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A fractal uncertainty principle for Bergman spaces and analytic wavelets ☆

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ABSTRACT

Motivated by results of Dyatlov on Fourier uncertainty principles for Cantor sets and by similar results of Knutsen for joint time-frequency representations (i.e., the short-time Fourier transform (STFT) with a Gaussian window, equivalent to Fock spaces), we suggest a general setting relating localization and uncertainty and prove, within this context, an uncertainty principle for Cantor sets in Bergman spaces on the unit disk, where the Cantor set is defined as a union of annuli that are equidistributed in the hyperbolic measure. The result can be written in terms of analytic Cauchy wavelets. As in the case of the STFT considered by Knutsen, our result consists of a two-sided bound for the norm of a localization operator involving the fractal dimension $\log 2/\log 3$ in the exponent. As in the STFT case and in Dyatlov's fractal uncertainty principle, the (hyperbolic) measure of the dilated iterates of the Cantor set in the disk tends to infinity, while the corresponding norm of the localization operator tends to zero.

1. Introduction

1.1. Fractal uncertainty principles for the Fourier transform

The uncertainty principle is a collection of statements in harmonic analysis, each of them quantifying in some form the fundamental duality between a function f and its Fourier transform $\mathcal{F}f$, which prevents

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both representations from being "simultaneously concentrated in small sets" [25]. Let us consider the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx \quad \text{for} \quad f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

and the h-dilated Fourier transform $\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$\mathcal{F}_h f(\xi) = h^{-1/2} \cdot \mathcal{F}f(h^{-1}\xi).$$

Several mathematical manifestations of the uncertainty principle consist of bounds on the norm of the operator which concentrates the energy of f in a set X and the energy of $\mathcal{F}f$ in a set Y. Following Dyatlov's definition in [9], one can resort to the h-dilated Fourier transform, and declare a pair of real h-dependent sets $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$ to satisfy an uncertainty principle with exponent $\beta > 0$ if, as $h \to 0$,

$$\left\|1_X \mathcal{F}_h 1_Y\right\|_{op} = O(h^\beta) \to 0, \tag{1.1}$$

where the operator $T = 1_X \mathcal{F}_h 1_Y$ acts via $Tf = 1_X \cdot \mathcal{F}_h[1_Y\, f]$ with 1_X denoting the indicator function of the set X. For instance, if $X, Y = [0, h]$ then the Hölder inequality, combined with the estimate $\|\mathcal{F}_h f\|_{L^\infty} \le h^{-1/2} \|f\|_{L^1}$ gives the uncertainty principle

$$\left\|1_X \mathcal{F}_h 1_Y\right\|_{op} = O(h^{\frac{1}{2}}) \to 0. \tag{1.2}$$

Given $R > 0$, set $h = R^{-2}$. Then

$$\left\|1_X \mathcal{F}_h 1_Y\right\|_{op} = \left\|1_{X/\sqrt{h}} \mathcal{F} 1_{Y/\sqrt{h}}\right\|_{op} = \left\|1_{RX} \mathcal{F} 1_{RY}\right\|_{op}. \tag{1.3}$$

Now, (1.2) becomes an uncertainty principle for the Fourier transform and the dilated sets RX and RY: as $R \to \infty$,

$$\left\|1_{RX} \mathcal{F} 1_{RY}\right\|_{op} = O(R^{-1}) \to 0.$$

Moreover, as $R \to \infty$, (1.1) becomes

$$\|1_X \mathcal{F}_h 1_Y\|_{op} = \left\|1_{X/\sqrt{h}} \mathcal{F} 1_{Y/\sqrt{h}}\right\|_{op} = \|1_{RX} \mathcal{F} 1_{RY}\|_{op} = O(R^{-2\beta}) \to 0.$$

We want to emphasize the following aspect of *the fractal uncertainty principle* [9,8,7]: it covers situations where RX and RY are close to having a fractal structure (they depend on R and approach fractals when $R \to \infty$), where their volume approaches ∞ as $R \to \infty$, but nevertheless RX and RY satisfy an uncertainty principle (their operator norm decays like $O(R^{-\beta})$ for some $\beta > 0$). This can roughly be described by saying that *no function can be localized close to a fractal set in both time and frequency.*

