot energies $\mathcal{E}$, the underlying energy space is a Sobolev-Slobodeckiĭ space $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for some $s \in (0, 1)$, $\rho > 1/s$. In the still abstract setting of the present section we regularise any such $\mathcal{E}$ by adding a power of the norm of the logarithmic strain Σ defined in (1.6) to form a total energy ϕ as in (1.7). It was shown in [27] that Σ is continuously differentiable on regular injective curves of that class.

Lemma 3.1 ([27, Proposition 5.1]) *For any* $s \in (0, 1)$ *and* $\rho \in \left(\frac{1}{s}, \infty\right)$ *one has*

$$\Sigma \in C^1(W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), W^{s,\rho}(\mathbb{R}/\mathbb{Z})).$$

Here, and from now on we use the following abbreviated notation: For a given space $\mathcal{F}$ of closed curves we denote from now on by $\mathcal{F}_{\mathrm{i}}$ the subset of all injective curves, by $\mathcal{F}_{\mathrm{r}}$ the regular curves, and we write $\mathcal{F}_{\mathrm{a}}$ for the arc length-parametrised curves contained in $\mathcal{F}$, and we combine these indices i, r and a as needed.

To quantify injectivity of closed curves γ, we define the *bi-Lipschitz constant* $\mathrm{BiLip}(\gamma)$ by means of

$$\mathrm{BiLip}(\gamma) := \inf_{\substack{x,y\in\mathbb{R}/\mathbb{Z}\\ x\neq y}} \frac{|\gamma(y) - \gamma(x)|}{|x - y|_{\mathbb{R}/\mathbb{Z}}}.$$

To apply the metric existence result, Theorem 2.1 of Sect. 2, we formulate in the present section a few general assumptions on the knot energy $\mathcal{E}$ so that the total energy ϕ in (1.7) satisfies Assumptions *(Φ1)*–*(Φ5)* of Sect. 2. In order to verify *(Φ4)*, however, we will need to consider a slightly smaller reflexive Banach space $\mathcal{C}$ that is compactly embedded in $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ to finally prove the abstract existence result of this section; see Theorem 3.2 below.

Assumptions on the knot energy $\mathcal{E}$. Let $s \in (0, 1)$ and $\rho \in (\frac{1}{s}, \infty)$.

(E1) $\mathcal{E}$ is non-negative and of class C^1 on $W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ (differentiability).

[8] Note that in the language of [18], we use the weight function $\varphi(s) := s^{\frac{\theta}{\beta}}$.

(E2) $\mathcal{E}(\gamma) \le \liminf_{k\to\infty} \in (\gamma_k)$ for all $\gamma_k, \gamma \in W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $\gamma_k \rightharpoonup \gamma$ as $k \to \infty$ (weak lower semi-continuity).

(E3) For all $c_1, c_2 > 0$ there is a constant $C = C(\mathcal{E}, c_1, c_2)$ such that for all curves $\gamma \in C^1_{\mathrm{i}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $c_1^{-1} \le |\gamma'| \le c_1$ and $\mathcal{E}(\gamma) \le c_2$ one has $\mathrm{BiLip}(\gamma) \ge C$ (uniform control of bi-Lipschitz constant).

Now we can state our abstract existence result for curves of maximal slope and solutions of the gradient flow equation.

Theorem 3.2. *Let $s \in (0,1)$, $\rho \in (\frac{1}{s}, \infty)$ and $\theta, \kappa \in (1,\infty)$. Suppose that $\mathcal{C} \subset W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is a compactly embedded reflexive Banach space and that the energy $\mathcal{E}$ satisfies Assumptions (E1)–(E3), and $\gamma_0 \in \mathscr{D}(\phi)$, where the total energy ϕ is defined as in (1.7). Then there exists $\boldsymbol{u} \in AC^\theta([0,\infty), (\mathcal{C}, \|\cdot\|_{\mathcal{C}}))$ with $\boldsymbol{u}(0) = \gamma_0$ and*

$$-D\phi[\boldsymbol{u}(t)] \in \mathfrak{J}_{\mathcal{C},\theta}(\boldsymbol{u}'(t)) \quad \text{for a.e. } t > 0, \tag{3.1}$$

where $\mathfrak{J}_{\mathcal{C},\theta} : \mathcal{C} \mapsto 2^{\mathcal{C}^*}$ is the θ-duality mapping defined in (1.8). If, in addition, $\mathcal{C}$ is strictly convex and has a Gâteaux-differentiable norm on $\mathcal{C} \setminus \{0\}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t) = -\mathfrak{J}^{-1}_{\mathcal{C},\theta}(D\phi[\boldsymbol{u}(t)]) \quad \text{for a.e. } t > 0. \tag{3.2}$$

Proof. We proceed as follows. In the first four steps we verify Assumptions (M), and (Φ1)–(Φ5) of Sect. 2.1 to obtain by virtue of the metric existence result, Theorem 2.1, a curve of maximal slope for the total energy ϕ with respect to its local slope $|\partial\phi|$. Then in a final step we apply the more specific results for Banach spaces presented in Sect. 2.2 to obtain the differential inclusion (3.1) and finally the gradient flow equation (3.2).

Step 1: Assumptions (M), (Φ2), (Φ3) are satisfied. The Sobolev-Slobodeckiĭ space $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ as well as the Banach space $\mathcal{C}$ are complete metric spaces with the metric induced by their respective norms. It follows from general Banach space theory (see, for example, [15, Propositions 3.3 & 3.5]) that these norms are weakly lower semi-continuous, and that the weak topology is Hausdorff, so that Assumption (M) for the complete metric space $\mathscr{S} := \mathcal{C}$ is satisfied (as well as for the larger space $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$). The total energy ϕ is coercive since $\mathcal{E}$ and the norm of Σ are non-negative, which verifies Assumption (Φ2). Moreover, $W^{1+s,\rho}$ for $\rho > 1$ is reflexive by Lemma A.1, and $\mathcal{C}$ is reflexive by assumption. So, the sequential weak compactness of bounded sets follows from general theory again; see, for example, [15, Theorem 3.17]. In particular, Assumption (Φ3) holds here.

Step 2: Assumption (Φ1) holds true. To prove this claim we first show that a sequence of bounded total energy cannot have weak cluster points outside of the set of injective and regular curves. This way, the energy may not jump to infinity. Afterwards, it is straight-forward to prove weak lower semi-continuity on $W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

Let $(\gamma_k)_{k\in\mathbb{N}} \subset W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be a sequence that converges weakly to a curve γ of class $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ as $k \to \infty$, and up to a further subsequence we may also assume that

$$\liminf_{k\to\infty} \phi(\gamma_k) < \infty, \quad \sup_{k\in\mathbb{N}} \phi(\gamma_k) < \infty, \quad \text{and} \quad \gamma_k \to \gamma \quad \text{in } C^1, \tag{3.3}$$

where we used Arzela-Ascoli's theorem in combination with the embedding result, Proposition A.3 (ii) in the appendix for $k_1 := 1$, $k_2 := 1$, $\rho_1 := \rho$, and an arbitrary $\mu \in \left(0, s - \frac{1}{\rho}\right)$, to obtain the C^1-convergence. The finiteness of $\phi(\gamma_k)$ implies that $\gamma_k \in W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for all $k \in \mathbb{N}$ by the very definition (1.7) of the total energy ϕ, and further that $\sup_{k\in\mathbb{N}} \|\Sigma(\gamma_k)\|_{W^{s,\rho}} < \infty$. Because of the embedding $W^{s,\rho} \hookrightarrow C^0$ (see part (ii) of Proposition A.3 for $k_1 := 0$, $\rho_1 := \rho$, $k_2 := 0$ and some $\mu \in (0, s - \frac{1}{\rho})$), we infer that $\sup_{k\in\mathbb{N}} \sup_{x\in\mathbb{R}/\mathbb{Z}} |\log|\gamma_k'(x)|| < \infty$. Consequently, there exists a constant c_1 such that

$$c_1^{-1} \le |\gamma_k'(x)| \le c_1 \quad \text{for all } x \in \mathbb{R}/\mathbb{Z}, \tag{3.4}$$

which together with the uniform total energy bound (3.3), in particular the resulting uniform energy bound $c_2 := \sup_{k\in\mathbb{N}} \in (\gamma_k) < \infty$, yields the uniform bound

$$C = C(\mathcal{E}, c_1, c_2)$$

on the bi-Lipschitz constant by means of Assumption (E3): $\inf_{k\in\mathbb{N}} \mathrm{BiLip}(\gamma_k) \ge C > 0$. By means of (3.4) and the convergence in (3.3) we obtain that γ is regular, and for distinct parameters $x, y \in \mathbb{R}/\mathbb{Z}$

$$|\gamma(x) - \gamma(y)| \overset{(3.3)}{=} \lim_{k\to\infty} |\gamma_k(x) - \gamma_k(y)| \ge \inf_{k\in\mathbb{N}} \mathrm{BiLip}(\gamma_k)|x - y|_{\mathbb{R}/\mathbb{Z}} > 0,$$

so that γ is injective, and therefore $\gamma \in W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

By Assumption (E2) the energy $\mathcal{E}$ is sequentially weakly lower semi-continuous on the set $W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. It remains to prove the same for the regularising logarithmic strain term, which also appears in the total energy ϕ; cf. (1.7). Indeed, writing it out for γ_k,

$$\|\Sigma(\gamma_k)\|^\kappa_{W^{s,\rho}} = \left(\int_{\mathbb{R}/\mathbb{Z}} |\Sigma(\gamma_k)(x)|^\rho \, \mathrm{d}x + \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\Sigma(\gamma_k)(x) - \Sigma(\gamma_k)(y)|^\rho}{|x - y|^{1+s\rho}} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{\kappa}{\rho}},$$

we can use the C^1-convergence in (3.3) to find pointwise convergence of $\Sigma(\gamma_k)$ to $\Sigma(\gamma)$ as $k \to \infty$, in order to apply Fatou's Lemma to obtain

$$\liminf_{k\to\infty} \|\Sigma(\gamma_k)\|^\kappa_{W^{s,\rho}} \ge \|\Sigma(\gamma)\|^\kappa_{W^{s,\rho}}.$$

Since $\mathcal{C}$ embeds continuously into $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, Assumption ($\Phi$1) is fulfilled.

Step 3: Assumption (Φ4) is satisfied. By definition of the total energy ϕ in (1.7) every sublevel set of ϕ is contained in $\mathcal{C}_{\mathrm{ir}}$, which according to Lemma A.2 in the appendix is an open subset of the full space $\mathcal{C}$. By Theorem 3.1 and Assumption (E1) the logarithmic strain Σ and the energy $\mathcal{E}$ are continuously Fréchet differentiable on $W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Furthermore, the choice $\kappa > 1$ and Lemma A.1 in the appendix for $k := 0$ imply that $\|\cdot\|^{\kappa}_{W^{s,\rho}}$ is also continuously differentiable. Hence, the total energy ϕ is of class $C^1(W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$. Since the canonical embedding

$$\iota\colon \mathcal{C} \to W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$$

is linear and bounded and since $\mathcal{C}_{\mathrm{ir}} \subset W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, we infer that ϕ is of class $C^1(\mathcal{C}_{\mathrm{ir}}, \mathbb{R})$. It follows from Proposition 2.4 that the local slope takes the form $|\partial\phi|(\gamma) = \|D\phi[\gamma]\|_{\mathcal{C}^*}$. To be precise, denoting by ι^* the adjoint of ι, we have $|\partial\phi|(\gamma) = \|D(\phi \circ \iota)[\gamma]\|_{\mathcal{C}^*} = \|\cdot\|_{\mathcal{C}^*} \circ \iota^* \circ D\phi \circ \iota[\gamma]$. The map $\|\cdot\|_{\mathcal{C}^*} \circ \iota^* \circ D\phi$ is continuous with respect to the norm-topology on $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Since $\mathcal{C}$ embeds compactly, i.e. since ι is a compact operator, effectively converting weakly convergent sequences in $\mathcal{C}$ into strongly convergent ones in $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, we obtain the weak sequential continuity of $|\partial\phi|(\cdot)$ on $\mathcal{C}_{\mathrm{ir}}$. In particular, it is also weakly sequentially lower semi-continuous, so that Assumption (Φ4) is satisfied.

Step 4: Assumption (Φ5) *holds.* Let $\boldsymbol{u} \in AC(I, \mathcal{C})$ such that $\phi(\boldsymbol{u}(t)) < \infty$ for all $t \in I$. By the previous step, the local slope $|\partial\phi|$ is weakly sequentially lower semi-continuous and therefore it is lower semicontinuous with respect to the strong topology. It follows that $|\partial\phi| \circ \boldsymbol{u}$ is lower semicontinuous and thus measurable. Since ϕ is continuously Fréchet-differentiable on $\mathcal{C}_{\mathrm{ir}}$, it is locally Lipschitz-continuous, whence $\phi \circ \boldsymbol{u}$ is locally absolutely continuous, and therefore differentiable a.e. on I. Furthermore, since the reflexive space $\mathcal{C}$ satisfies the Radon-Nikodym property; see [2, Corollary 1.2.7] $\boldsymbol{u}$ is also differentiable a.e. on I. Therefore, by the chain rule and Proposition 2.4

$$\begin{aligned} |\phi \circ \vec{u}(t) - \phi \circ \vec{u}(s)| &= \left| \int_s^t \frac{\mathrm{d}}{\mathrm{d}r} \phi \circ \vec{u}(r)\, \mathrm{d}r \right| = \left| \int_s^t \langle D\phi[\vec{u}(r)], \vec{u}'(r) \rangle_{\mathcal{C}^* \times \mathcal{C}}\, \mathrm{d}r \right| \\ &\leq \int_s^t \|D\phi[\vec{u}(r)]\|_{\mathcal{C}^*} \left\| \vec{u}'(r) \right\|_{\mathcal{C}} \mathrm{d}r = \int_s^t |\partial\phi|(\vec{u}(r)) |\vec{u}'|(r)\, \mathrm{d}r \end{aligned}$$

for all $s, t \in I$ with $s < t$. This is the defining inequality (2.1) for the strong upper gradient $g := |\partial\phi|$. Thus, Assumption (Φ5) is verified.

Step 5: Proof of (3.1) *and* (3.2). The previous steps have established the validity of Assumptions (M), and (Φ1)–(Φ5) so that Theorem 2.1 for the choice $\mathscr{S} := \mathcal{C}$ yields a curve of maximal slope for the total energy ϕ with respect to its local slope $|\partial\phi|$ starting at $\gamma_0 \in \mathscr{D}(\phi)$. Since $\mathcal{C}$ is reflexive it has the Radon-Nikodym property [2, Corollary 1.2.7], part (i) of Proposition 2.5 implies the validity of the differential inclusion (3.1). Under the stronger assumption that $\mathcal{C}$ is strictly convex and has a Gâteaux-differentiable norm on $\mathcal{C} \setminus \{0\}$, part (ii) of the same proposition yields the gradient flow equation (3.2). □

The parameter $\kappa > 1$ was introduced to render the last step of the previous proof possible: If $\kappa > 1$ then the total energy ϕ is continuously differentiable and the curve of maximal slope is actually a gradient flow. For the limiting case $\kappa = 1$, it is still possible to prove the existence of a curve of maximal slope, as shown in the following proposition.

Proposition 3.3. *Let $s \in (0,1)$, $\rho \in (\frac{1}{s}, \infty)$, $\theta \in (1,\infty)$, and $\kappa = 1$. Suppose that $\mathcal{C} \subset W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is a compactly embedded reflexive Banach space, that the energy $\mathcal{E}$ satisfies Assumptions (E1)–(E3), and $\gamma_0 \mathcal{E} \mathscr{D}(\phi)$, where the total energy ϕ is defined as in (1.7). Then there exists a θ-curve of maximal slope $\boldsymbol{u} \in AC^\theta([0,\infty),(\mathcal{C},\|\cdot\|_{\mathcal{C}}))$ for ϕ with respect to its strong upper gradient $|\partial\phi|$ starting at γ_0. Furthermore, the local slope at $\gamma \in \mathcal{C}$ takes the form*

$$|\partial\phi|(\gamma) = \sup_{\substack{v \in \mathcal{C} \\ \|v\|_{\mathcal{C}}=1}} \begin{cases} \left(\langle D\mathcal{E}[\gamma], v\rangle_{\mathcal{C}^*\times\mathcal{C}} - \|D\Sigma[\gamma]v\|_{\mathcal{A}}\right)^+, & \Sigma(\gamma)=0, \\ \langle D\mathcal{E}[\gamma], v\rangle_{\mathcal{C}^*\times\mathcal{C}} + \left\langle \frac{\mathfrak{J}_{\mathcal{A},2}(\Sigma(\gamma))}{\|\Sigma(\gamma)\|_{\mathcal{A}}}, D\Sigma[\gamma]v\right\rangle_{\mathcal{A}^*\times\mathcal{A}}, & \Sigma(\gamma)\neq 0, \end{cases} \tag{3.5}$$

where $\mathcal{A} := W^{s,\rho}(\mathbb{R}/\mathbb{Z})$.

Proof. The assumptions (M) and (Φ1)–(Φ3) follow exactly as in in the first two steps of Theorem 3.2.