For illustration and motivation, consider the dilated real Cantor set $C(R; \mathbb{R}) \subset [0, R]$ defined as

$$C(R; \mathbb{R}) = \bigcap_{n=1}^{\infty} C_n(R; \mathbb{R}) = \bigcap_{n=1}^{\infty} RC_n,$$

where C_n is the n-th step in the iterative construction of the Cantor set, where C_{n+1} is obtained from C_n by noting that C_n is a finite union of intervals, from each of which one removes the middle third to obtain

C_{n+1}; for instance, $C_0 = [0,1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, etc. The set $C_n(R;\mathbb{R})$ thus consists of 2^n disjoint intervals $I_{n,j} \subset [0,R]$, each with measure $\mathrm{vol}(I_{n,j}) = 3^{-n}R$. Taking n such that $\mathrm{vol}(I_{n,j}) \asymp 1/R$, then $3^{n/2} \asymp R$ and

$$\mathrm{vol}(C_n(R;\mathbb{R})) = 2^n \,\mathrm{vol}(I_{n,j}) \asymp R^{-1+2\frac{\ln 2}{\ln 3}} \to \infty,$$

as $R \to \infty$. Using this volume bound and Hölder's inequality as in (1.2), leads to

$$\left\| 1_{C_n(R;\mathbb{R})} \mathcal{F} 1_{C_n(R;\mathbb{R})} \right\|_{op} \lesssim R^{-1+2\frac{\ln 2}{\ln 3}} \to \infty,$$

therefore not enough to assure that $X = Y = C(R;\mathbb{R})$ satisfy an uncertainty principle according to Dyatlov's definition (see also Equation (1.3)). However, from Example 2.6 and Theorems 2.12 and 2.13 in [9] it follows that there exists an exponent $\beta > 0$ such that, as $R \to \infty$, (1.1) holds for $RX, RY = C_n(R;\mathbb{R})$ and $h = R^{-2}$. Since $R \asymp 3^{n/2}$, as $n \to \infty$, also $R \to \infty$ and Equation (1.3) implies that

$$\left\| 1_{C_n(R;\mathbb{R})} \mathcal{F} 1_{C_n(R;\mathbb{R})} \right\|_{op} = O(R^{-2\beta}) \to 0. \tag{1.4}$$

1.2. Uncertainty and localization

The time-frequency localization operator of the previous paragraph suggests a construction in a general Lebesgue space $L^2(\Lambda)$ (with Λ being a metric measure space). If $L^2(\Lambda)$ has a reproducing kernel $K(z,w)$ (for a subspace of $L^2(\Lambda)$), one can define a localization/Toeplitz operator P_Ω mapping the function $f \in L^2(\Lambda)$ to a smooth function $P_\Omega f \in L^2(\Lambda)$ essentially concentrated in a bounded region $\Omega \subset \Lambda$. The operator P_Ω is explicitly defined as

$$(P_\Omega f)(w) = \int_\Omega f(z)\overline{K(z,w)}d\mu(z).$$

In this context, we say that $\Omega = \Omega(R)$ $(R > 0)$ satisfies an uncertainty principle if, as $R \to \infty$,

$$\|P_\Omega\|_{op} = O(R^{-\beta}) \quad \text{for some } \beta > 0.$$

As in the previous paragraph, one can look for bounds of $\|P_\Omega\|_{op}$ when Ω is a fractal set. In [18], the case of the Fock space setting has been considered, in the equivalent formulation provided by the short-time-Fourier transform [12, Chapter 3]. For the analogy with our results, it will be convenient to rephrase the results in the Fock space. This corresponds to the choice of the measure $d\mu(z) = e^{-\pi|z|^2}dz$ on $\Lambda = \mathbb{C}$ (where dz is Lebesgue measure), of the kernel

$$K(z,w) = K_{Fock}(z,w) = e^{\pi z\overline{w}},$$

and of $\Omega \subset \mathbb{C}$ as the n-th iterate $C_n(R;\mathbb{C})$ of the *planar Cantor set*, i.e.,

$$C_n(R;\mathbb{C}) := \{z \in \mathbb{C} \colon |z|^2 \in C_n(R^2;\mathbb{R})\}. \tag{1.5}$$

The set $C_n(R;\mathbb{C})$ is a disjoint union of 2^n annuli $I^{\mathbb{C}}_{n,j}$, each of measure $\mathrm{vol}(I^{\mathbb{C}}_{n,j}(R^2)) = \frac{\pi R^2}{3^n}$, so that for each annulus we consider a $1/3^n$ part of the initial disk with area πR^2. For this measure to be well distributed among $[0, \pi R^2]$, one takes $\mathrm{vol}(I^{\mathbb{C}}_{n,j}(R^2)) = 1/(\pi R^2)$, yielding $(\pi R^2)^2 \asymp 3^n$ and

$$\mathrm{vol}(C_n(R;\mathbb{C})) \asymp R^{-2+4\frac{\ln 2}{\ln 3}} \to \infty,$$

as $R \to \infty$. Moreover, as $R \to \infty$, it is proven in [18, Corollary 4.1] that

$$\left\| P_{C_n(R;\mathbb{C})} \right\|_{op} \asymp R^{-2+2\frac{\ln 2}{\ln 3}} \to 0.$$

Thus, in the Fock case, the measure of the dilates of the iterated Cantor set tends to ∞, while the norm of the operator tends to zero.

Small operator norms facilitate recovery in signal analysis problems [2]. The operator norm of P_Ω for the Fock case discussed above is maximized when Ω is a disk [23], while there exist sets Ω with infinite Lebesgue measure such that the operator norm of P_Ω is arbitrarily small [10]. In the wavelet case, the operator norm is maximized when the localization domain is a pseudohyperbolic disk [24].

We will show in this paper that, in the case of the disk, we have a similar situation: one can define *a Cantor set in the disk, whose hyperbolic measure tends to infinity*, and an associated Toeplitz operator, *whose operator norm tends to zero.*

Our contributions are organized as follows: A disk version of the Cantor set is considered in the next section. In the same section, the main result on the fractal uncertainty principle on Bergman spaces is stated and translated to the language of analytic wavelets. More details and proofs are given in the following two sections, with the most technical estimates delegated to the last section of the paper.

2. Fractal uncertainty principles for the Bergman space

In this paper, we consider the reproducing kernel of the *weighted analytic Bergman space* associated to the measure $dA_\alpha(z) = 2\alpha\,(1-|z|^2)^{2\alpha-1} dA(z)$ on the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where $\alpha \in (0, \infty)$ and where $dA(z) = \frac{dz}{\pi}$, with dz denoting the planar Lebesgue measure. As shown in [13, Pages 4 and 5], this reproducing kernel is explicitly given by

$$\mathcal{K}_{\mathbb{D}}^{\alpha}(z,w) = \frac{1}{(1-z\overline{w})^{2\alpha+1}} = \sum_{n=0}^{\infty} e_n^\alpha(z)\overline{e_n^\alpha(w)}, \tag{2.1}$$

where

$$e_n^\alpha(z) = \sqrt{\gamma_n} z^n, \quad \text{and} \quad \gamma_n = \gamma_n^\alpha := \frac{\Gamma(n+1+2\alpha)}{\Gamma(n+1)\Gamma(1+2\alpha)} = \frac{1}{2\alpha}[B(n+1,2\alpha)]^{-1}.$$

Given this reproducing kernel, we consider the associated localization operator

$$(P^{(\alpha)}_{C_n(R;\mathbb{D})} f)(w) = \int\limits_{C_n(R;\mathbb{D})} f(z)\,\overline{\mathcal{K}_{\mathbb{D}}^{\alpha}(z,w)} dA_\alpha(z), \tag{2.2}$$

where the fractal localization region is now the following disk version of the iterates of the Cantor set

$$C_n(R;\mathbb{D}) := \left\{ z \in \mathbb{D} \ : \ \frac{|z|^2}{1-|z|^2} \in C_n(R;\mathbb{R}) \right\}. \tag{2.3}$$