We now show that the local slope takes the form (3.5). If $\Sigma(\gamma) \neq 0$, then the total energy $\phi = \mathcal{E} + \|\Sigma\|_{\mathcal{A}}$ is differentiable at γ and we can apply Proposition 2.4. Notice that $D\|\cdot\|_{\mathcal{A}}[\zeta] = \frac{\mathfrak{J}_{\mathcal{A},2}(\zeta)}{\|\zeta\|_{\mathcal{A}}}$ for every $\zeta \in \mathcal{A}\setminus\{0\}$. Let us now assume that $\Sigma(\gamma) = 0$. By differentiability, we have

$$\begin{aligned} \phi(\eta) &= \mathcal{E}(\eta) + \|\Sigma(\eta)\|_{\mathcal{A}} \\ &= \mathcal{E}(\gamma) + \langle D\mathcal{E}[\gamma], \eta-\gamma\rangle_{\mathcal{C}^*\times\mathcal{C}} + \|\Sigma(\gamma) + D\Sigma[\gamma](\eta-\gamma)\|_{\mathcal{A}} + o\big(\|\gamma-\eta\|_{\mathcal{C}}\big) \\ &= \phi(\gamma) + \langle D\mathcal{E}[\gamma], \eta-\gamma\rangle_{\mathcal{C}^*\times\mathcal{C}} + \|D\Sigma[\gamma](\eta-\gamma)\|_{\mathcal{A}} + o\big(\|\gamma-\eta\|_{\mathcal{C}}\big) \end{aligned}$$

as $\|\gamma-\eta\|_{\mathcal{C}} \to 0$, and therefore,

$$\begin{aligned} \frac{\big(\phi(\gamma)-\phi(\eta)\big)^+}{\|\gamma-\eta\|_{\mathcal{C}}} &= \left(-\langle D\mathcal{E}[\gamma], \frac{\eta-\gamma}{\|\eta-\gamma\|_{\mathcal{C}}}\rangle_{\mathcal{C}^*\times\mathcal{C}} - \left\| D\Sigma[\gamma]\left(\frac{\eta-\gamma}{\|\eta-\gamma\|_{\mathcal{C}}}\right)\right\|_{\mathcal{A}}\right)^+ + o(1) \\ &\le \sup_{v\in\mathcal{C}, \|v\|_{\mathcal{C}}=1} \big(\langle D\mathcal{E}[\gamma], -v\rangle_{\mathcal{C}^*\times\mathcal{C}} - \|D\Sigma[\gamma]v\|_{\mathcal{A}}\big)^+ + o(1) \end{aligned}$$

as $\|\gamma-\eta\|_{\mathcal{C}} \to 0$. Choosing a maximising sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ for the supremum and setting $\eta_n := \gamma + \frac{v_n}{n}$, we infer that

$$\begin{aligned} \limsup_{\eta\to\gamma} \frac{(\phi(\gamma)-\phi(\eta))^+}{\|\gamma-\eta\|_{\mathcal{C}}} &\ge \limsup_{n\to\infty} \big(\langle D\mathcal{E}[\gamma], -v_n\rangle_{\mathcal{C}^*\times\mathcal{C}} - \|D\Sigma[\gamma]v_n\|_{\mathcal{A}}\big)^+ \\ &= \sup_{v\in\mathcal{C}, \|v\|_{\mathcal{C}}=1} \big(\langle D\mathcal{E}[\gamma], -v\rangle_{\mathcal{C}^*\times\mathcal{C}} - \|D\Sigma[\gamma]v\|_{\mathcal{A}}\big)^+. \end{aligned}$$

Changing v to $-v$ completes the representation formula (3.5) for the local slope.

Next, we prove that the local slope is sequentially weakly lower semicontinuous and thus satisfies Assumption (Φ4). Let $(\gamma_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ that converges weakly to $\gamma \in \mathcal{C}$.

Then, by compact embedding, γ_n converges strongly to γ in $\mathcal{W} := W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. By Assumption (E1) and Theorem 3.1 we have

$$\lim_{n\to\infty} \left(\|\Sigma(\gamma_n)-\Sigma(\gamma)\|_{\mathcal{A}}+\|D\Sigma[\gamma_n]-D\Sigma[\gamma]\|_{\mathcal{L}(\mathcal{W},\mathcal{A})}+\|D\mathcal{E}[\gamma_n]-D\mathcal{E}[\gamma]\|_{\mathcal{W}^*}\right)=0. \tag{3.6}$$

We begin with the case where $\Sigma(\gamma) = 0$. Let $v \in \mathcal{C}$ with $\|v\|_{\mathcal{C}} = 1$. We have by (3.6)

$$\left(\langle D\mathcal{E}[\gamma], v\rangle_{\mathcal{C}^*\times\mathcal{C}} - \|D\Sigma[\gamma]v\|_{\mathcal{A}}\right)^+ = \lim_{n\to\infty} \left(\langle D\mathcal{E}[\gamma_n], v\rangle_{\mathcal{C}^*\times\mathcal{C}} - \|D\Sigma[\gamma_n]v\|_{\mathcal{A}}\right)^+. \tag{3.7}$$

If $\Sigma(\gamma_n) = 0$, it follows from (3.5) that

$$\left(\langle D\mathcal{E}[\gamma_n], v\rangle_{\mathcal{C}^*\times\mathcal{C}} - \|D\Sigma[\gamma_n]v\|_{\mathcal{A}}\right)^+ \le |\partial\phi|(\gamma_n). \tag{3.8}$$

If $\Sigma(\gamma_n) \neq 0$, we use the identity $\left\|\mathfrak{J}_{\mathcal{A},2}(\zeta)\right\|_{\mathcal{A}^*} = \|\zeta\|_{\mathcal{A}}$ for all $\zeta \in \mathcal{A}$ to compute

$$\begin{aligned}
&\left(\langle D\mathcal{E}[\gamma_n], v\rangle_{\mathcal{C}^*\times\mathcal{C}} - \|D\Sigma[\gamma_n]v\|_{\mathcal{A}}\right)^+ \\
&= \left(\langle D\mathcal{E}[\gamma_n], v\rangle_{\mathcal{C}^*\times\mathcal{C}} - \frac{\left\|\mathfrak{J}_{\mathcal{A},2}(\Sigma(\gamma_n))\right\|_{\mathcal{A}^*}}{\|\Sigma(\gamma_n)\|_{\mathcal{A}}} \|D\Sigma[\gamma_n]v\|_{\mathcal{A}}\right)^+ \\
&\le \left(\langle D\mathcal{E}[\gamma_n], v\rangle_{\mathcal{C}^*\times\mathcal{C}} + \langle\frac{\mathfrak{J}_{\mathcal{A},2}(\Sigma(\gamma_n))}{\|\Sigma(\gamma_n)\|_{\mathcal{A}}}, D\Sigma[\gamma_n]v\rangle_{\mathcal{A}^*\times\mathcal{A}}\right)^+ \overset{(3.5)}{\le} |\partial\phi|(\gamma_n).
\end{aligned} \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we obtain the inequality

$$(\langle D\mathcal{E}[\gamma], v\rangle_{\mathcal{C}^*\times\mathcal{C}} - \|D\Sigma[\gamma]v\|_{\mathcal{A}})^+ \le \liminf_{n\to\infty} |\partial\phi|(\gamma_n).$$

Taking the supremum over all $v \in \mathcal{C}$ with $\|v\|_{\mathcal{C}} = 1$ on the left-hand side, we infer the lower semicontinuity. The case where $\Sigma(\gamma) \neq 0$ follows directly from the continuity properties (3.6), since $\Sigma(\gamma_n) \neq 0$ for all sufficiently large $n \in \mathbb{N}$. This proves that Assumption (Φ4) is satisfied.

Finally, we show that the local slope is a strong upper gradient, i.e. we verify Assumption (Φ5). Since $\mathcal{E}$ and Σ are continuously Fréchet-differentiable, they are locally Lipschitz-continuous and so is ϕ. In particular, $\phi\circ\boldsymbol{u}$ is absolutely continuous for every $\boldsymbol{u} \in AC([a, b], \mathcal{C})$. It follows from [1, Theorem 1.2.5], [1, Definition 1.2.2], and [1, Definition 1.2.1] that $|\partial\phi|$ is a strong upper gradient for ϕ. Therefore, Assumption (Φ5) is satisfied. The existence of a curve of maximal slope now follows as in step 5 of Theorem 3.2. □

4. Gradient flows for various knot energies

To verify Assumptions (E1)–(E3) of Theorem 3.2 for the three energy families $E^{\alpha,p}$, $\text{intM}^{(p,q)}$, and $\text{TP}^{(p,q)}$, the following two general results turn out to be useful. The first one yields sequential lower semi-continuity for a multiple integral functional, whereas the second guarantees uniform control over the bi-Lipschitz constants. Recall that we use the indices i, r, and a on a function space to denote its subsets consisting of injective, regular, and arc length-parametrised curves, respectively.

Lemma 4.1. *Let $N \in \mathbb{N}$, $\rho \in (1, \infty)$, $s \in (\frac{1}{\rho}, 1)$, and*

$$\mathcal{E} : W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to [0, \infty],\ \gamma \mapsto \int \cdots \int_{(\mathbb{R}/\mathbb{Z})^N} e(\gamma; x_1, \ldots, x_N)\,\mathrm{d}x_N \ldots \mathrm{d}x_1.$$

Suppose the integrand $e : W^{1+s,\rho}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \times (\mathbb{R}/\mathbb{Z})^N \to [0, \infty]$ satisfies

(e1) *$e(\gamma; x_1, \ldots, x_N) < \infty$ and*

(e2) *$C^1_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to [0, \infty)$, $\gamma \mapsto e(\gamma; x_1, \ldots, x_N)$ is continuous*

for almost all $x_1, \ldots, x_N \in \mathbb{R}/\mathbb{Z}$. Then, the energy $\mathcal{E}$ satisfies Assumption (E2).

Proof. Up to taking subsequences we may assume that $\gamma_k \rightharpoonup \gamma$ in $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and $\gamma_k \to \gamma$ in $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ as $k \to \infty$ (by the embedding result, Proposition A.3), and that $\lim_{k\to\infty} \mathcal{E}(\gamma_k) = \liminf_{k\to\infty} \mathcal{E}(\gamma_k) < \infty$. The strong convergence in C^1 implies by virtue of assumption (e2) the pointwise convergence $e(\gamma_k; x_1, \ldots, x_N) \to e(\gamma; x_1, \ldots, x_N)$ as $k \to \infty$ for a.e. $x_1, \ldots, x_N \in \mathbb{R}/\mathbb{Z}$. Now apply Fatou's lemma to conclude. □

Lemma 4.2. *Let $\mathcal{E} : W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to [0, \infty]$ be invariant under reparametrisation and positively d-homogeneous for some $d \in \mathbb{R}$, and suppose that the following holds: For every $M < \infty$, there is a constant $C = C(M, d, s, \rho) > 0$ such that for all $\gamma \in C^1_{\mathrm{ia}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $\mathcal{E}(\gamma) \le M$ one has* BiLip $\gamma \ge C$. *Then, $\mathcal{E}$ satisfies Assumption (E3).*

Proof. Let $c_1, c_2 > 0$ and $\gamma \in C^1_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ such that

$$c_1^{-1} \le |\gamma'| \le c_1 \quad \text{on } \mathbb{R}/\mathbb{Z}, \quad \text{and} \quad \mathcal{E}(\gamma) \le c_2. \tag{4.1}$$

Then consider the arc length parametrisation $\Gamma \in W^{1,\infty}_{\mathrm{ia}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ of the rescaled curve $\tilde{\gamma} := \frac{\gamma}{\mathcal{L}(\gamma)}$, where $\mathcal{L}(\gamma) > 0$ denotes the length of γ. Then

$$\mathcal{E}(\Gamma) = \mathcal{E}(\tilde{\gamma}) = \mathcal{L}(\gamma)^{-d}\mathcal{E}(\gamma).$$

Using (4.1), we may estimate the first factor by $\tilde{c}_1 := \max\left\{c_1^d, c_1^{-d}\right\}$, allowing us to set $M := \tilde{c}_1 c_2$ in order to use our assumption to obtain a constant

$$b = b(c_1, c_2, d, s, \rho) > 0$$

independent of the specific curve γ such that BiLip $\Gamma \ge b$. Now, [27, Lemma B.1] in combination with (4.1) yields that BiLip $\tilde{\gamma} \ge bc_1^{-1}$, and consequently

$$\operatorname{BiLip} \gamma \ge bc_1^{-1}\mathcal{L}(\gamma) \ge bc_1^{-2} =: \tilde{C}(c_1, c_2, d, s, \rho).$$

Hence, Assumption (E3) is verified with the constant $C(\mathcal{E}, c_1, c_2) := \tilde{C}$. □

With these ingredients we can now apply Theorem 3.2 to prove our central long-time existence result, Theorem 1.1, for the total energy ϕ defined in (1.7) for the knot energies $\mathcal{E} \in \{E^{\alpha,p}, \mathrm{intM}^{(p,q)}, \mathrm{TP}^{(p,q)}\}$ in the respective ranges of parameters α, p and q.

Proof of Theorem 1.1. In all three scenarios (i)–(iii) the Banach space $\mathcal{B}$ is chosen as the energy space of the respective knot energy $\mathcal{E}$, as pointed out in the second of Remarks 1.3 of the introduction. These Sobolev-Slobodeckiĭ spaces are all of the type $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for some $s \in (0, 1)$ and $\rho \in (\frac{1}{s}, \infty)$ as considered in Theorem 3.2. Moreover, the slightly smaller Sobolev-Slobodeckiĭ space $\mathcal{C}$ is in each of the cases (i)–(iii) a reflexive Banach space compactly embedded in $\mathcal{B}$ according to Lemma A.1 and Proposition A.3. In addition, Lemma A.1 for $k := 1$ guarantees that the respective norm $\|\cdot\|_{\mathcal{C}}$ on the smaller Banach space $\mathcal{C} \subset \mathcal{B}$ is continuously Fréchet differentiable away from 0, which according to Theorem 3.2 yields that the differential inclusion (3.1) (if it holds at all) reduces to the gradient flow equation (3.2), which corresponds to (1.9) for each choice (i), (ii), and (iii). The C^1-regularity in time follows from the continuity in time of the right-hand side of (1.9): The curve $\boldsymbol{u}$ is absolutely continuous in time with values in $\mathcal{C}_{\mathrm{ir}}$, and the differential $D\phi$ is continuous on $\mathcal{C}_{\mathrm{ir}}$, once Assumption (E1) is verified. In addition, Lemma A.1 implies that the duality mapping $\mathfrak{J}_{\mathcal{C},\theta}$ is a homeomorphism from $\mathcal{C}$ onto the dual space $\mathcal{C}^*$. To summarise, the right-hand side of (1.9) is the composition of continuous mappings and therefore continuous in time.

So, it remains to establish (3.1), and for that it suffices to check that the respective knot energy $\mathcal{E}$ satisfies Assumptions (E1)–(E3) in each case (i)–(iii).

(i) O'Hara's knot energies $E^{\alpha,p}$ are non-negative by definition, and they are continuously differentiable on $W_{\mathrm{ir}}^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ in the range of parameters α, p given in Theorem 1.1. This is shown in Theorem 5.1 in Sect. 5, thus verifying Assumption (E1) for $s := \frac{\alpha p-1}{2p}$ and $\rho := 2p$.

The integrand of $E^{\alpha,p}$ in (1.1) obviously satisfies the assumptions (e1) and (e2) of Lemma 4.1 for $N = 2$, $s := \frac{\alpha p-1}{2p}$ and $\rho := 2p$ (which implies $s > \frac{1}{\rho}$ since $\alpha p > 2$). Therefore, Assumption (E2) is verified.

To verify Assumption (E3) for $E^{\alpha,p}$ assume first that $\gamma \in C^1_{\mathrm{ia}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $E^{\alpha,p}(\gamma) \le M$, which by means of [5, Theorem 1.1][9] implies that γ is of class $W^{1+\frac{\alpha p-1}{2p},2p}$. Then, we may approximate γ with respect to the $W^{1+\frac{\alpha p-1}{2p},2p}$-norm by curves $\gamma_k \in C^\infty_{\mathrm{ia}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ (see [28, Theorem 4.2]), and use the continuity of $E^{\alpha,p}$ implied by the upcoming Theorem 5.1 to assume without loss of generality that $E^{\alpha,p}(\gamma_k) \le E^{\alpha,p}(\gamma) + 1 \le M + 1$ for all $k \in \mathbb{N}$. Now, to these smooth arc length-parametrised approximants we can apply [32, Theorem 2.3] to obtain a uniform bi-Lipschitz constant, at least for the restricted parameter range $\alpha \in (\frac{2}{p}, 2]$. But even if $\alpha > 2$, we may use O'Hara's work. For curves $\eta \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ parametrised by arc length, one has $|\Delta\eta| \le d_\eta \le \frac{1}{2}$. Consequently, both, the integrand of $E^{\alpha,p}$ and the

[9]Note again the unfortunate misprint in that reference.

energy $E^{\alpha,p}$ itself are non-decreasing in α.[10] This means that if $E^{\alpha,p}(\eta) \leq M+1$ for $\alpha \geq 2$, so is $E^{2,p}$ and we may use the bi-Lipschitz constant obtained for that energy. O'Hara's bi-Lipschitz estimate yields a constant $K = K(\alpha, p, M+1)$ independent of γ such that

$$K|x-y|_{\mathbb{R}/\mathbb{Z}} \leq |\gamma_k(x) - \gamma_k(y)| \xrightarrow[k\to\infty]{} |\gamma(x)-\gamma(y)|.$$

This, together with the energy's invariance under reparametrisation and its positive $(2-\alpha p)$-homogeneity, implies that $E^{\alpha,p}$ satisfies the suppositions of Lemma 4.2 and thus also Assumption (E3).