We will show (see Proposition 3.1) that $C_n(R;\mathbb{D})$ is a disjoint union of 2^n annuli $D_\ell^{(n)}(0,R)$, each of hyperbolic measure $\mu_{\mathbb{D}}\big(D_\ell^{(n)}(0,R)\big) = \frac{R}{3^n}$. Here, the hyperbolic measure $\mu_{\mathbb{D}}$ is given by

$$\mu_{\mathbb{D}}(M) := \int\limits_M (1-|z|^2)^{-2}\, dA(z) \quad \text{for } M \subset \mathbb{D} \text{ Borel measurable}, \tag{2.4}$$

so that the hyperbolic measure of the disk $D(0,r) = \{w \in \mathbb{C} : |w| < r\}$ with $r \in [0,1)$ is given by

$$\mu_{\mathbb{D}}(D(0,r)) = \int_{D(0,r)} (1-|z|^2)^{-2} dA(z) = 2\int_0^r \frac{s}{(1-s^2)^2}\, ds = \frac{r^2}{1-r^2}. \tag{2.5}$$

Thus, as in the previous examples, for this measure to be well distributed among $[0,R]$ we take n such that $\mu_{\mathbb{D}}\big(D_\ell^{(n)}(0,R)\big) \asymp 1/R$, leading to $3^n \asymp R^2$ and to $R^{2\frac{\ln 2}{\ln 3}} \asymp 2^n$, so that

$$\mu_{\mathbb{D}}\big(C_n(R;\mathbb{D})\big) = R \cdot \left(\frac{2}{3}\right)^n \asymp R^{2\frac{\ln 2}{\ln 3}-1} \to \infty,$$

as $R, n \to \infty$. So far, everything is perfectly tuned with our model Fourier and time-frequency/Fock cases. However, the analogue of the conditions (1.4) only holds in the asymptotic case. The bounds on the non-asymptotic case depend on the size of R.

Theorem 2.1. *Given $\alpha \in (0,\infty)$, there are constants $0 < C_1 \leq C_2 < \infty$ (which only depend on α) such that the operator norm of the time-scale localization operator $P^{(\alpha)}_{C_n(R;\mathbb{D})}$ satisfies for all $n \in \mathbb{N}$ and $R \in (0,\infty)$ the estimate*

$$C_1 \begin{cases} \left(\frac{2}{3}\right)^n R, & \text{if } 0 < R \leq 1 \\ \left(\frac{2}{3}\right)^n R^{1-\frac{\ln 2}{\ln 3}}, & \text{if } 1 \leq R \leq 3^n \\ 1, & \text{if } R \geq 3^n \end{cases} \leq \left\| P^{(\alpha)}_{C_n(R;\mathbb{D})} \right\|_{op} \leq C_2 \begin{cases} \left(\frac{2}{3}\right)^n R, & \text{if } 0 < R \leq 1, \\ \left(\frac{2}{3}\right)^n R^{1-\frac{\ln 2}{\ln 3}}, & \text{if } 1 \leq R \leq 3^n, \\ 1, & \text{if } R \geq 3^n. \end{cases}$$

Furthermore, if R is chosen so that $R^2 \asymp 3^n$, then

$$\left\| P^{(\alpha)}_{C_n(R;\mathbb{D})} \right\|_{op} \asymp \left(\frac{2}{3}\right)^{\frac{n}{2}} \asymp R^{\frac{\ln 2}{\ln 3}-1} \to 0,$$

as $R \to \infty$.

We note that the term

$$\delta = \delta_{C(R;\mathbb{R})} = \frac{\ln 2}{\ln 3}$$

appearing in the exponent of $R^{-1+\frac{\ln 2}{\ln 3}}$ is the Hausdorff dimension of the Cantor set. The uncertainty principles in [9] consider more general fractal sets and the results are obtained in terms of their Hausdorff dimensions. See [19,20] for new developments in this direction in the planar case. As in the joint time-frequency case, following the suggestion in the comments after [18, Corollary 4.1], where the bound $R^{-2+2\delta}$ is obtained for the planar Cantor set, this opens interesting problems, if one considers more general fractal sets and seeks bounds of the associated Toeplitz operator in terms of the Hausdorff dimension of the sets.