(ii) Assumption (E1) is satisfied by means of [13, Theorem 3]. The integrand of $\mathrm{intM}^{(p,q)}$,

$$e(\gamma; x_1, x_2, x_3) := \frac{|(\gamma(x_1)-\gamma(x_2)) \wedge (\gamma(x_3)-\gamma(x_2))|^q}{|\gamma(x_1)-\gamma(x_2)|^p|\gamma(x_2)-\gamma(x_3)|^p|\gamma(x_1)-\gamma(x_3)|^p}|\gamma'(x_1)||\gamma'(x_2)||\gamma'(x_3)|$$

is non-negative and continuous in γ with respect to the C^1-topology for all pairwise distinct $x_1, x_2, x_3 \in \mathbb{R}/\mathbb{Z}$. Thus, Assumption (E2) is verified for $\mathrm{intM}^{(p,q)}$ by means of Lemma 4.1 for $N := 3$, $\rho := q$ and $s = \frac{3p-2}{q} - 2$.

The energy $\mathrm{intM}^{(p,q)}$ is positively $(3+2q-3p)$-homogeneous and invariant under reparametrisations, which we can combine with [13, Proposition 2.1] and Lemma 4.2 to conclude that Assumption (E3) holds as well.

(iii) The C^1-regularity of the tangent-point energies $\mathrm{TP}^{(p,q)}$ was first stated in [12, Remark 3.1], for a proof see [44, Satz 7.4] or [39, Section 3.2], so, Assumption (E1) holds for $\mathrm{TP}^{(p,q)}$.

Applying Lemma 4.1 with $N := 2$, $\rho := q$, and $s := \frac{p-1}{q} - 1$ to the integrand of $\mathrm{TP}^{(p,q)}$, $e(\gamma; x_1, x_2) := |P^{\perp}_{\gamma'(x_1)}(\gamma(x_2)-\gamma(x_1))|^q|\gamma(x_2)-\gamma(x_1)|^{-p}|\gamma'(x_2)||\gamma'(x_1)|$ we infer Assumption (E2) for $\mathrm{TP}^{(p,q)}$.

Finally, $\mathrm{TP}^{(p,q)}$ is invariant under reparametrisations and positively $(q-p+2)$-homogeneous. Therefore, we can combine [12, Proposition 2.7] with Lemma 4.2 for $s := \frac{p-1}{q} - 1$ and $\rho := q$ to conclude that Assumption (E3) is satisfied for $\mathrm{TP}^{(p,q)}$ as well. □

Proof of Corollary 1.2. For simplicity we set $\mathcal{C} := \mathcal{C}_\varepsilon$. Note that for our choice of $\mathcal{C}$, (1.9) is in fact well-defined, as by Lemma A.1 the duality mapping $\mathfrak{J}_{\mathcal{C},\theta}$ is a homeomorphism between $\mathcal{C}$ and $\mathcal{C}^*$ (and in particular single-valued). Combining the continuity of $\mathfrak{J}^{-1}_{\mathcal{C},\theta}$ with the continuity of $\boldsymbol{u}$ and $D\phi$ (see the proof of Theorem 1.1 for the latter) we obtain that $\boldsymbol{u}'$ is almost everywhere equal to the continuous function

$$\boldsymbol{v} : [0,\infty) \to \mathcal{C},\, t \mapsto -\mathfrak{J}^{-1}_{\mathcal{C},\theta}(D\phi[\boldsymbol{u}(t)]).$$

[10] To see this, observe that $\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(t^{-\alpha} - s^{-\alpha}\right) = -\alpha\left(t^{-\alpha-1} - s^{-\alpha-1}\right) \geq 0$ for $t \geq s > 0$.

It remains to establish that $\boldsymbol{u}'(t)$ exists for all t and is equal to $\boldsymbol{v}(t)$. Since $\boldsymbol{u}$ is absolutely continuous on compact intervals and $\mathcal{C}$ is reflexive, we have the fundamental theorem of calculus [2, Corollary 1.2.7, Definition 1.2.5, Proposition 1.2.3], so that

$$\frac{1}{h}(\boldsymbol{u}(t+h)-\boldsymbol{u}(t)) = \frac{1}{h}\int_t^{t+h} \boldsymbol{u}'(r)\,\mathrm{d}r = \frac{1}{h}\int_t^{t+h} \boldsymbol{v}(r)\,\mathrm{d}r.$$

As $\boldsymbol{v}$ is continuous, we can immediately infer that the right-hand side converges to $\boldsymbol{v}(t)$ as $h \to 0$ and hence $\boldsymbol{u}$ is of class $C^1([0,\infty),\mathcal{C})$.

We continue by proving that the energy is non-increasing along the flow. According to Lemma A.1 the Banach space $\mathcal{C}$ is reflexive in each of the three cases (i)–(iii) of Theorem 1.1. The θ-duality mapping $\mathfrak{J}_{\mathcal{C},\theta} : \mathcal{C} \to 2^{\mathcal{C}^*}$ is a duality map *with weight* $\varphi(s) := s^{\frac{\theta}{\beta}}$ in the language of [18, Definition I.4.1]. Therefore, we may apply [18, Corollary II.3.5] to identify the inverse $\mathfrak{J}_{\mathcal{C},\theta}^{-1}$ with the duality mapping on $\mathcal{C}^*$ with weight $\varphi^{-1}(s) = s^{\frac{\beta}{\theta}}$, which is just $\mathfrak{J}_{\mathcal{C}^*,\beta}$. Thus, by the chain rule and (1.9),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi(\boldsymbol{u}(t)) &= \langle D\phi[\boldsymbol{u}(t)], \boldsymbol{u}'(t)\rangle_{\mathcal{C}^*\times\mathcal{C}} \overset{(1.9)}{=} -\langle D\phi[\boldsymbol{u}(t)], \mathfrak{J}_{\mathcal{C},\theta}^{-1}(D\phi[\boldsymbol{u}(t)])\rangle_{\mathcal{C}^*\times\mathcal{C}}\\ &= -\langle D\phi[\boldsymbol{u}(t)], \mathfrak{J}_{\mathcal{C}^*,\beta}(D\phi[\boldsymbol{u}(t)])\rangle_{\mathcal{C}^*\times\mathcal{C}^{**}} = -\|D\phi[\boldsymbol{u}(t)]\|_{\mathcal{C}^*}^{\beta} \le 0\end{aligned}$$

for all $t \ge 0$. Hence $\phi\circ\boldsymbol{u}$ is non-increasing, so that [1, Proposition 1.4.1] implies that $\boldsymbol{u}$ is a θ-curve of maximal slope with respect to the weak upper gradient[11] $\|D\phi[\boldsymbol{u}(t)]\|_{\mathcal{C}^*}$. In fact, by Step 4 of Theorem 3.2 whose prerequisites were verified in the proof of Theorem 1.1, it is even a strong upper gradient. Finally, $\mathcal{C}$ continuously embeds into $C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ in all three cases of Theorem 1.1, and thus $\boldsymbol{u}$ is a C^1-isotopy. All curves $\boldsymbol{u}(t)$ are embedded because $\phi(\boldsymbol{u}(t)) < \infty$ for all $t \ge 0$ and so, by [4,35], $[\boldsymbol{u}(t)] = [\gamma_0]$ for all $t \ge 0$, i.e. the knot class is preserved along the flow. □

The following corollary shows that the gradient flows obtained in Theorem 1.1 for each $\varepsilon > 0$ admit a converging subsequence as $\varepsilon \to 0$.

Corollary 4.3. *Let $\theta,\kappa \in (1,\infty)$, ϕ and the spaces $\mathcal{B}$ and $\mathcal{C}_\varepsilon$ for $\mathcal{E} > 0$ be as in Theorem 1.1, and $\gamma_0 \in \mathcal{B}_{\mathrm{ir}}$. Then for any sequence $(\gamma_{0,\varepsilon})_\varepsilon \subset \mathcal{C}_\varepsilon$ with*

$$\gamma_{0,\varepsilon} \xrightarrow[(\varepsilon\to 0)]{} \gamma_0 \text{ in } \mathcal{B}, \quad \text{and} \quad \phi(\gamma_{0,\varepsilon}) \xrightarrow[(\varepsilon\to 0)]{} \phi(\gamma_0) \tag{4.2}$$

and for corresponding solutions $\boldsymbol{u}_\varepsilon \in C^1([0,\infty),(\mathcal{C}_\varepsilon,\|\cdot\|_{\mathcal{C}_\varepsilon}))$ of the gradient flow equation (1.9) with $\boldsymbol{u}_\varepsilon(0) = \gamma_{0,\varepsilon}$, there exists a subsequence $\varepsilon_k \to 0$ and a curve $\boldsymbol{u}^* \in AC^\theta([0,\infty),(\mathcal{B},\|\cdot\|_{\mathcal{B}}))$ such that $\boldsymbol{u}^*(0) = \gamma_0$, $\boldsymbol{u}_{\varepsilon_k}(t) \xrightarrow[(k\to\infty)]{} \boldsymbol{u}^*(t)$ in $\mathcal{B}$ and $\phi(\boldsymbol{u}^*(t)) \le \phi(\gamma_0)$ for all $t \ge 0$.

[11] As discussed in the beginning of Sect. 2.1, in our setting our notion of a curve of maximal slope is equivalent to the one presented in [1].

Remark 4.4. Since $\phi \in C^0\big(W_{\mathrm{ir}}^{1+s,\rho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)\big)$, we can choose arbitrary convolutions $\gamma_{0,\varepsilon} \in C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ with $\|\gamma_0 - \gamma_{0,\varepsilon}\|_{W^{1+s,\rho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} < \varepsilon$ to secure Assumption (4.2). Furthermore, since the weak convergence in $\mathcal{B} = W^{1+s,\rho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ implies strong convergence in $C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ and since $C^1_{\mathrm{ir}} \subset C^1$ is an open subset, see Lemma A.2, it follows from (4.2) and $\gamma_0 \in \mathcal{B}_{\mathrm{ir}} \subset C^1_{\mathrm{ir}}$ that for all sufficiently small $\varepsilon > 0$ the curves $\gamma_{0,\varepsilon}$ are injective and regular. Therefore, we may use Theorem 1.1 to secure the existence of solutions of (1.9) starting at $\gamma_{0,\varepsilon}$.

Proof of Corollary 4.3. The claim is a consequence of Proposition 2.2, whose prerequisites we now check. As was shown in step 1 of the proof of Theorem 3.2, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ with its weak topology satisfies Assumption (M). Moreover, for all curves $\gamma \in W^{1+s+\varepsilon}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ we have $\|\gamma\|_{W^{1+s,\rho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} \le \|\gamma\|_{W^{1+s+\varepsilon,\rho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)}$. In addition, as was shown in the proof of Theorem 1.1, ϕ satisfies the Assumptions (E1)–(E3) and thus, by steps 1 and 2 of the proof of Theorem 3.2, it also satisfies the Assumptions (Φ1)–(Φ3) with $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ as the underlying metric space. Furthermore, since we are only interested in small $\varepsilon > 0$, in view of (4.2) we may assume that

$$\sup_{\varepsilon>0} \phi(\gamma_{0,\varepsilon}) < \infty \quad \text{and} \quad \sup_{\varepsilon>0} \|\gamma_{0,\varepsilon}\|_{\mathcal{B}} < \infty.$$

Moreover, by Corollary 1.2, $\boldsymbol{u}_\varepsilon$ is a θ-curve of maximal slope for ϕ with respect to the strong upper gradient $|\partial\phi|_{\mathcal{C}_\varepsilon} = \|D\phi\|_{\mathcal{C}_\varepsilon^*}$ starting at $\gamma_{0,\varepsilon}$.

Identifying $(\mathscr{S}_0, d_0)$ with $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ let σ be the weak topology on $\mathcal{B}$. Furthermore, for $\varepsilon > 0$ let $(\mathscr{S}_\varepsilon, d_\varepsilon) := (\mathcal{C}_\varepsilon, \|\cdot\|_{\mathcal{C}_\varepsilon})$, $g_\varepsilon := |\partial\phi|_{\mathcal{C}_\varepsilon}$, and $u_{0,\varepsilon} := \gamma_{0,\varepsilon}$. Then these satisfy all the assumptions of Proposition 2.2 with $c_0 = 1$ which concludes the proof. Notice that since $\phi(\boldsymbol{u}^*(t)) \ge 0$ and $\phi(\gamma_0) < \infty$, it follows that $\boldsymbol{u}^*$ is actually absolutely continuous and not only locally absolutely continuous. □

5. O'Hara's knot energies are continuously differentiable

Let us rewrite $E^{\alpha,p}$ as

$$E^{\alpha,p}(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^2} e_\alpha^p(\gamma; x, y)|\gamma'(x)||\gamma'(y)|\,\mathrm{d}x\,\mathrm{d}y, \quad e_\alpha(\gamma; x, y) = \frac{1}{|\Delta\gamma|^\alpha} - \frac{1}{d_\gamma^\alpha}, \tag{5.1}$$

where $\Delta\gamma := \Delta\gamma(x, y) := \gamma(x) - \gamma(y)$ and $d_\gamma := d_\gamma(x, y)$ is the intrinsic distance between $\gamma(x)$ and $\gamma(y)$, i.e. the length of the shortest arc of γ connecting $\gamma(x)$ and $\gamma(y)$. Note that O'Hara defined the energy for curves parametrised by arc length only and had $|x - y|_{\mathbb{R}/\mathbb{Z}}^{-\alpha}$ as the second term. Using $d_\gamma^{-\alpha}$ instead is a sensible generalisation as the energy is then invariant under reparametrisations, cf. [33, Remark 4.1.1(6)].

Our method of proof for continuous differentiability of $E^{\alpha,p}$ is inspired by [34, Section 3]. There are several arguments which carry over completely, especially in the proofs of Claims 4 and 5; we include these in our proof for the reader's convenience.

Several technical results needed in the proof of Theorem 5.1 below, namely Lemmata 5.2–5.6 are deferred to the end of this section.

Theorem 5.1. *Let $p \geq 1$, $\alpha > 0$, $2 < \alpha p < 2p+1$. Then, $E^{\alpha,p}$ is continuously Fréchet-differentiable on $W_{\text{ir}}^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$.*

In the following, we will use the *derivative with respect to arc length* $D_\gamma \eta(u) := \frac{\eta'(u)}{|\gamma'(u)|}$ and the interval $I_\gamma(x,y)$ parametrising the arc of γ where $d_\gamma(x,y)$ is attained. To be more precise, $I_\gamma(x,y)$ is the interval containing x and one of $y-1, y, y+1$ such that $d_\gamma(x,y) = \int_{I_\gamma(x,y)} |\gamma'(t)|\,\mathrm{d}t$. This is well-defined whenever $d_\gamma(x,y) < \frac{L}{2}$, i.e. for almost all $x, y \in \mathbb{R}$. When there is no risk of confusion, we omit the arguments x and y. Lastly, we need the *minimal velocity* $v_\gamma := \operatorname{essinf}_{x\in\mathbb{R}/\mathbb{Z}} |\gamma'(x)|$.

Proof of Theorem 5.1. For $p=1$, this was already proved in [11, Proposition 2.1], so we may assume $p>1$. For notational convenience we denote $W^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ by $\mathcal{B}$ in the following proof.

We use the chain rule (see, for example, [45, Proposition 4.10] for the general Banach space version) to prove our statement. As outer function, we choose a geometric L^p-norm additionally depending on γ:

$$\|\cdot\|^p_{L^p_\cdot((\mathbb{R}/\mathbb{Z})^2)} : \mathcal{B}_{\text{ir}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n) \times L^p((\mathbb{R}/\mathbb{Z})^2) \to \mathbb{R},$$
$$(\gamma, g) \mapsto \iint_{(\mathbb{R}/\mathbb{Z})^2} |g(x,y)|^p |\gamma'(x)||\gamma'(y)|\,\mathrm{d}y\,\mathrm{d}x.$$

We consider the integrand e_α in (5.1) as a mapping from $\mathcal{B}_{\text{ir}}$ to $L^p((\mathbb{R}/\mathbb{Z})^2)$ for the inner function. Then, $E^{\alpha,p}(\gamma) = \|\cdot\|^p_{L^p_\gamma} \circ e_\alpha(\gamma)$ and it suffices to show that both functions are C^1. Lemma 5.2 together with the embedding result, part (ii) of Proposition A.3, take care of the outer function, so we only need to look at e_α. In order to do this, define

$$F_k(\gamma;\eta_1,\ldots,\eta_k) := \delta^k(\gamma \mapsto e_\alpha(\gamma)(x,y))(\eta_1,\ldots,\eta_k),$$
$$G_k(\gamma;\eta_1,\ldots,\eta_k) := \int_{I_\gamma} \delta^k(\gamma \mapsto |\gamma'(t)|)(\eta_1,\ldots,\eta_k)\,\mathrm{d}t.$$

Here, δ^k denotes the *k-th variation*. As for all $x \in \mathbb{R}/\mathbb{Z}$ there exists exactly one $y \neq x \in \mathbb{R}/\mathbb{Z}$ such that $I_\gamma(x,y)$ is not well-defined, we work on

$$\Sigma := \left\{(x,y) \in (\mathbb{R}/\mathbb{Z})^2 \mid x \neq y \text{ and } d_\gamma(x,y) < \tfrac{\mathcal{L}(\gamma)}{2}\right\}$$

which is open (since $\gamma \in \mathcal{B} \hookrightarrow C^1$) and is the same as $(\mathbb{R}/\mathbb{Z})^2$ up to a set of measure 0. For all $\gamma \in \mathcal{B}_{\text{ir}}$ and $(x,y) \in \Sigma$, there is an open neighbourhood $U(x,y) \subseteq \mathcal{B}_{\text{ir}}$ of γ such that $\eta \mapsto I_\eta(x,y)$ is constant on $U(x,y)$. This means that the G_k exist for all such (x,y) and a simple calculation shows that

$$G_1(\gamma;\eta_1) = \int_{I_\gamma} \langle D_\gamma\gamma, D_\gamma\eta_1\rangle |\gamma'|\,\mathrm{d}\tau,$$
$$G_2(\gamma;\eta_1,\eta_2) = \int_{I_\gamma} (\langle D_\gamma\eta_1, D_\gamma\eta_2\rangle - \langle D_\gamma\gamma, D_\gamma\eta_1\rangle\langle D_\gamma\gamma, D_\gamma\eta_2\rangle)|\gamma'|\,\mathrm{d}t.$$

To shorten notation, we left out the t-dependencies in these terms and will also do this in the following when there is no risk of confusion. Recall that $d_\gamma(x, y) = \int_{I_\gamma} |\gamma'(t)|\,\mathrm{d}t$. For fixed η and $|\tau| < 1$ sufficiently small, the estimate $v_{\gamma+\tau\eta} \geq \frac{1}{2} v_\gamma$ holds and so

$$|D_{\gamma+\tau\eta}\tilde{\eta}(t)| = \left|\frac{\tilde{\eta}'(t)}{|\gamma'(t) + \tau\eta'(t)|}\right| \leq |D_\gamma \tilde{\eta}| \left|\frac{\gamma'(t)}{|\gamma'(t) + \tau\eta'|}\right| \leq |D_\gamma \tilde{\eta}| \, \|\gamma'\|_{L^\infty} \frac{2}{v_\gamma}.$$

This, together with Lemma A.4, enables us to find integrable majorants for the τ-derivatives of $|\gamma_\tau'| := |\gamma'(t) + \tau\eta'(t)|$: They only consist of sums of inner products of $D_{\gamma_\tau}\gamma_\tau$, $D_{\gamma_\tau}\eta$ and $|\gamma_\tau'|$ (whose derivatives again fit the pattern). Consequently, $\delta^k d_\gamma = G_k(\gamma)$ and we may calculate for $(x, y) \in \Sigma$, that

$$F_1(\gamma; \eta_1) = \alpha \left(\frac{1}{d_\gamma^{\alpha+1}} G_1(\gamma; \eta_1) - \frac{1}{|\Delta\gamma|^{\alpha+2}} \langle \Delta\gamma, \Delta\eta_1 \rangle \right)$$

and

$$\begin{aligned} F_2(\gamma; \eta_1, \eta_2) = {} & \alpha(\alpha+2) \frac{1}{|\Delta\gamma|^{\alpha+4}} \langle \Delta\gamma, \Delta\eta_1 \rangle \langle \Delta\gamma, \Delta\eta_2 \rangle - \alpha \frac{\langle \Delta\eta_1, \Delta\eta_2 \rangle}{|\Delta\gamma|^{\alpha+2}} \\ & - \alpha(\alpha+1) \frac{1}{d_\gamma^{\alpha+2}} G_1(\gamma; \eta_1) G_1(\gamma; \eta_2) + \alpha \frac{1}{d_\gamma^{\alpha+1}} G_2(\gamma; \eta_1, \eta_2). \end{aligned}$$

Up until now, these are only pointwise limits and we still need to show that F_1 is indeed the Fréchet-derivative of e_α and also continuous with respect to γ. Let us first show that F_1 is a valid candidate for a derivative.

Claim 1. There is $C = C(\gamma) > 0$ which continuously depends on γ such that $\|F_1(\gamma; \eta)\|_{L^p} \leq C \|\eta\|_{\mathcal{B}}$ for all $\eta \in \mathcal{B}$.

In order to show this, decompose F_1 as

$$\frac{\alpha}{d_\gamma^{\alpha+2}} \left(d_\gamma G_1(\gamma; \eta_1) - \langle \Delta\gamma, \Delta\eta_1 \rangle \right) - \alpha \left(\frac{1}{|\Delta\gamma|^{\alpha+2}} - \frac{1}{d_\gamma^{\alpha+2}} \right) \langle \Delta\gamma, \Delta\eta_1 \rangle.$$

Lemma 5.5 gives us a fitting upper bound for the first term, Lemma 5.3 gives one for the second term when we choose $\varphi = 0$, $\psi = \alpha$, $\tilde{\eta}_1 := \gamma$ and $\tilde{\eta}_2 := \eta_1$ (note that $L_1 = L_2 = \frac{\Delta}{d_\gamma}$).

We will later need a similar bound for F_2.

Claim 2. There is $\Xi = \Xi(\gamma) > 0$ depending continuously on γ such that

$$\|F_2(\gamma; \eta_1, \eta_2)\|_{L^p} \leq \Xi \, \|\eta_1\|_{\mathcal{B}} \, \|\eta_2\|_{\mathcal{B}} \, .$$

The central ingredient is once again the right decomposition. Rewrite F_2 as

$$\alpha(\alpha+2)\langle\Delta\gamma,\Delta\eta_1\rangle\langle\Delta\gamma,\Delta\eta_2\rangle\left(\frac{1}{|\Delta\gamma|^{\alpha+4}}-\frac{1}{d_\gamma^{\alpha+4}}\right) \tag{5.2}$$

$$-\alpha\langle\Delta\eta_1,\Delta\eta_2\rangle\left(\frac{1}{|\Delta\gamma|^{\alpha+2}}-\frac{1}{d_\gamma^{\alpha+2}}\right) \tag{5.3}$$

$$+\frac{\alpha}{d_\gamma^{\alpha+2}}\big(G_1(\gamma;\eta_1)G_1(\gamma;\eta_2)+d_\gamma G_2(\gamma;\eta_1,\eta_2)-\langle\Delta\eta_1,\Delta\eta_2\rangle\big) \tag{5.4}$$

$$+\alpha(\alpha+2)\frac{1}{d_\gamma^{\alpha+4}}\big(\langle\Delta\gamma,\Delta\eta_1\rangle\langle\Delta\gamma,\Delta\eta_2\rangle-d_\gamma G_1(\gamma;\eta_1)d_\gamma G_1(\gamma;\eta_2)\big). \tag{5.5}$$

We can use Lemma 5.3 with $\psi=\alpha+2$, $\varphi=0$, $\tilde{\eta}_1=\tilde{\eta}_3=\gamma$, $\tilde{\eta}_2=\eta_1$ and $\tilde{\eta}_4=\eta_2$ to deal with (5.2) and the same Lemma with $\psi=\alpha$, $\varphi=0$, $\tilde{\eta}_1=\eta_1$ and $\tilde{\eta}_2=\eta_2$ to find an upper bound for (5.3). In order to take care of (5.4), let us define $H:=\int_{I_\gamma}\langle D_\gamma\eta_1,D_\gamma\eta_2\rangle|\gamma'|\,\mathrm{d}t$. Then,

$$\begin{aligned}(5.4)=\frac{\alpha}{d_\gamma^{\alpha+2}}\big(d_\gamma H-\langle\Delta\eta_1,\Delta\eta_2\rangle\big)+\frac{\alpha}{d_\gamma^{\alpha+2}}\big(&G_1(\gamma;\eta_1)G_1(\gamma;\eta_2)\\ &-d_\gamma\left(H-G_2(\gamma;\eta_1,\eta_2)\right)\big).\end{aligned}$$

The first term is again bounded above via Lemma 5.5, so let us take a look at the second one. Define $\varphi_i:=\langle D_\gamma\gamma,D_\gamma\eta_i\rangle$. Then, $H-G_2(\gamma;\eta_1,\eta_2)=\int_{I_\gamma}\varphi_1\varphi_2|\gamma'|\,\mathrm{d}t$ and thus the second term is equal to

$$\begin{aligned}&\frac{\alpha}{d_\gamma^{\alpha+2}}\left(\int_{I_\gamma}\varphi_1(s)|\gamma'(s)|\,\mathrm{d}s\int_{I_\gamma}\varphi_2(t)|\gamma'(t)|\,\mathrm{d}t-\iint_{I_\gamma^2}\varphi_1(t)\varphi_2(t)|\gamma'(s)||\gamma'(t)|\,\mathrm{d}t\,\mathrm{d}s\right)\\ &=\frac{\alpha}{2d_\gamma^{\alpha+2}}\left(\iint_{I_\gamma^2}(\varphi_1(s)-\varphi_1(t))\varphi_2(t)|\gamma'(s)||\gamma'(t)|\,\mathrm{d}t\,\mathrm{d}s\right.\\ &\quad\left.+\iint_{I_\gamma^2}(\varphi_1(t)-\varphi_1(s))\varphi_2(s)|\gamma'(s)||\gamma'(t)|\,\mathrm{d}t\,\mathrm{d}s\right)\\ &=-\frac{\alpha}{2d_\gamma^{\alpha+2}}\iint_{I_\gamma^2}\Delta\varphi_1(s,t)\Delta\varphi_2(s,t)|\gamma'(s)||\gamma'(t)|\,\mathrm{d}t\,\mathrm{d}s\end{aligned}$$

If we can estimate this by terms controlled via Lemma 5.6, we have an upper bound for the L^p-norm of (5.4). To achieve this, first apply the Hölder inequality (with

$p = q = 2$) and our usual upper bound for the line elements to reduce the problem to bounding

$$
\begin{aligned}
&\frac{1}{d_\gamma^{\alpha+2}} \iint_{I_\gamma^2} |\Delta\varphi_i(s,t)|^2 \,\mathrm{d}t\,\mathrm{d}s \\
&= \frac{1}{d_\gamma^{\alpha+2}} \iint_{I_\gamma^2} |\langle D_\gamma\gamma(t), D_\gamma\eta_i(t)\rangle - \langle D_\gamma\gamma(s), D_\gamma\eta_i(s)\rangle|^2 \,\mathrm{d}t\,\mathrm{d}s \\
&= \frac{1}{d_\gamma^{\alpha+2}} \iint_{I_\gamma^2} |\langle D_\gamma\gamma(t), D_\gamma\eta_i(t) - D_\gamma\eta_i(s)\rangle + \langle D_\gamma\gamma(t) - D_\gamma\gamma(s), D_\gamma\eta_i(s)\rangle|^2 \,\mathrm{d}t\,\mathrm{d}s \\
&\le \frac{2}{d_\gamma^{\alpha+2}} \iint_{I_\gamma^2} |\langle D_\gamma\gamma(t), D_\gamma\eta_i(t) - D_\gamma\eta_i(s)\rangle|^2 + |\langle D_\gamma\gamma(t) - D_\gamma\gamma(s), D_\gamma\eta_i(s)\rangle|^2 \,\mathrm{d}t\,\mathrm{d}s \\
&\le 2\left\|D_\gamma\gamma\right\|_{L^\infty}^2 \frac{1}{d_\gamma^{\alpha+2}} \iint_{I_\gamma^2} |\Delta D_\gamma\eta_i(s,t)|^2 \,\mathrm{d}t\,\mathrm{d}s + \\
&\quad 2\left\|D_\gamma\eta_i\right\|_{L^\infty}^2 \frac{1}{d_\gamma^{\alpha+2}} \iint_{I_\gamma^2} |\Delta D_\gamma\gamma(s,t)|^2 \,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

Each summand in the last line fits the pattern of Lemma 5.6, so we are finished with bounding (5.4).

In dealing with (5.5), again the important technique is creative rewriting. In this case, we sum up two terms which differ only by exchanging η_1 and η_2:

$$
\begin{aligned}
&\frac{\alpha(\alpha+2)}{2}\left(\langle\frac{\Delta\gamma}{d_\gamma}, \frac{\Delta\eta_1}{d_\gamma}\rangle + \frac{G_1(\gamma;\eta_1)}{d_\gamma}\right)\cdot\frac{1}{d_\gamma^{\alpha+2}}\left(\langle\Delta\gamma, \Delta\eta_2\rangle - d_\gamma G_1(\gamma;\eta_2)\right) \\
&\quad + \frac{\alpha(\alpha+2)}{2}\left(\langle\frac{\Delta\gamma}{d_\gamma}, \frac{\Delta\eta_2}{d_\gamma}\rangle + \frac{G_1(\gamma;\eta_2)}{d_\gamma}\right)\cdot\frac{1}{d_\gamma^{\alpha+2}}\left(\langle\Delta\gamma, \Delta\eta_1\rangle - d_\gamma G_1(\gamma;\eta_1)\right) \\
&= \alpha(\alpha+2)\frac{1}{d_\gamma^{\alpha+4}}\left(\langle\Delta\gamma, \Delta\eta_1\rangle\langle\Delta\gamma, \Delta\eta_2\rangle - d_\gamma^2 G_1(\gamma;\eta_1)G_2(\gamma;\eta_2)\right) = (5.5).
\end{aligned}
$$

The first factor of each summand on the left-hand side is bounded above by $2\frac{\|\eta_i'\|_{L^\infty}}{v_\gamma}$, because of the upcoming estimate (5.20) and the fact that

$$
|G_1(\gamma;\eta_i)| = \left|\int_{I_\gamma}\langle\gamma'(t), \frac{\eta_i'(t)}{|\gamma'(t)|}\rangle\,\mathrm{d}t\right| \le \frac{\left\|\eta_i'\right\|_{L^\infty}}{v_\gamma} d_\gamma.
$$

The second factor of each summand on the left-hand side is L^p-bounded because of Lemma 5.5 and thus, also (5.5) is L^p-bounded.

Now let us really prove that F_1 is the Fréchet-derivative.

Claim 3. *$F_1(\gamma,\eta)$ is the Fréchet-derivative of $e_\alpha^p(\gamma)$ in direction $\eta \in \mathcal{B}$.*

From now on, fix $\gamma \in \mathcal{B}_{\mathrm{ir}}$ and $\varepsilon > 0$ such that for all η with $\|\eta\|_{\mathcal{B}} < \varepsilon$ one has not only that $\gamma + \eta \in \mathcal{B}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, but also the following:

$$
\Xi(\gamma+\eta) < 2\Xi(\gamma) \tag{5.6}
$$

$$
\left\|\eta'\right\|_{L^\infty} < \min\left\{\frac{\mathrm{BiLip}(\gamma)}{2}, \frac{v_\gamma}{12}, \mathcal{L}(\gamma)\right\}. \tag{5.7}
$$

Furthermore, let $\gamma_t := \gamma + t\eta_1$ and decompose $\Sigma = U(\eta_1) \cup V(\eta_1)$ with

$$U(\eta_1) := \left\{(x, y) \in \Sigma | \text{for all } t \in [0, 1], I_{\gamma_t}(x, y) = I_\gamma(x, y)\right\} \text{ and}$$
$$V(\eta_1) := \left\{(x, y) \in \Sigma | \text{there exists } t \in [0, 1] \text{ such that } I_{\gamma_t}(x, y) \neq I_\gamma(x, y)\right\}.$$

On $U(\eta_1)$, $e_\alpha(\gamma_t)$ is differentiable with respect to t because d_{γ_t} is differentiable as long as I_{γ_t} is fixed. By Taylor's theorem, we have for $(x, y) \in U(\eta_1)$

$$|e_\alpha(\gamma + \eta_1) - e_\alpha(\gamma) - F_1(\gamma; \eta_1)| = \left| \int_0^1 (1 - t) F_2(\gamma_t; \eta_1, \eta_1)\, \mathrm{d}t \right|,$$

so taking the L^p-norm on $U(\eta_1)$ yields, with Jensen's inequality, Tonelli's variant of Fubini's theorem, Theorem 2 and (5.6):

$$\begin{aligned}
&\iint_{U(\eta_1)} |e_\alpha(\gamma + \eta) - e_\alpha(\gamma) - F_1(\gamma; \eta_1)|^p \,\mathrm{d}y\, \mathrm{d}y \\
&\leq \iint_{U(\eta_1)} \left(\int_0^1 (1 - t)|F_2(\gamma_t; \eta_1, \eta_1)|\, \mathrm{d}t \right)^p \mathrm{d}y\, \mathrm{d}x \\
&\leq \int_0^1 (1 - t)^p \iint_{U(\eta_1)} |F_2(\gamma_t; \eta_1, \eta_1)|^p \,\mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t \\
&\leq \int_0^1 (1 - t)^p\, \Xi(\gamma_t)^p\, \|\eta_1\|_{\mathcal{B}}^{2p}\, \mathrm{d}t \\
&\leq 2^p\, \Xi(\gamma)^p \int_0^1 (1 - t)^p\, \mathrm{d}t\, \|\eta_1\|_{\mathcal{B}}^{2p} \leq 2^p\, \Xi(\gamma)^p\, \|\eta_1\|_{\mathcal{B}}^{2p}.
\end{aligned} \tag{5.8}$$

Instead of trying to show that the same holds true on the "bad" set $V(\eta_1)$, we will show in Theorem 4 that there, $e_\alpha(\gamma)(x, y)$ is locally Lipschitz continuous with respect to γ and has a Lipschitz constant that is uniform in (x, y). If that is the case,

$$|F_1(\gamma; \eta_1)(x, y)| = \lim_{h \to 0} \left| \frac{e_\alpha(\gamma + h\eta_1)(x, y) - e_\alpha(\gamma)(x, y)}{h} \right| \leq L_{e_\alpha} \|\eta_1\|_{\mathcal{B}}$$

and consequently

$$\begin{aligned}
&\iint_{V(\eta_1)} |e_\alpha(\gamma + \eta_1) - e_\alpha(\gamma) - F_1(\gamma; \eta_1)|^p \,\mathrm{d}y\, \mathrm{d}x \\
&\leq C(p) \iint_{V(\eta_1)} |e_\alpha(\gamma + \eta_1) - e_\alpha(\gamma)|^p + |F_1(\gamma; \eta_1)|^p \,\mathrm{d}y\, \mathrm{d}x \\
&\leq C(p) \iint_{V(\eta_1)} 2L_{e_\alpha}^p \|\eta_1\|_{\mathcal{B}}^p \,\mathrm{d}y\, \mathrm{d}x.
\end{aligned}$$

If we can furthermore show that $|V(\eta_1)| \leq C(\gamma) \|\eta_1\|_{\mathcal{B}}$, see Theorem 5, we obtain

$$\iint_{V(\eta_1)} |e_\alpha(\gamma + \eta_1) - e_\alpha(\gamma) - F_1(\gamma; \eta_1)|^p \,\mathrm{d}y\, \mathrm{d}x \leq 2C(p)C(\gamma)L_{e_\alpha}^p \|\eta_1\|_{\mathcal{B}}^{p+1}$$

and can use this together with (5.8) to show that

$$\|e_\alpha(\gamma+\eta_1)-e_\alpha(\gamma)-F_1(\gamma;\eta_1)\|_{L^p}\leq \widetilde{C}(\gamma)\,\|\eta_1\|_{\mathcal{B}}^{1+\frac{1}{p}},$$

which is enough for Fréchet-differentiability.