2.1. Fractal uncertainty principle for analytic wavelets

In this section we outline how our result can be written in terms of analytic wavelets. We will use the basic notation for $\mathcal{H}^2(\mathbb{C}^+)$, the Hardy space in the upper half plane $\mathbb{C}^+$, as the space of analytic functions $f : \mathbb{C}^+ \to \mathbb{C}$ such that

$$\sup_{0<s<\infty} \int_{-\infty}^{\infty} |f(x+is)|^2 dx < \infty.$$

To simplify the computations it is often convenient to use the equivalent definition (since the Paley-Wiener theorem (see [6] or [27, Theorem 19.2]) shows up to canonical identifications that $\mathcal{F}(\mathcal{H}^2(\mathbb{C}^+)) = L^2(0,\infty)$)

$$\mathcal{H}^2(\mathbb{C}^+) = \left\{ f \in L^2(\mathbb{R}) : (\mathcal{F}f)(\xi) = 0 \text{ for almost all } \xi < 0 \right\}.$$

The *wavelet transform* of a function $f \in \mathcal{H}^2(\mathbb{C}^+)$ with *mother wavelet* $\psi \in \mathcal{H}^2(\mathbb{C}^+)$, such that its *admissibility constant* $C_\psi = 2\pi \cdot \|\mathcal{F}\psi\|^2_{L^2(\mathbb{R}^+, t^{-1}dt)}$ is finite, is defined as

$$W_\psi f(z) := \int_{\mathbb{R}} f(t)\overline{s^{-\frac{1}{2}}\psi(s^{-1}(t-x))}dt = \sqrt{s}\int_0^\infty \widehat{f}(\xi)\overline{\widehat{\psi}(s\xi)}e^{ix\xi}d\xi, \quad z = x + is \in \mathbb{C}^+. \tag{2.6}$$

The *analytic wavelets* are the functions ψ_0^α defined via their Fourier transforms by

$$(\mathcal{F}\psi_0^\alpha)(\xi) = \xi^{\frac{\alpha}{2}} \cdot e^{-\xi} \cdot 1_{(0,\infty)}(\xi), \quad \xi \in \mathbb{R}.$$

As proven recently in [14], $W_\psi f(z)$ leads to analytic (Bergman) phase spaces only for this special choice of ψ (up to a phase factor). Nevertheless, it is customary to refer to $W_\psi f(z)$ as the analytic wavelet transform, due to the discard of negative frequencies. We will write

$$d\mu^+(z) = (\mathrm{Im} z)^{-2}\, dz,$$

where dz is the Lebesgue measure on $\mathbb{C}^+$. The orthogonality relations for the wavelet transform

$$\int_{\mathbb{C}^+} W_{\psi_1} f_1(z)\overline{W_{\psi_2} f_2(z)}d\mu^+(z) = 2\pi \cdot \langle \mathcal{F}\psi_1, \mathcal{F}\psi_2\rangle_{L^2(\mathbb{R}^+, t^{-1}dt)} \cdot \langle f_1, f_2\rangle_{\mathcal{H}^2(\mathbb{C}^+)} \tag{2.7}$$

are valid for all $f_1, f_2 \in \mathcal{H}^2(\mathbb{C}^+)$ and all admissible $\psi_1, \psi_2 \in \mathcal{H}^2(\mathbb{C}^+)$; see [4, Proposition 2.4.1]. Then, setting $\psi_1 = \psi_2 = \psi$ and $f_1 = f_2$ in (2.7), gives

$$\int_{\mathbb{C}^+} |W_\psi f(z)|^2\, d\mu^+(z) = C_\psi \cdot \|f\|^2_{\mathcal{H}^2(\mathbb{C}^+)},$$

showing that the continuous wavelet transform provides an (up to a constant factor) isometric inclusion

$$W_\psi : \quad \mathcal{H}^2(\mathbb{C}^+) \to L^2(\mathbb{C}^+, \mu^+).$$

Writing $z = x + is$ and setting $\psi_1 = \psi_2 = \psi$ and $f_2 = \pi(z)\psi$, where

$$[\pi(z)]\psi(t) := s^{-\frac{1}{2}} \cdot \psi(s^{-1}(t-x)),$$

in (2.7), then for every $f \in \mathcal{H}^2(\mathbb{C}^+)$, one has

$$W_\psi f(z) = \frac{1}{C_\psi}\int_{\mathbb{C}^+} W_\psi f(w)\langle \pi(w)\psi, \pi(z)\psi\rangle d\mu^+(w), \quad z \in \mathbb{C}^+.$$