Claim 4. There is a constant $L_{e_\alpha}=L_{e_\alpha}(\alpha,\gamma,n,p)$ such that

$$\|e_\alpha(\gamma+\eta)-e_\alpha(\gamma)\|_{L^\infty}\leq L_{e_\alpha}\,\|\eta\|_{\mathcal{B}}$$

for all $\|\eta\|_{\mathcal{B}}<\varepsilon$. Here,

$$\begin{aligned} &L_{e_\alpha}(\gamma)\\ &=C_E(\alpha,n,p)\left(\alpha\left(\frac{\mathcal{L}(\gamma)}{3\,\|\gamma'\|_{L^\infty}}\right)^{-\alpha}\left(\frac{BiLip\,(\gamma)}{2}\right)^{-\alpha-1}+\alpha\left(\frac{\mathcal{L}(\gamma)}{6}\right)^{-\alpha-1}\frac{2}{v_\gamma}\mathcal{L}(\gamma)\right).\end{aligned}\tag{5.9}$$

The key ingredient is the local Lipschitz continuity of the intrinsic distance d_γ as a mapping from $\mathcal{B}_{\text{ir}}$ to $L^\infty((\mathbb{R}/\mathbb{Z})^2)$. Since Σ has full measure, it suffices to look at $(x,y)\in\Sigma$ and differentiate between two cases: The first case is $I_\gamma=I_{\tilde\gamma}$ (setting $\tilde\gamma=\gamma+\eta$ with $\|\eta\|_{\mathcal{B}}<\varepsilon$). Then we may simply compute

$$|d_{\tilde\gamma}(x,y)-d_\gamma(x,y)|=\left|\int_{I_\gamma}|\tilde\gamma'(t)|-|\gamma'(t)|\,\mathrm{d}t\right|\leq|I_\gamma|\,\left\|\eta'\right\|_{L^\infty}\leq\left\|\eta'\right\|_{L^\infty}.$$

The other case is when the shortest connections do not match, i.e. $I_\gamma\neq I_{\tilde\gamma}$. In this case, we need more precise control of the integrands, so let us first find bounds for them. We know that $||\tilde\gamma'(t)|-|\gamma'(t)||\leq\left\|\eta'\right\|_{L^\infty}$ and so

$$|\tilde\gamma'(t)|\geq|\gamma'(t)|-\left\|\eta'\right\|_{L^\infty}\geq|\gamma'(t)|-\frac{|\gamma'(t)|}{v_\gamma}\left\|\eta'\right\|_{L^\infty}.$$

Performing the same estimates for an upper bound, we arrive at the fact that

$$|\gamma'(t)|\left(1-\frac{1}{v_\gamma}\left\|\eta'\right\|_{L^\infty}\right)\leq|\tilde\gamma'(t)|\leq|\gamma'(t)|\left(1+\frac{1}{v_\gamma}\left\|\eta'\right\|_{L^\infty}\right).$$

Then, we use the fact that the connection between $\tilde\gamma(x)$ and $\tilde\gamma(y)$ via I_γ is longer than the one via $I_{\tilde\gamma}$ to estimate

$$\begin{aligned}\int_{I_{\tilde\gamma}}|\gamma'(t)|\left(1-\frac{1}{v_\gamma}\left\|\eta'\right\|_{L^\infty}\right)\mathrm{d}t&\leq\int_{I_{\tilde\gamma}}|\tilde\gamma'(t)|\,\mathrm{d}t\\ &\leq\int_{I_\gamma}|\tilde\gamma'(t)|\,\mathrm{d}t\leq\int_{I_\gamma}|\gamma'(t)|\left(1+\frac{1}{v_\gamma}\left\|\eta'\right\|_{L^\infty}\right)\mathrm{d}t\end{aligned}\tag{5.10}$$

which implies

$$\int_{I_{\tilde\gamma}}|\gamma'(t)|\leq\int_{I_\gamma}|\gamma'(t)|\,\mathrm{d}t+\int_{I_\gamma\cup I_{\tilde\gamma}}\frac{1}{v_\gamma}\left\|\eta'\right\|_{L^\infty}|\gamma'(t)|\,\mathrm{d}t=\int_{I_\gamma}|\gamma'(t)|\,\mathrm{d}t+\frac{1}{v_\gamma}\left\|\eta'\right\|_{L^\infty}\mathcal{L}(\gamma).\tag{5.11}$$

Consequently, this time using the fact that $I_{\tilde{\gamma}}$ parametrises the longer connection between $\gamma(x)$ and $\gamma(y)$, we have

$$\left|\int_{I_{\tilde{\gamma}}} |\gamma'(t)| - \int_{I_\gamma} |\gamma'(t)|\,\mathrm{d}t\right| = \int_{I_{\tilde{\gamma}}} |\gamma'(t)|\,\mathrm{d}t - \int_{I_\gamma} |\gamma'(t)|\,\mathrm{d}t \overset{(5.11)}{\leq} \frac{1}{v_\gamma} \left\|\eta'\right\|_{L^\infty} \mathcal{L}(\gamma).$$

Furthermore, we can use the definition of $\tilde{\gamma}$ and the fact that $1 \leq \frac{|\gamma'(t)|}{v_\gamma}$ to obtain

$$\begin{aligned}\left|\int_{I_{\tilde{\gamma}}} |\tilde{\gamma}'(t)| - |\gamma'(t)|\,\mathrm{d}t\right| &\leq \int_{I_{\tilde{\gamma}}} ||\tilde{\gamma}'(t)| - |\gamma'(t)||\,\mathrm{d}t \\ &\leq \int_{I_{\tilde{\gamma}}} \left\|\eta'\right\|_{L^\infty} \frac{1}{v_\gamma} |\gamma'(t)|\,\mathrm{d}t \leq \frac{1}{v_\gamma} \left\|\eta'\right\|_{L^\infty} \mathcal{L}(\gamma).\end{aligned}$$

Combining the last two estimates yields

$$\begin{aligned}&|d_{\tilde{\gamma}}(x,y) - d_\gamma(x,y)| \\ &= \left|\int_{I_{\tilde{\gamma}}} |\tilde{\gamma}'(t)|\,\mathrm{d}t - \int_{I_\gamma} |\gamma'(t)|\,\mathrm{d}t\right| \\ &\leq \left|\int_{I_{\tilde{\gamma}}} |\tilde{\gamma}'(t)| - |\gamma'(t)|\,\mathrm{d}t\right| + \left|\int_{I_{\tilde{\gamma}}} |\gamma'(t)|\,\mathrm{d}t - \int_{I_\gamma} |\gamma'(t)|\,\mathrm{d}t\right| \\ &\leq \frac{2}{v_\gamma} \left\|\eta'\right\|_{L^\infty} \mathcal{L}(\gamma),\end{aligned} \tag{5.12}$$

so we have proven the local Lipschitz property of d_γ.

To apply this result to e_α, we first provide a simple local Lipschitz estimate for $x \mapsto x^{-\alpha}$. Assume $x, x+h > 0$, then

$$\begin{aligned}|x^{-\alpha} - (x+h)^{-\alpha}| &= \left|h \int_0^1 -\alpha(x+th)^{-\alpha-1}\,\mathrm{d}t\right| \\ &\leq |\alpha| \max\left\{|x|^{-\alpha-1}, |x+h|^{-\alpha-1}\right\} |h|.\end{aligned} \tag{5.13}$$

Rewriting e_α a bit, we obtain

$$\begin{aligned}&|e_\alpha(\tilde{\gamma})(x,y) - e_\alpha(\gamma)(x,y)| \\ &= \Big|\frac{1}{|\Delta\tilde{\gamma}|^\alpha} - \frac{1}{d_{\tilde{\gamma}}^\alpha} - \frac{1}{|\Delta\gamma|^\alpha} + \frac{1}{d_\gamma^\alpha}\Big| \\ &\leq |x-y|_{\mathbb{R}/\mathbb{Z}}^{-\alpha} \Big|\Big(\frac{|x-y|_{\mathbb{R}/\mathbb{Z}}}{|\Delta\tilde{\gamma}|}\Big)^\alpha - \Big(\frac{|x-y|_{\mathbb{R}/\mathbb{Z}}}{|\Delta\gamma|}\Big)^\alpha\Big| + \Big|\frac{1}{d_\gamma^\alpha} - \frac{1}{d_{\tilde{\gamma}}^\alpha}\Big|.\end{aligned} \tag{5.14}$$

In order to use (5.13), we need to make sure that the $\tilde{\gamma}$-terms do not veer too far from their γ-counterparts. For the Δ-terms, consider that for all $k \in \mathbb{Z}$, we have

$$\begin{aligned}&\left|\frac{\Delta\gamma}{|x-y|_{\mathbb{R}/\mathbb{Z}}}-\frac{\Delta\tilde{\gamma}}{|x-y|_{\mathbb{R}/\mathbb{Z}}}\right|\\&=\left|\int_0^1 \tilde{\gamma}'(x+t(y+k-x))-\gamma'(x+t(y+k-x))\,\mathrm{d}t\right|\frac{|y+k-x|}{|y-x|_{\mathbb{R}/\mathbb{Z}}}\\&\leq \left\|\eta'\right\|_{L^\infty}\frac{|y+k-x|}{|y-x|_{\mathbb{R}/\mathbb{Z}}}.\end{aligned}$$

Taking the minimum over all k, the fraction on the right-hand side becomes 1 and so we obtain $\left\|\eta'\right\|_{L^\infty}$ as an upper bound. By (5.7), this means that

$$\left|\frac{\Delta\tilde{\gamma}}{|x-y|_{\mathbb{R}/\mathbb{Z}}}\right| \geq \left|\frac{\Delta\gamma}{|x-y|_{\mathbb{R}/\mathbb{Z}}}\right| - \frac{1}{2}\operatorname{BiLip}(\gamma) \geq \frac{1}{2}\operatorname{BiLip}(\gamma),$$

so the first part of (5.14) is bounded above by

$$|x-y|_{\mathbb{R}/\mathbb{Z}}^{-\alpha}\alpha\left(\frac{\operatorname{BiLip}(\gamma)}{2}\right)^{-\alpha-1}\left\|\eta'\right\|_{L^\infty}. \tag{5.15}$$

In order to apply (5.13) to the d_γ-terms of (5.14), we need a lower bound for $d_{\tilde{\gamma}}(x, y)$. To achieve this, assume for the moment that $d_\gamma(x, y) \geq \frac{\mathcal{L}(\gamma)}{3}$, which we will prove in (5.17). Then, by (5.12) and (5.7),

$$\begin{aligned}d_{\tilde{\gamma}}(x,y) &= d_\gamma(x,y) - \big(d_\gamma(x,y)-d_{\tilde{\gamma}}(x,y)\big) \geq d_\gamma(x,y) - \frac{2}{v_\gamma}\left\|\eta'\right\|_{L^\infty}\mathcal{L}(\gamma)\\&\geq \frac{\mathcal{L}(\gamma)}{3} - \frac{2}{v_\gamma}\left\|\eta'\right\|_{L^\infty}\mathcal{L}(\gamma) \geq \frac{\mathcal{L}(\gamma)}{6}.\end{aligned}$$

Applying (5.13) and (5.12), we obtain that the second part of (5.14) is bounded above by

$$\alpha\left(\frac{\mathcal{L}(\gamma)}{6}\right)^{-\alpha-1}\frac{2}{v_\gamma}\left\|\eta'\right\|_{L^\infty}\mathcal{L}(\gamma). \tag{5.16}$$

The last thing we need for Lipschitz continuity of e_α on $V(\eta_1)$ is that x and y cannot get too close. The tuple (x, y) is in $V(\eta_1)$ if and only if $I_\gamma(x, y) \neq I_{\gamma_t}(x, y)$, so it suffices to establish that this cannot happen when $|x-y|_{\mathbb{R}/\mathbb{Z}}$ is small.

Assume $d_\gamma(x, y) \leq \frac{\mathcal{L}(\gamma)}{3}$. We will show that this is impossible thus establishing (5.17) below, since

$$|x-y|_{\mathbb{R}/\mathbb{Z}} = \left|\int_x^y \frac{|\gamma'(t)|}{|\gamma'(t)|}\,\mathrm{d}t\right| \geq \frac{1}{\|\gamma'\|_{L^\infty}}\left|\int_x^y |\gamma'(t)|\,\mathrm{d}t\right| \geq \frac{1}{\|\gamma'\|_{L^\infty}}d_\gamma(x,y).$$

Under our assumption and by means of (5.10) (recall that $\tilde{\gamma} = \gamma + \eta$) and (5.7),

$$\int_{I_\gamma}|\gamma'(t)+\eta'(t)|\,\mathrm{d}t \leq \int_{I_\gamma}|\gamma'(t)|\,\mathrm{d}t\left(1+\frac{\left\|\eta'\right\|_{L^\infty}}{v_\gamma}\right) \leq \frac{\mathcal{L}(\gamma)}{3}\left(1+\frac{\left\|\eta'\right\|_{L^\infty}}{v_\gamma}\right) < \frac{\mathcal{L}(\gamma)}{2}.$$

Thus, $I_\gamma = I_{\tilde\gamma}$ for all $\tilde\gamma \in B_\varepsilon(\gamma)$, so $(x, y) \notin V(\eta_1)$ a contradiction. Thus,

$$d_\gamma(x, y) > \frac{\mathcal{L}(\gamma)}{3} \text{ and } |x - y|_{\mathbb{R}/\mathbb{Z}} > \frac{\mathcal{L}(\gamma)}{3\,\|\gamma'\|_{L^\infty}} \text{ for all } (x, y) \in V(\eta_1). \quad (5.17)$$

Combining the upper bounds (5.15) and (5.16) for the right-hand side of (5.14) with (5.17), we obtain

$$\begin{aligned}
&|x - y|_{\mathbb{R}/\mathbb{Z}}^{-\alpha}\alpha\left(\frac{\mathrm{BiLip}(\gamma)}{2}\right)^{-\alpha-1} + \alpha\left(\frac{\mathcal{L}(\gamma)}{6}\right)^{-\alpha-1}\frac{2}{v_\gamma}\mathcal{L}(\gamma)\\
&\overset{(5.17)}{<} \alpha\left(\frac{\mathcal{L}(\gamma)}{3\,\|\gamma'\|_{L^\infty}}\right)^{-\alpha}\left(\frac{\mathrm{BiLip}(\gamma)}{2}\right)^{-\alpha-1} + \alpha\left(\frac{\mathcal{L}(\gamma)}{6}\right)^{-\alpha-1}\frac{2}{v_\gamma}\mathcal{L}(\gamma) =: \widetilde{L}_{e_\alpha}(\gamma)
\end{aligned}$$

as Lipschitz constant for $\gamma \mapsto e_\alpha(\gamma)$ as a mapping from $W^{1,\infty}$ to L^∞. Finally, using the embedding from Proposition A.3 (ii) to estimate $\|\eta'\|_{L^\infty}$ by $C_E(\alpha, n, p)\,\|\eta\|_{\mathcal{B}}$ (with $k_1 = k_2 = 1$, $s_1 = \frac{\alpha p-1}{2p} - 1$, $\rho_1 = 2p$ and some $\mu \in (0, s_1 - \frac{1}{\rho_1})$) yields $L_{e_\alpha}(\alpha, \gamma, n, p) = C_E(\alpha, n, p)\widetilde{L}_{e_\alpha}(\gamma)$.

To wrap up the proof of Theorem 3, we need that $V(\eta_1)$ is small.

Claim 5. $|V(\eta_1)| \leq \frac{6}{\mathrm{BiLip}(\gamma)}\,\|\eta_1'\|_{L^\infty}$.

For $(x, y) \in V(\eta_1)$, there is $t \in [0, 1]$ such that the intrinsic distance is parametrised over $I_{\tilde\gamma}$ instead of I_γ, so let us take a closer look at the corresponding integral. The fundamental theorem of calculus yields

$$\int_{I_{\tilde\gamma}} |\gamma_t'(s)|\,\mathrm{d}s = \int_{I_{\tilde\gamma}} |\gamma'(s)| + t\int_0^1 \langle D_{\gamma_{\tau t}}\gamma_{\tau t}(s), \eta_1'(s)\rangle\,\mathrm{d}\tau\,\mathrm{d}s$$

and so, for the fitting t,

$$\begin{aligned}
d_{\gamma_t}(x, y) &= \int_{I_{\tilde\gamma}} |\gamma_t'(s)|\,\mathrm{d}s \geq \int_{I_{\tilde\gamma}} |\gamma'(s)| - t\left|\int_0^1 \langle D_{\gamma_{\tau t}}\gamma_{\tau t}(s), \eta_1'(s)\rangle\,\mathrm{d}\tau\right|\,\mathrm{d}s\\
&\geq \int_{I_{\tilde\gamma}} |\gamma'(s)|\,\mathrm{d}s - \|\eta_1'\|_{L^\infty} \geq \frac{\mathcal{L}(\gamma)}{2} - \|\eta_1'\|_{L^\infty}.
\end{aligned}$$

Note that for each $(x, y) \in V(\eta_1)$, there is exactly one $\tilde{x} = \tilde{x}(x, t)$ such that $d_{\gamma_t}(x, \tilde{x}) = \frac{1}{2}\mathcal{L}(\gamma_t)$ and because it does not matter which arc of γ_t we travel through to get from $\gamma_t(x)$ to $\gamma_t(\tilde{x})$, $d_{\gamma_t}(x, \tilde{x}) = d_{\gamma_t}(x, y) + d_{\gamma_t}(\tilde{x}, y)$. Thus, $d_{\gamma_t}(x, y) = \frac{1}{2}\mathcal{L}(\gamma_t) - d_{\gamma_t}(\tilde{x}, y)$. Consequently,

$$\|\eta_1'\|_{L^\infty} \geq \frac{1}{2}\mathcal{L}(\gamma) - d_{\gamma_t}(x, y) = \frac{1}{2}\left(\mathcal{L}(\gamma) - \mathcal{L}(\gamma_t)\right) + d_{\gamma_t}(\tilde{x}, y)$$

and so

$$\begin{aligned}
d_{\gamma_t}(\tilde{x}, y) \leq \|\eta_1'\|_{L^\infty} + \frac{1}{2}\left(\mathcal{L}(\gamma_t) - \mathcal{L}(\gamma)\right) &= \|\eta_1'\|_{L^\infty}\\
&+ \frac{1}{2}\int_0^1 |\gamma_t'(s)| - |\gamma'(s)|\,\mathrm{d}s \leq \frac{3}{2}\,\|\eta_1'\|_{L^\infty}.
\end{aligned}$$

By (5.7), $d_{\gamma_t}(\tilde{x}, y) \geq v_{\gamma_t}|\tilde{x} - y|_{\mathbb{R}/\mathbb{Z}} \geq \frac{1}{2}\operatorname{BiLip}(\gamma)|\tilde{x} - y|_{\mathbb{R}/\mathbb{Z}}$, and therefore

$$|\tilde{x} - y|_{\mathbb{R}/\mathbb{Z}} \leq \frac{3}{\operatorname{BiLip}(\gamma)} \left\|\eta_1'\right\|_{L^\infty} \tag{5.18}$$

for $(x, y) \in V(\eta_1)$, $\tilde{x} = \tilde{x}(x)$. Let us set $V_1(x) := \{y \in [0, 1] | (x, y) \in V(\eta_1)\}$ and $B_r^{\mathbb{R}/\mathbb{Z}}(x) := \left\{y \in [0, 1] | |x - y|_{\mathbb{R}/\mathbb{Z}} < r\right\}$. Note that

$$|B_r^{\mathbb{R}/\mathbb{Z}}(x)| = \left|B_r^{\mathbb{R}/\mathbb{Z}}\left(\frac{1}{2}\right)\right| \leq \left|B_r\left(\frac{1}{2}\right)\right| = 2r.$$

We may use this in combination with (5.18) to estimate

$$\begin{aligned}|V(\eta_1)| &= \int_0^1 |V_1(x)| \,\mathrm{d}x \leq \int_0^1 |B^{\mathbb{R}/\mathbb{Z}}_{\frac{3}{\operatorname{BiLip}(\gamma)}\|\eta_1'\|_{L^\infty}}(\tilde{x}(x))| \,\mathrm{d}x \\ &= \int_0^1 \frac{6}{\operatorname{BiLip}(\gamma)} \left\|\eta_1'\right\|_{L^\infty} \mathrm{d}x = \frac{6}{\operatorname{BiLip}(\gamma)} \left\|\eta_1'\right\|_{L^\infty}.\end{aligned}$$

Next, we prove that F_1 is continuous in γ.

Claim 6. The mapping

$$\mathcal{B}_{\mathrm{ir}} \to \mathcal{L}\Big(\mathcal{B}, L^p((\mathbb{R}/\mathbb{Z})^2)\Big), \gamma \mapsto F_1(\gamma, \cdot)$$

is continuous.

It suffices to show that $\|F_1(\gamma + \eta_1; \eta_2) - F_1(\gamma; \eta_2)\|_{L^p((\mathbb{R}/\mathbb{Z})^2)} \leq \|\eta_2\|_{\mathcal{B}}\, o(1)$ as $\eta_1 \to 0$, the Landau symbol being uniform in η_2. Let us once again split the domain of integration into $U(\eta_1)$ and $V(\eta_1)$. On the former, we can use that $t \mapsto F_1(\gamma_t; \eta_2)$ is differentiable, as well as Jensen's inequality, Tonelli's variant of Fubini's theorem, Theorem 2 and (5.6):

$$\begin{aligned}&\iint_{U(\eta_1)} |F_1(\gamma + \eta_1; \eta_2) - F_1(\gamma, \eta_2)|^p \,\mathrm{d}y\,\mathrm{d}x = \iint_{U(\eta_1)} \left|\int_0^1 F_2(\gamma_t; \eta_2; \eta_1)\,\mathrm{d}t\right|^p \mathrm{d}y\,\mathrm{d}x \\ &\leq \iint_{U(\eta_1)} \int_0^1 |F_2(\gamma_t; \eta_2; \eta_1)|^p \,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \iint_{U(\eta_1)} |F_2(\gamma_t; \eta_2; \eta_1)|^p \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \Xi^p(\gamma_t) \|\eta_1\|_{\mathcal{B}}^p \|\eta_2\|_{\mathcal{B}}^p \,\mathrm{d}t \leq (2\Xi(\gamma))^p \|\eta_1\|_{\mathcal{B}}^p \|\eta_2\|_{\mathcal{B}}^p.\end{aligned}$$

On $V(\eta_1)$ we may use Theorem 4 and Theorem 5. Note that because of (5.7) and (5.9), $L_{e_\alpha}(\gamma + \eta) \leq C L_{e_\alpha}(\gamma)$ for all $\|\eta\|_{\mathcal{B}} < \varepsilon$ and we can thus estimate

$$\begin{aligned}&\iint_{V(\eta_1)} |F_1(\gamma + \eta_1; \eta_2) - F_1(\gamma, \eta_2)|^p \,\mathrm{d}y\,\mathrm{d}x \\ &\leq C(p) \iint_{V(\eta_1)} |F_1(\gamma + \eta_1; \eta_2)|^p + |F_1(\gamma, \eta_2)|^p \,\mathrm{d}y\,\mathrm{d}x \\ &\leq C(p)(C^p + 1) L_{e_\alpha}^p \frac{6}{\operatorname{BiLip}(\gamma)} \left\|\eta_1'\right\|_{L^\infty} \|\eta_2\|_{\mathcal{B}}^p.\end{aligned}$$

□

Lemma 5.2. *Let $p > 1$ and $\mathcal{O} := \left\{\gamma \in W^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) | v_\gamma > 0\right\}$. Then, the map*

$$\|\cdot\|^p_{L^p_\cdot((\mathbb{R}/\mathbb{Z})^2)} : \mathcal{O} \times L^p((\mathbb{R}/\mathbb{Z})^2) \to \mathbb{R},$$
$$(\gamma, g) \mapsto \iint_{(\mathbb{R}/\mathbb{Z})^2} |g(x, y)|^p |\gamma'(x)||\gamma'(y)| \, \mathrm{d}y \, \mathrm{d}x$$

is continuously differentiable.

The authors are indebted to the anonymous referee for pointing out an error in the original version of this lemma and for giving the idea for the following, simpler proof. Note that each regular C^1-curve, and consequently each $W_{\mathrm{ir}}^{1+\frac{\alpha p-1}{2p},2p}$-curve, is in $\mathcal{O}$ enabling us to use this lemma in the proof of Theorem 5.1.

Proof. First note that $\mathcal{O}$ is an open subset of $W^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and that we can write

$$\|g\|^p_{L^p_\gamma((\mathbb{R}/\mathbb{Z})^2,\mathbb{R})} := \langle |g|^p, |\gamma'| \otimes |\gamma'| \rangle_{L^1,L^\infty}$$
$$:= \iint_{(\mathbb{R}/\mathbb{Z})^2} |g(x, y)|^p (|\gamma'| \otimes |\gamma'|)(x, y) \, \mathrm{d}y \, \mathrm{d}x$$

for $(|\gamma'| \otimes |\gamma'|)(x, y) := |\gamma'(x)||\gamma'(y)|$. As both

$$\langle \cdot, \cdot \rangle_{L^1,L^\infty} : L^1((\mathbb{R}/\mathbb{Z})^2, \mathbb{R}) \times L^\infty((\mathbb{R}/\mathbb{Z})^2, \mathbb{R}) \to \mathbb{R}$$

and

$$\cdot \otimes \cdot : L^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \times L^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \to L^\infty((\mathbb{R}/\mathbb{Z})^2, \mathbb{R})$$

are bilinear and bounded, they are smooth. By the chain- and product rule for Banach spaces (see, for example, [45, Propositions 4.10 and 4.11]), it suffices to establish continuous differentiability of the mappings

$$L^p((\mathbb{R}/\mathbb{Z})^2, \mathbb{R}) \to L^1((\mathbb{R}/\mathbb{Z})^2, \mathbb{R}), \, g \mapsto |g|^p \text{ and } \mathcal{O} \to L^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}), \, \gamma \mapsto |\gamma'|.$$

Checking continuous differentiability of the former is straight-forward, so let us deal with the latter.

Consider $\gamma \in \mathcal{O}$ and $\eta \in W^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ such that $\|\eta'\|_{L^\infty} \leq \frac{v_\gamma}{2}$. Then, we may estimate the pointwise difference between $|\gamma' + \eta'|$ and its first-order approximation, omitting the x-dependence:

$$\Big| |\gamma' + \eta'| - |\gamma'| - \langle D_\gamma \gamma, \eta' \rangle \Big| = \left| \int_0^1 \langle D_{\gamma+t\eta}(\gamma + t\eta) - D_\gamma \gamma, \eta' \rangle \, \mathrm{d}t \right|$$
$$\leq \|\eta'\|_{L^\infty} \int_0^1 |D_{\gamma+t\eta}(\gamma + t\eta) - D_\gamma \gamma| \, \mathrm{d}t.$$

It suffices to show that the integrand is of order $o(1)$ as $\|\eta\|_{W^{1,\infty}}$ goes to 0 to have Fréchet differentiability. To see that this is the case, we rewrite it as

$$\left|\frac{|\gamma'|(\gamma'+t\eta')-|\gamma'+t\eta'|\gamma'}{|\gamma'+t\eta'||\gamma'|}\right| \\ \leq 2v_\gamma^{-2}||\gamma'|(\gamma'+t\eta')-|\gamma'|\gamma'+|\gamma'|\gamma'-|\gamma'+t\eta'|\gamma'| \leq 4v_\gamma^{-2}t|\gamma'||\eta'| \tag{5.19}$$

which behaves exactly as required.

To establish continuous differentiability, we may recycle (5.19) with $t = 1$ and some $W^{1,\infty}$-small perturbation η satisfying the same bound as above. This is because we then obtain

$$\left\|\langle D_\gamma\gamma,(\cdot)'\rangle-\langle D_{\gamma+\eta}(\gamma+\eta),(\cdot)'\rangle\right\|_{\mathcal{L}(W^{1,\infty},L^\infty)} \leq \left\|D_\gamma\gamma-D_{\gamma+\eta}(\gamma+\eta)\right\|_{L^\infty} \\ \leq 4v_\gamma^{-2}\left\|\gamma'\right\|_{L^\infty}\left\|\eta'\right\|_{L^\infty}$$

which tends to 0 as $\|\eta\|_{W^{1,\infty}} \to 0$. □

Lemma 5.3. *Let* $\alpha > 0$*,* $p \geq 1$*,* $2 < \alpha p < 2p+1$*,* $k \in \mathbb{N}$*,* $\varphi \in \mathbb{R}$*,* $\psi > 0$*, and* $\gamma \in W_{\text{ir}}^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$*. Furthermore, let* $b\colon (\mathbb{R}^n)^k \to \mathbb{R}$ *be a* k*-multilinear map and*

$$B_\gamma^{\varphi,\psi} : (W^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n))^k \to \mathbb{R}, (\eta_1,\ldots,\eta_k) \mapsto \\ \iint_{(\mathbb{R}/\mathbb{Z})^2}\left|b(L_1\eta_1,\ldots,L_k\eta_k)\frac{d_\gamma(x,y)^{\varphi+\psi+2-\alpha}}{|\Delta\gamma|^\varphi}\cdot\left(\frac{1}{|\Delta\gamma|^{2+\psi}}-\frac{1}{d_\gamma^{2+\psi}}\right)\right|^p \mathrm{d}y\,\mathrm{d}x$$

where $L_i u \in \left\{\frac{\Delta}{d_\gamma}, \eta\mapsto D_\gamma\eta(x), \eta\mapsto D_\gamma\eta(y)\right\}$*. Then,* $B_\gamma^{\varphi,\psi}$ *is well-defined and*

$$|B_\gamma^{\varphi,\psi}(\eta_1,\ldots,\eta_k)| \leq \Xi(\gamma)\,\|b\|_{\mathcal{L}^k(\mathbb{R}^n,\mathbb{R})}^p\prod_{i=1}^k\left\|\eta_i'\right\|_{L^\infty}^p.$$

Here, $\Xi(\gamma) > 0$ *continuously depends on* γ *with respect to the* $W^{1+\frac{\alpha p-1}{2p},2p}$*-norm.*

Remark 5.4. This as well as Lemma 5.5 and Lemma 5.6 are also valid for the case $\alpha p = 2$ if one considers the intersection $W^{1+\frac{\alpha p-1}{2p},2p} \cap W^{1,\infty}$. We do not make the effort to include this case as we do not want to deal with the more complicated space in Theorem 5.1.

Proof. Similarly to the proof of Theorem 5.1, we write $\mathcal{B}$ for $W^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ and $\mathcal{A}$ instead of $W^{\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ in the following. Let us begin by bounding the b-term:

$$|b(L_1\eta_1,\ldots,L_k\eta_k)| \leq \|b\|_{\mathcal{L}^k(\mathbb{R}^n,\mathbb{R})}\prod_{i=1}^k|L_i\eta_i|.$$

All $|L_i\eta_i|$ may be bounded above by $\left\|D_\gamma\eta_i\right\|_{L^\infty} \le \frac{1}{v_\gamma}\left\|\eta_i'\right\|_{L^\infty}$, which is clear for the second and third option for L_i. For the first one, we calculate

$$d_\gamma^{-1}|\Delta\eta| = d_\gamma^{-1}\left|\int_{I_\gamma}\eta'(t)\frac{|\gamma'(t)|}{|\gamma'(t)|}\,\mathrm{d}t\right| \le d_\gamma^{-1}\left\|D_\gamma\eta\right\|_{L^\infty}\left|\int_{I_\gamma}|\gamma'(t)|\,\mathrm{d}t\right| = \left\|D_\gamma\eta\right\|_{L^\infty}. \tag{5.20}$$

We continue by defining a function ζ which helps reduce the integrand to its most relevant content: $\zeta\colon(0,\infty)\to\mathbb{R}, r\mapsto r^{2+\varphi}\frac{r^{2+\psi}-1}{r^2-1}$. With this, we may rewrite the second part of the integrand:

$$\frac{d_\gamma(x,y)^{\varphi+\psi+2-\alpha}}{|\Delta\gamma|^\varphi}\cdot\left(\frac{1}{|\Delta\gamma|^{2+\psi}}-\frac{1}{d_\gamma^{2+\psi}}\right) = \frac{1}{2}\zeta\left(\frac{d_\gamma}{|\Delta\gamma|}\right)\frac{1}{d_\gamma^{2+\alpha}}(2d_\gamma^2-2|\Delta\gamma|^2).$$

The last difference can be written in terms of the unit tangents $\tau_\gamma = \frac{\gamma'}{|\gamma'|}$:

$$\begin{aligned} 2d_\gamma^2 - 2|\Delta\gamma|^2 &= \iint_{I_\gamma^2}(|\tau_\gamma(s)|^2+|\tau_\gamma(t)|^2)|\gamma'(s)||\gamma'(t)|\,\mathrm{d}t\,\mathrm{d}s \\ &\quad -2\iint_{I_\gamma^2}\langle\tau_\gamma(s),\tau_\gamma(t)\rangle|\gamma'(s)||\gamma'(t)|\,\mathrm{d}t\,\mathrm{d}s \\ &= \iint_{I_\gamma^2}|\tau_\gamma(s)-\tau_\gamma(t)|^2|\gamma'(s)||\gamma'(t)|\,\mathrm{d}t\,\mathrm{d}s. \end{aligned}$$

Let us show that ζ is bounded. By L'Hôpital's rule, it is continuous in $r=1$ and for all other r, too.