Thus, the range of the wavelet transform

$$\mathcal{W}_\psi := \left\{ F \in L^2(\mathbb{C}^+, \mu^+) \ : \ F = W_\psi f,\ f \in \mathcal{H}^2(\mathbb{C}^+) \right\}$$

is a closed subspace of $L^2(\mathbb{C}^+, \mu^+)$, with reproducing kernel

$$k_\psi(z,w) = \frac{1}{C_\psi}\langle \pi(w)\psi, \pi(z)\psi\rangle_{\mathcal{H}^2(\mathbb{C}^+)} = \frac{1}{C_\psi} W_\psi\psi(w^{-1}z), \quad \text{and} \quad k_\psi(z,z) = \frac{\|\psi\|_2^2}{C_\psi}.$$

Here, the multiplication (and inversion) on $\mathbb{C}^+$ is *not* the usual multiplication inherited from $\mathbb{C}$, but stems from identifying $\mathbb{C}^+ \cong \mathbb{R} \times (0, \infty)$ with the $ax + b$ group, so that $(x + is)(y + iv) = x + sy + isv$.

For $z = x + is, w = y + iv \in \mathbb{C}^+$, the kernel $k_{\psi_0^{2\alpha}}(z,w)$ is given by[1]

$$k_{\psi_0^{2\alpha}}(z,w) = \frac{1}{C_{\psi_0^{2\alpha}}} W_{\psi_0^{2\alpha}}\psi_0^{2\alpha}(w^{-1}\cdot z) = \frac{2^{2\alpha}\alpha}{\pi} \cdot \left(\frac{\sqrt{\operatorname{Im} z \operatorname{Im} w}}{-i(z-\overline{w})}\right)^{2\alpha+1}. \tag{2.8}$$

This is a multiple of the reproducing kernel of the Bergman space

$$A^{2\alpha-1}(\mathbb{C}^+) := \left\{ f : \mathbb{C}^+ \to \mathbb{C} \text{ holomorphic } : \int_{\mathbb{C}^+} |f(z)|^2 \cdot \operatorname{Im}(z)^{2\alpha-1} dz < \infty \right\}.$$

Therefore, $(\operatorname{Im}\cdot)^{-\alpha-1/2}\mathcal{W}_{\psi_0^{2\alpha}} : \mathcal{H}^2(\mathbb{C}^+) \to A^{2\alpha-1}(\mathbb{C}^+)$ is an (up to a constant factor) isometric inclusion. Moreover, $A^{2\alpha-1}(\mathbb{C}^+)$ is conformally equivalent to the Bergman space $A^{2\alpha-1}(\mathbb{D})$ on the unit disk under the transformation

$$z \in \mathbb{C}^+ \mapsto \xi(z) = \frac{z-i}{z+i} \in \mathbb{D};$$

see [6]. It follows that $P^{(\alpha)}_{C_n(R;\mathbb{D})}$ can be unitarily mapped to (a constant multiple of) the operator

$$\big(P_{\xi^{-1}(C_n(R;\mathbb{D}))} f\big)(w) = \int_{\xi^{-1}(C_n(R;\mathbb{D}))} f(z) \cdot \overline{k_{\psi_0^{2\alpha}}(z,w)} dz. \tag{2.9}$$

Thus, Theorem 2.1 is equivalent to an uncertainty principle for wavelet representations on the Cantor set of $\mathbb{C}^+$ defined by $\xi^{-1}(C_n(R;\mathbb{D}))$. The analyzing wavelets $\psi_0^{2\alpha}$ are the only ones leading to an analytic structure [14]. Considering more general classes of analyzing wavelets, as those leading to a polyanalytic decomposition in [29,1,15], or the slightly different ones connected to Maass forms and hyperbolic Landau levels [22,21], may be a natural extension of the problem we have considered here. The Toeplitz operator (2.9) has been first considered by Daubechies and Paul [3] for $n = 0$, i.e., for $C_0(R;\mathbb{D})$ The approach based on double orthogonality that we use is due to Seip [28] and provides some insight on why the approach using circular symmetric sets in the plane and in the disk considerably simplifies the problem. With square time-frequency regions, as required by the Fourier transform approach of Dyatlov, one has no access to explicit eigenvalue formulas. The general eigenvalue problem of Gabor and wavelet localization operators has been considered in [5].