Now, let us bound the argument of ζ: Setting I as the interval in $\mathbb{R}/\mathbb{Z}$ connecting x and y with length at most $\frac{1}{2}$ (so $|I| = |x-y|_{\mathbb{R}/\mathbb{Z}}$), we obtain that

$$d_\gamma(x,y) = \int_{I_\gamma}|\gamma'(s)|\,\mathrm{d}s \le \int_I|\gamma'(s)|\,\mathrm{d}s \le |I|\left\|\gamma'\right\|_{L^\infty} = |x-y|_{\mathbb{R}/\mathbb{Z}}\left\|\gamma'\right\|_{L^\infty}$$

and thus

$$\frac{\mathrm{BiLip}(\gamma)}{\|\gamma'\|_{L^\infty}} \le \frac{\mathrm{BiLip}(\gamma)|x-y|_{\mathbb{R}/\mathbb{Z}}}{d_\gamma} \le \frac{|\Delta\gamma|}{d_\gamma} \overset{(5.20)}{\le} \frac{\left\|\gamma'\right\|_{L^\infty}}{v_\gamma}.$$

Consequently, we may bound $|\zeta(\frac{|\Delta\gamma|}{d_\gamma})|$ in terms of a constant C_ζ that continuously depends on $\mathrm{BiLip}(\gamma)$, v_γ and $\left\|\gamma'\right\|_{L^\infty}$. It only remains to find an upper estimate for the L^p-Norm of

$$\frac{1}{d_\gamma^{2+\alpha}}\iint_{I_\gamma^2}|\tau_\gamma(s)-\tau_\gamma(t)|^2|\gamma'(s)||\gamma'(t)|\,\mathrm{d}t\,\mathrm{d}s.$$

Seeing as we may bound $|\gamma'(\cdot)|$ by $\left\|\gamma'\right\|_{L^\infty}$, let us concentrate on the remaining integral,

$$\iint_{(\mathbb{R}/\mathbb{Z})^2}\left(\frac{1}{d_\gamma^{2+\alpha}}\iint_{I_\gamma^2}|\tau_\gamma(s)-\tau_\gamma(t)|^2\,\mathrm{d}t\,\mathrm{d}s\right)^p\mathrm{d}y\,\mathrm{d}x$$

which we can bound above by

$$C(\alpha, n, p) v_\gamma^{-(\alpha+8)p-2} \left\|\gamma'\right\|_{L^\infty}^2 \left\|\gamma'\right\|_{\mathcal{A}}^{4p}$$

according to Lemma 5.6. Combining our estimates, we arrive at

$$\Xi(\gamma) = v_\gamma^{-(\alpha+8)p-k-2} C_\zeta(\mathrm{BiLip}(\gamma), v_\gamma, \|\gamma'\|_{L^\infty}) \left\|\gamma'\right\|_{L^\infty}^{2p+2} C(\alpha, n, p) \|\gamma'\|_{\mathcal{A}}^{4p}.$$

All the γ-dependent parameters of Ξ are continuously dependent on γ with respect to the $\mathcal{B}$-norm. □

Lemma 5.5. *Let $\alpha > 0$, $p \geq 1$, $2 < \alpha p < 2p + 1$ and $\gamma \in W_{\mathrm{ir}}^{1+\frac{\alpha p-1}{2p}, 2p}$. Then, the map $B_\gamma : (W^{1+\frac{\alpha p-1}{2p}, 2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))^2 \to \mathbb{R}$ sending (η_1, η_2) to the double integral*

$$\iint_{(\mathbb{R}/\mathbb{Z})^2} \left| \left(\frac{2}{d_\gamma(x, y)} \int_{I_\gamma(x,y)} \langle D_\gamma \eta_1(s), D_\gamma \eta_2(s) \rangle |\gamma'(s)| \,\mathrm{d}s \right.\right.$$
$$\left.\left. - 2\left\langle \frac{\Delta\eta_1}{d_\gamma}, \frac{\Delta\eta_2}{d_\gamma} \right\rangle (x, y) \right) \cdot \frac{1}{d_\gamma(x, y)^\alpha} \right|^p \mathrm{d}y \,\mathrm{d}x$$

is well-defined and satisfies

$$|B_\gamma(\eta_1, \eta_2)|$$
$$\leq C(\alpha, n, p) \left\|\gamma'\right\|_{L^\infty}^{2p+2} v_\gamma^{-(8+\alpha)p-2} \left\|\gamma'\right\|_{W^{\frac{\alpha p-1}{2p}, 2p}}^{2p} \left\|\eta_1'\right\|_{W^{\frac{\alpha p-1}{2p}, 2p}}^p \left\|\eta_2'\right\|_{W^{\frac{\alpha p-1}{2p}, 2p}}^p.$$

Proof. As above, we use the shorthand $\mathcal{A}$ for $W^{\frac{\alpha p-1}{2p}, 2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Note that

$$\frac{2}{d_\gamma} \int_{I_\gamma} \langle D_\gamma \eta_1(s), D_\gamma \eta_2(s) \rangle |\gamma'(s)| \,\mathrm{d}s$$
$$= \frac{1}{d_\gamma^2} \iint_{I_\gamma^2} \big(\langle D_\gamma \eta_1(s), D_\gamma \eta_2(s) \rangle + \langle D_\gamma \eta_1(t), D_\gamma \eta_2(t) \rangle \big) |\gamma'(s)||\gamma'(t)| \,\mathrm{d}t \,\mathrm{d}s$$

and

$$2\left\langle \frac{\Delta\eta_1}{d_\gamma}, \frac{\Delta\eta_2}{d_\gamma} \right\rangle$$
$$= \frac{1}{d_\gamma^2} \iint_{I_\gamma^2} \big(\langle D_\gamma \eta_1(s), D_\gamma \eta_2(t) \rangle + \langle D_\gamma \eta_1(t), D_\gamma \eta_2(s) \rangle \big) |\gamma'(s)||\gamma'(t)| \,\mathrm{d}t \,\mathrm{d}s.$$

With these identities, we may rewrite the integrand and then bound it via the Cauchy-Schwarz-inequality as follows:

$$\frac{1}{d_\gamma^{(2+\alpha)p}} \left(\iint_{I_\gamma^2} \langle \Delta D_\gamma \eta_1, \Delta D_\gamma \eta_2 \rangle (s, t) |\gamma'(s)||\gamma'(t)| \,\mathrm{d}t \,\mathrm{d}s \right)^p$$
$$\leq \frac{\|\gamma'\|_{L^\infty}^{2p}}{d_\gamma^{(2+\alpha)p}} \prod_{i=1}^2 \left(\iint_{I_\gamma^2} |\Delta D_\gamma \eta_i(s, t)|^2 \,\mathrm{d}t \,\mathrm{d}s \right)^{\frac{p}{2}}$$

Integrating over $(\mathbb{R}/\mathbb{Z})^2$ and using Cauchy-Schwarz again, we arrive at a new bound to which we can apply Lemma 5.6:

$$\begin{aligned} &\|\gamma'\|_{L^\infty}^{2p} \prod_{i=1}^{2} \left(\iint_{(\mathbb{R}/\mathbb{Z})^2} \frac{1}{d_\gamma^{(2+\alpha)p}} \left(\iint_{I_\gamma^2} |\Delta D_\gamma \eta_i(s,t)|^2 \,\mathrm{d}t\,\mathrm{d}s \right)^p \mathrm{d}y\,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\quad \le C(\alpha,n,p) \|\gamma'\|_{L^\infty}^{2p+2} v_\gamma^{-(8+\alpha)p-2} \|\gamma'\|_{\mathcal{A}}^{2p} \|\eta_1'\|_{\mathcal{A}}^{p} \|\eta_2'\|_{\mathcal{A}}^{p}. \end{aligned}$$

□

Lemma 5.6. *Let $\alpha > 0$, $\beta \in (2, \alpha+2]$, $p \ge 1$, $2 < \alpha p < 2p+1$, $\gamma \in W_{\mathrm{ir}}^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ and $\eta \in W^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. Then,*

$$\begin{aligned} &\iint_{(\mathbb{R}/\mathbb{Z})^2} \left(\frac{1}{d_\gamma^\beta} \iint_{I_\gamma^2} |\Delta D_\gamma \eta(s,t)|^2 \,\mathrm{d}t\,\mathrm{d}s \right)^p \mathrm{d}y\,\mathrm{d}x \\ &\quad \le C v_\gamma^{-(\beta+6)p-2} \|\gamma'\|_{L^\infty}^2 \|\gamma'\|_{W^{\frac{\alpha p-1}{2p},2p}}^{2p} \|\eta'\|_{W^{\frac{\alpha p-1}{2p},2p}}^{2p} \end{aligned}$$

for some constant $C = C(\alpha,\beta,n,p) > 0$.

Proof. We once more employ the shorthands $\mathcal{B}$ for $W^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ and $\mathcal{A}$ instead of $W^{\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. We prove this statement via substitutions with the inverse arc length function $a_\gamma := \mathcal{L}(\gamma,\cdot)^{-1}$, where $\mathcal{L}(\gamma,\cdot) : \mathbb{R} \to \mathbb{R}$, $s \mapsto \int_0^s |\gamma'(t)|\,\mathrm{d}t$. As $\mathcal{L}(\gamma,\cdot)$ is in $W^{1+\frac{\alpha p-1}{2p},2p}(\mathbb{R}/\mathbb{Z},\mathbb{R})$, see [27, Lemma B.2], it is also in $C^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$. Since it is strictly increasing and its derivative is equal to $|\gamma'(x)| \ge v_\gamma > 0$, $a_\gamma \in C^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ with $a_\gamma'(x) = \frac{1}{\mathcal{L}(\gamma,\cdot)'(a_\gamma(x))} = \frac{1}{|\gamma'(a_\gamma(x))|}$ and so

$$|a_\gamma'(x)| \le \frac{1}{v_\gamma}. \tag{5.21}$$

Using integration by substitution, our bound on a_γ' (5.21), and periodicity of the outermost integrand, we obtain that

$$\begin{aligned} &\iint_{(\mathbb{R}/\mathbb{Z})^2} \left(d_\gamma(x,y)^{-\beta} \iint_{I_\gamma(x,y)^2} |D_\gamma\eta(s) - D_\gamma\eta(t)|^2 \,\mathrm{d}t\,\mathrm{d}s \right)^p \mathrm{d}y\,\mathrm{d}x \\ &\le v_\gamma^{-2} \iint_{(\mathbb{R}/L\mathbb{Z})^2} \left(d_\gamma(a_\gamma(\tilde{x}), a_\gamma(\tilde{y}))^{-\beta} \iint_{I_\gamma(a_\gamma(\tilde{x}),a_\gamma(\tilde{y}))^2} |D_\gamma\eta(s) - D_\gamma\eta(t)|^2 \,\mathrm{d}t\,\mathrm{d}s \right)^p \mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x} \\ &= v_\gamma^{-2} \int_0^L \int_{\tilde{x}-\frac{L}{2}}^{\tilde{x}+\frac{L}{2}} \left(d_\gamma(a_\gamma(\tilde{x}), a_\gamma(\tilde{y}))^{-\beta} \iint_{I_\gamma(a_\gamma(\tilde{x}),a_\gamma(\tilde{y}))^2} |D_\gamma\eta(s) - D_\gamma\eta(t)|^2 \,\mathrm{d}t\,\mathrm{d}s \right)^p \mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}. \end{aligned} \tag{5.22}$$

Although it may not seem this way at first, this representation is in fact simpler, as our parametrisation fits both d_γ and I_γ: Let $\tilde{x}, \tilde{y} \in \mathbb{R}$. Then, there is $k \in \mathbb{Z}$ such that

$$\begin{aligned} d_\gamma(a_\gamma(\tilde{x}), a_\gamma(\tilde{y})) &= \left| \int_{a_\gamma(\tilde{x})}^{a_\gamma(\tilde{y})+k} |\gamma'(s)| \,\mathrm{d}s \right| = |\mathcal{L}(\gamma, a_\gamma(\tilde{y}) + k) - \mathcal{L}(\gamma, a_\gamma(\tilde{x}))| \\ &= |\mathcal{L}(\gamma, a_\gamma(\tilde{y})) + kL - \mathcal{L}(\gamma, a_\gamma(\tilde{x}))| = |\tilde{y} + kL - \tilde{x}|. \end{aligned}$$

As $d_\gamma(a_\gamma(\tilde{x}), a_\gamma(\tilde{y})) \le \frac{L}{2}$, we obtain that

$$d_\gamma(a_\gamma(\tilde{x}), a_\gamma(\tilde{y})) = |\tilde{y} + kL - \tilde{x}| = |\tilde{y} - \tilde{x}|_{\mathbb{R}/L\mathbb{Z}}.$$

In particular, if $|\tilde{x} - \tilde{y}| < \frac{L}{2}$, we have that $I_\gamma(a_\gamma(\tilde{x}), a_\gamma(\tilde{y})) = [a_\gamma(\tilde{x}), a_\gamma(\tilde{y})]$, employing the short-hand notation $[a, b] := [b, a]$ if $b < a$.

Thus, we may rewrite our integral from (5.22) as

$$v_\gamma^{-2} \int_0^L \int_{\tilde{x}-\frac{L}{2}}^{\tilde{x}+\frac{L}{2}} \left(|\tilde{x} - \tilde{y}|^{-\beta} \iint_{[a_\gamma(\tilde{x}), a_\gamma(\tilde{y})]^2} |D_\gamma \eta(s) - D_\gamma \eta(t)|^2 \,\mathrm{d}t \,\mathrm{d}s \right)^p \mathrm{d}\tilde{y} \,\mathrm{d}\tilde{x}.$$

First substituting $\tilde{y}$ by $y = \tilde{y} - \tilde{x}$ and then (s, t) by $(a_\gamma(\tilde{x} + \theta_1 y), a_\gamma(\tilde{x} + \theta_2 y))$, we obtain by virtue of (5.21)

$$\begin{aligned} & v_\gamma^{-2} \int_0^L \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(|y|^{-\beta} \iint_{[a_\gamma(\tilde{x}), a_\gamma(\tilde{x}+y)]^2} |D_\gamma \eta(s) - D_\gamma \eta(t)|^2 \,\mathrm{d}t \,\mathrm{d}s \right)^p \mathrm{d}y \,\mathrm{d}\tilde{x} \\ & \quad \le v_\gamma^{-2-2p} \int_0^L \int_{-\frac{L}{2}}^{\frac{L}{2}} \\ & \left(|y|^{-\beta} \iint_{[0,1]^2} |D_\gamma \eta(a_\gamma(\tilde{x} + \theta_1 y)) - D_\gamma \eta(a_\gamma(\tilde{x} + \theta_2 y))|^2 |y|^2 \,\mathrm{d}\theta_2 \,\mathrm{d}\theta_1 \right)^p \mathrm{d}\tilde{y} \,\mathrm{d}\tilde{x}. \end{aligned}$$

For the moment dropping the factor v_γ^{-2-2p}, employing Jensen's inequality, Tonelli's variant of Fubini's theorem, the substitution $u = \tilde{x} + \theta_2 y$ as well as the L-periodicity of the integrand with respect to u, we obtain the new estimate

$$\begin{aligned} & \int_0^L \int_{-\frac{L}{2}}^{\frac{L}{2}} |y|^{(2-\beta)p} \iint_{[0,1]^2} |D_\gamma \eta(a_\gamma(\tilde{x} + \theta_1 y)) - D_\gamma \eta(a_\gamma(\tilde{x} + \theta_2 y))|^{2p} \,\mathrm{d}\theta_2 \,\mathrm{d}\theta_1 \,\mathrm{d}\tilde{y} \,\mathrm{d}\tilde{x} \\ & \quad = \iint_{[0,1]^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_0^L |y|^{(2-\beta)p} |D_\gamma \eta(a_\gamma(u + (\theta_1 - \theta_2) y)) - D_\gamma \eta(a_\gamma(u))|^{2p} \,\mathrm{d}u \,\mathrm{d}y \,\mathrm{d}\theta_2 \,\mathrm{d}\theta_1. \end{aligned}$$

Two further substitutions, θ_2 by $\vartheta = \theta_1 - \theta_2$ and y by $w = \vartheta y$ yield a new upper bound. Note that we dropped the θ_1-integral as its domain has measure 1 and nothing depends on θ_1 after enlarging the domain of integration for ϑ.

$$\begin{aligned}
&\int_0^1 \int_{\theta_1-1}^{\theta_1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_0^L |y|^{(2-\beta)p} |D_\gamma \eta(a_\gamma(u+\vartheta y)) - D_\gamma \eta(a_\gamma(u))|^{2p} \,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}\vartheta\,\mathrm{d}\theta_1 \\
&\le \int_{-1}^1 \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_0^L |y|^{(2-\beta)p} |D_\gamma \eta(a_\gamma(u+\vartheta y)) - D_\gamma \eta(a_\gamma(u))|^{2p} \,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}\vartheta \\
&= \int_{-1}^1 \int_{-|\vartheta|\frac{L}{2}}^{|\vartheta|\frac{L}{2}} \int_0^L \left|\frac{w}{\vartheta}\right|^{(2-\beta)p} |D_\gamma \eta(a_\gamma(u+w)) - D_\gamma \eta(a_\gamma(u))|^{2p} \vartheta^{-1} \,\mathrm{d}u\,\mathrm{d}w\,\mathrm{d}\vartheta \\
&\le \int_{-1}^1 |\vartheta|^{(\beta-2)p-1}\,\mathrm{d}\vartheta \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_0^L |w|^{(2-\beta)p} |D_\gamma \eta(a_\gamma(u+w)) - D_\gamma \eta(a_\gamma(u))|^{2p} \,\mathrm{d}u\,\mathrm{d}w \\
&= C(\beta,p) \left[(D_\gamma \eta)\circ a_\gamma\right]^{2p}_{\frac{(\beta-2)p-1}{2p},2p} \le \widetilde{C}(\alpha,\beta,p) \left[(D_\gamma \eta)\circ a_\gamma\right]^{2p}_{\frac{\alpha p-1}{2p},2p}.
\end{aligned}$$

For the definition of the Gagliardo seminorm $[\cdot]_{s,\rho}$, see Appendix A. In the last line, we used that $2 < \beta \le \alpha + 2$. Using Lemma A.5 and Lemma A.4, we may estimate

$$\begin{aligned}
\left[(D_\gamma \eta)\circ a_\gamma\right]_{\frac{\alpha p-1}{2p},2p,\mathbb{R}/L\mathbb{Z}} &\le \left\|a_\gamma'\right\|_{C^0}^{\frac{1}{2p}+\frac{(\beta-2)p-1}{2p}} v_{a_\gamma}^{-\frac{1}{p}} \left[D_\gamma \eta\right]_{\frac{\alpha p-1}{2p},2p,\mathbb{R}/\mathbb{Z}} \\
&\le C(\beta,n,p) \left\|a_\gamma'\right\|_{C^0}^{\frac{(\beta-2)p}{2p}} v_{a_\gamma}^{-\frac{1}{p}} v_\gamma^{-2} \left\|\gamma'\right\|_{\mathcal{A}} \left\|\eta'\right\|_{\mathcal{A}}
\end{aligned}$$

Seeing as $v_{a_\gamma} = \left\|\gamma'\right\|_{L^\infty}^{-1}$ and $\left\|a_\gamma'\right\|_{C^0} = v_\gamma^{-1}$ and taking into account the factor v_γ^{-2-2p} we dropped along the way, we arrive at

$$\overline{C}(\alpha,\beta,n,p) v_\gamma^{-(\beta+6)p-2} \left\|\gamma'\right\|_{L^\infty}^2 \left\|\gamma'\right\|_{\mathcal{A}}^{2p} \left\|\eta'\right\|_{\mathcal{A}}^{2p}$$

as our final upper bound. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A. Properties of low-order Sobolev-Slobodeckiĭ Functions

In the following, we collect some basic statements concerning Sobolev-Slobodeckiĭ spaces. Recall that for $k \in \mathbb{N}$, $s \in (0,1)$, $\rho \geq 1$ and $l > 0$, the Sobolev-Slobodeckiĭ space $W^{k+s,\rho}(\mathbb{R}/l\mathbb{Z},\mathbb{R}^n)$ is the space of all l-periodic $W^{k,\rho}_{\text{loc}}(\mathbb{R},\mathbb{R}^n)$-functions f such that the *Gagliardo seminorm* of the highest order derivative $f^{(k)}$,

$$\left[f^{(k)}\right]_{s,\rho,\mathbb{R}/l\mathbb{Z}} := \left(\int_0^l \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{|f^{(k)}(u+w) - f^k(u)|^\rho}{|w|^{1+s\rho}} \,\mathrm{d}w\,\mathrm{d}u\right)^{\frac{1}{\rho}}$$

is finite. If there is no concern of confusion, we omit the domain and simply write $[\cdot]_{s,\rho}$.

Lemma A.1 (Uniform convexity) *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $s \in (0,1)$, and $\rho, \theta \in (1,\infty)$. The space $\mathcal{W} := W^{k+s,\rho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ is uniformly convex and reflexive. Furthermore its norm and the norm of its dual space $\mathcal{W}^*$ are continuously Fréchet-differentiable except at the origin. Consequentially, the θ-duality mapping $\mathfrak{J}_{\mathcal{W},\theta}$ maps $\mathcal{W}$ homeomorphically onto $\mathcal{W}^*$.*

Proof. In order to show all the claims, we exploit the fact that $\mathcal{W}$ is isometrically isomorphic to a subspace of

$$Z := \left(\bigoplus_{j=0}^{k} L^\rho(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)\right) \oplus L^\rho\big((\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z},\nu),\mathbb{R}^n\big)$$

equipped with the ρ-norm $|(a_0,\ldots,a_k,a_{k+1})|_\rho^\rho = \sum_{j=0}^{k+1}|a_j|^\rho$ for the direct sums and where ν is the measure given by

$$\nu(A) = \iint_A \frac{\mathrm{d}x\mathrm{d}y}{|x-y|_{\mathbb{R}/\mathbb{Z}}}, \quad \text{for Borel subsets } A \subset \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}.$$

In the following, we abbreviate $L^\rho := L^\rho(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, and

$$L^\rho(\nu) := L^\rho\big((\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z},\nu),\mathbb{R}^n\big).$$

Let the fractional difference quotient $\Delta^s_{\text{Höl}} : W^{s,\rho} \to L^\rho(\nu)$ be given by

$$\Delta^s_{\text{Höl}} g(x, y) := \frac{g(x) - g(y)}{|x - y|^s_{\mathbb{R}/\mathbb{Z}}}.$$

The map $\Psi : \mathcal{W} \to Z$ defined by $\Psi(f) := \big(\Psi_0(f), \dots, \Psi_{k+1}(f)\big)$ with $\Psi_j(f) = f^{(j)}$ for $j \in \{0, \dots, k\}$ and $\Psi_{k+1}(f) := \Delta^s_{\text{Höl}} f^{(k)}$ is a linear isometry, since

$$\begin{aligned}\|f\|^\rho_{\mathcal{W}} &= \left(\sum_{j=0}^{k} \left\|f^{(j)}\right\|^\rho_{L^\rho}\right) + \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|f^{(k)}(x) - f^{(k)}(y)|^\rho}{|x - y|^{1+s\rho}_{\mathbb{R}/\mathbb{Z}}} \mathrm{d}x\mathrm{d}y \\ &= \left(\sum_{j=0}^{k} \left\|f^{(j)}\right\|^\rho_{L^\rho}\right) + \left\|\Delta^s_{\text{Höl}} f^{(k)}\right\|^\rho_{L^\rho(\nu)} = \|\Psi(f)\|^\rho_Z.\end{aligned}$$

We deduce from the following two results that Z is uniformly convex. Firstly, in [19, Theorem 2 & p. 507] it is shown for non-negative measures $\tilde{\mu}$, numbers $1 < \tilde{\rho} < \infty$, and Banach spaces $\tilde{\mathcal{B}}$ that $L^{\tilde{\rho}}(\tilde{\mu}; \tilde{\mathcal{B}})$ is uniformly convex if $\tilde{\mathcal{B}}$ is. The space $\tilde{\mathcal{B}} := \mathbb{R}^n$ endowed with the Euclidean 2-norm satisfies this condition. Secondly, an immediate consequence of [19, Theorem 3 & p. 504] is that $l^{\tilde{\rho}}$-direct sums of finitely many uniformly convex Banach spaces are uniformly convex. It is straightforward to check that uniform convexity is inherited by subspaces and preserved by linear isometries. Therefore, $\Psi(\mathcal{W})$ and $\mathcal{W}$ are uniformly convex.

Next, we show the Fréchet-differentiability of $\|\cdot\|_{\mathcal{W}}$ on $\mathcal{W} \setminus \{0\}$. Since Ψ is a bounded linear operator, it is Fréchet-differentiable. It remains to investigate $\|\cdot\|_Z$. In [29, Theorem 2.5] it is shown for every non-negative measure $\tilde{\mu}$, $1 < \tilde{\rho} < \infty$, and Banach space $\tilde{\mathcal{B}}$, that the norm on $L^{\tilde{\rho}}(\tilde{\mu}, \tilde{\mathcal{B}})$ is Fréchet-differentiable except at 0 if and only if the norm on $\tilde{\mathcal{B}}$ is Fréchet-differentiable except at 0. Obviously, $\tilde{\mathcal{B}} := \mathbb{R}^n$ with the Euclidean 2-norm satisfies this condition. Since $\tilde{\rho} > 1$, the map $\|\cdot\|^{\tilde{\rho}}_{L^{\tilde{\rho}}(\tilde{\mu}, \tilde{\mathcal{B}})}$ is differentiable everywhere. We infer that the map

$$Z \ni (g_0, \dots, g_{k+1}) \mapsto \|(g_0, \dots, g_{k+1})\|^\rho_Z = \left(\sum_{j=0}^{k} \|g_j\|^\rho_{L^\rho}\right) + \|g_{k+1}\|^\rho_{L^\rho(\nu)}$$

is differentiable.

The remaining claims follow from general results on Banach spaces. Since $\mathcal{W}$ is uniformly convex, it is reflexive by the Milman-Pettis Theorem; see, for example, [18, Theorem II.2.9]. Another consequence of the uniform convexity of $\mathcal{W}$ is that the norm of its dual space is Fréchet-differentiable away from 0, see [18, Theorem II.2.13]. By, [20, Corollary 8.5], both norms are actually continuously Fréchet-differentiable. Finally, [18, Corollary II.3.15] asserts that the duality mapping $\mathfrak{J}_{\mathcal{W},\theta}$ is a homeomorphism between $\mathcal{W}$ and its dual $\mathcal{W}^*$. □

Lemma A.2 (Injective regular curves form open subsets.) *The function space $C^1_{\text{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is open in $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with respect to the C^1-topology. In particular, $W^{1+s,\rho}_{\text{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \subset W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is open with respect to its norm* *topology, as well as* $\mathcal{B}_{\mathrm{ir}} \subset \mathcal{B}$ *for any Banach space* $\mathcal{B}$ *continuously embedded in* $W^{1+s,\rho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

Proof. The statement for C^1-curves was shown in [27, Lemma B.3]. For the Sobolev-Slobodeckiĭ spaces and therefore for Banach spaces continuously embedded in those, this claim follows from the following standard embedding result, Proposition A.3. □

The following proposition gathers well-known embedding results for Sobolev-Slobodeckiĭ spaces. See, for example, the appendix of [31] for a proof based on the Besov space theory presented in [43].

Proposition A.3. (Embedding Theorem) *Let* $k_1, k_2 \in \mathbb{N}_0$, $s_1, s_2 \in (0, 1)$, $\mu \in (0, 1]$, *and* $1 < \rho_1, \rho_2 < \infty$.

(i) If $k_1 + s_1 - (k_2 + s_2) > \max\left\{\frac{1}{\rho_1} - \frac{1}{\rho_2}, 0\right\}$, *then the identity operator*

$$\mathrm{id} : W^{k_1+s_1,\rho_1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^d) \to W^{k_2+s_2,\rho_2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^d)$$

is compact.

(ii) If $k_1 + s_1 - (k_2 + \mu) > \frac{1}{\rho_1}$, *then the identity operator*

$$\mathrm{id} : W^{k_1+s_1,\rho_1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^d) \to C^{k_2,\mu}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^d)$$

is compact.

Now we estimate the fractional Sobolev norm of the derivative with respect to arc length $D_\gamma \eta$ that is repeatedly used in the calculations of Sect. 5

Lemma A.4. *Let* $l > 0$, $\rho \in (1, \infty)$, $s \in (\frac{1}{\rho}, 1)$ *and* $\gamma, \eta \in W^{1+s,\rho}(\mathbb{R}/l\mathbb{Z}, \mathbb{R}^n)$ *with* $v_\gamma > 0$. *Then, the differential operator* $D_\gamma \eta := \eta'/|\gamma'|$ *satisfies*

$$\left\| D_\gamma \eta \right\|_{W^{s,\rho}} \leq C v_\gamma^{-2} \left\| \gamma' \right\|_{W^{s,\rho}} \left\| \eta' \right\|_{W^{s,\rho}}$$

for some $C = C(n, s, \rho) > 0$.

Proof. First, we show that with γ', also $|\gamma'|$ is in $W^{s,\rho}$. For the L^ρ-norm, this is clear immediately, so let us look at the Gagliardo-seminorm:

$$\begin{aligned} \left[|\gamma'|\right]_{s,\rho}^{\rho} &= \int_{\mathbb{R}/l\mathbb{Z}} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{||g'(x+h)| - |g'(x)||^\rho}{|h|^{1+s\rho}} \,\mathrm{d}h \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}/l\mathbb{Z}} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{|g'(x+h) - g'(x)|^\rho}{|h|^{1+s\rho}} \,\mathrm{d}h \,\mathrm{d}x = \left[\gamma'\right]_{s,\rho}^{\rho}. \end{aligned}$$

Now, we need only use the fact that $\left\| \frac{1}{|\gamma'|} \right\|_{W^{s,\rho}} \leq v_\gamma^{-2} \left\| |\gamma'| \right\|_{W^{s,\rho}}$, see [27, Lemma A.6], and the product rule for fractional Sobolev functions which embed into C^1, see, for example, [27, Proposition A.5], to obtain the desired bound. □

We need to know how fractional Sobolev functions behave under reparametrisations. To that goal, we prove a simpler variant of [27, Lemma A.4].

Lemma A.5. *Let* $s \in (0,1)$, $\rho \geq 1$ *and* $l, L > 0$. *Furthermore, let* $f \in W^{s,\rho}(\mathbb{R}/l\mathbb{Z}, \mathbb{R}^n)$ *and* $g \in C^1_{\text{loc}}(\mathbb{R},\mathbb{R})$ *such that* g *is injective,* $g(x+L) = g(x) + l$ *for all* $x \in \mathbb{R}$ *as well as* $v_g = \inf_{x\in[0,L]}|g'(x)| > 0$. *Then,*

$$\begin{aligned} \|f\circ g\|_{L^\rho(\mathbb{R}/L\mathbb{Z},\mathbb{R}^n)} &\leq v_g^{-\frac{1}{\rho}} \|f\|_{L^\rho(\mathbb{R}/l\mathbb{Z},\mathbb{R}^n)} \qquad \textit{and} \\ [f\circ g]_{s,\rho,\mathbb{R}/L\mathbb{Z}} &\leq \|g'\|_{C^0}^{\frac{1}{\rho}+s} v_g^{-\frac{2}{\rho}} [f]_{s,\rho,\mathbb{R}/l\mathbb{Z}}. \end{aligned}$$

Proof. The first bound is the simpler one. We only need to use the change of variables formula and an estimate:

$$\begin{aligned} \int_{\mathbb{R}/L\mathbb{Z}} |f(g(x))|^\rho \,\mathrm{d}x &= \int_{\mathbb{R}/L\mathbb{Z}} |f(g(x))|^\rho \frac{|g'(x)|}{|g'(x)|} \,\mathrm{d}x \leq v_g^{-1} \int_{\mathbb{R}/L\mathbb{Z}} |f(g(x))|^\rho |g'(x)| \,\mathrm{d}x \\ &= v_g^{-1} \int_{\mathbb{R}/l\mathbb{Z}} |f(y)|^\rho \,\mathrm{d}y. \end{aligned}$$

For the second estimate, the idea is similar, but we first need to bound the denominator. In order to do this, we first prove that $|g(x)-g(v)|_{\mathbb{R}/l\mathbb{Z}} \leq \|g'\|_{C^0} |x-v|_{\mathbb{R}/L\mathbb{Z}}$. Seeing as we consider the l-periodic distance on the left-hand side and $g(y+L) = g(y)+l$, we may replace y by $y+kL$ for any integer k. Then,

$$\begin{aligned} |g(x)-g(y)|_{\mathbb{R}/l\mathbb{Z}} &\leq |g(x)-g(y+kL)| \\ &= \Big|\int_x^{y+kL} g'(\tau)\,\mathrm{d}\tau\Big| \leq |y+kL-x| \,\|g'\|_{C^0}. \end{aligned}$$

Taking the minimum over all $k \in \mathbb{Z}$ yields the estimate.

With this and another change of variables, we may calculate

$$\begin{aligned} &\iint_{[0,L]^2} \frac{|f(g(x))-f(g(y))|^\rho}{|x-y|_{\mathbb{R}/L\mathbb{Z}}^{1+s\rho}} \,\mathrm{d}y\,\mathrm{d}x \\ &\leq \|g'\|_{C^0}^{1+s\rho} \iint_{[0,L]^2} \frac{|f(g(x))-f(g(y))|^\rho}{|g(x)-g(y)|_{\mathbb{R}/l\mathbb{Z}}^{1+s\rho}} \,\mathrm{d}y\,\mathrm{d}x \\ &\leq \|g'\|_{C^0}^{1+s\rho} \iint_{[0,L]^2} \frac{|f(g(x))-f(g(y))|^\rho}{|g(x)-g(y)|_{\mathbb{R}/l\mathbb{Z}}^{1+s\rho}} \frac{|g'(x)||g'(y)|}{v_g^2} \,\mathrm{d}y\,\mathrm{d}x \\ &= \|g'\|_{C^0}^{1+s\rho} v_g^{-2} \iint_{[g^{-1}(0),g^{-1}(L)]^2} \frac{|f(\tilde{x})-f(\tilde{y})|^\rho}{|\tilde{x}-\tilde{y}|_{\mathbb{R}/l\mathbb{Z}}^{1+s\rho}} \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x}. \end{aligned}$$

Note that our assumptions on g imply its surjectivity. Finally, we may use that $g^{-1}(L) = g^{-1}(0)+l$ and periodicity to see that the double integral on the right-hand side is indeed the desired seminorm. □

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