[1] To see this, first note that $\int_0^\infty x^{z-1}e^{-sx}\,dx = \frac{\Gamma(z)}{s^z}$ for all $s, z \in \mathbb{C}$ with $\operatorname{Re} s, \operatorname{Re} z > 0$; this can be seen by keeping z fixed and verifying the identity for $s \in (0, \infty)$; this is enough, since both sides of the identity are holomorphic functions. Combining this formula with the definition of $\psi_0^{2\alpha}$ and the right-hand side of Equation (2.6) easily implies $W_{\psi_0^{2\alpha}}\psi_0^{2\alpha}(x+is) = s^{\frac{1}{2}+\alpha}\int_0^\infty \xi^{2\alpha}e^{-(1+s-ix)\xi}\,d\xi = s^{\frac{1}{2}+\alpha}\frac{\Gamma(2\alpha+1)}{(1+s-ix)^{2\alpha+1}}$. Furthermore, directly from the definition of C_ψ and of $\psi_0^{2\alpha}$, we see $C_{\psi_0^{2\alpha}} = 2\pi\int_0^\infty \xi^{2\alpha}e^{-2\xi}\frac{d\xi}{\xi} = \frac{\pi}{2^{2\alpha-1}}\Gamma(2\alpha)$. From this, Equation (2.8) follows easily.

3. The Cantor set

3.1. The Cantor set in the line

The usual Cantor set $C \subset [0,1]$ is defined as $C = \bigcap_{n=1}^{\infty} C_n$, where each of the sets C_n is a finite union of closed intervals which are iteratively constructed by the usual operation of "removing the middle third" of each of the intervals; see e.g. [26, Section 2.44]. For instance,

$$C_0 = [0,1], \qquad C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1], \quad \text{and} \quad C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

Now, given $R > 0$, we consider the dilated real Cantor set $C(R;\mathbb{R}) = R \cdot C \subset [0,R]$. Therefore, we have

$$C(R;\mathbb{R}) = \bigcap_{n=1}^{\infty} C_n(R;\mathbb{R}),$$

where

$$C_n(R;\mathbb{R}) := R \cdot C_n \quad \text{with} \quad C_n = \biguplus_{a \in \{0,2\}^n} \left[\sum_{j=1}^{n} \frac{a_j}{3^j}, \quad 3^{-n} + \sum_{j=1}^{n} \frac{a_j}{3^j} \right]. \tag{3.1}$$

To simplify the notation, let $\Omega^{(n)} := \{0,2\}^n$, and for $a = (a_1, \dots, a_n) \in \Omega^{(n)}$ define

$$b_a = b_a^{(n)} := \sum_{j=1}^{n} \frac{a_j}{3^j},$$

so that

$$C_n = \biguplus_{a \in \{0,2\}^n} I_a^{(n)} \qquad \text{with} \qquad I_a^{(n)} := \left[b_a^{(n)}, 3^{-n} + b_a^{(n)} \right]. \tag{3.2}$$

For analyzing the size of the endpoints b_a, it is convenient to introduce for $a \in \Omega^{(n)} \setminus \{0\}$ the *order* of its index as

$$\omega_a := \min \big\{ \ell \in \{1, \dots, n\} : a_\ell \neq 0 \big\},$$

which gives rise to the restricted index sets

$$\Omega_m^{(n)} := \Big\{ a \in \Omega^{(n)} \setminus \{0\} \quad : \quad \omega_a = m \Big\} \quad \text{for} \quad m \in \{1, \dots, n\}.$$

It is straightforward to verify that $\Omega_m^{(n)} = \{0\}^{m-1} \times \{2\} \times \{0,2\}^{n-m}$, and hence

$$\# \Omega_m^{(n)} = 2^{n-m}. \tag{3.3}$$

We will frequently use the estimate

$$\frac{2}{3^m} \leq b_a = \sum_{j=m}^{n} \frac{a_j}{3^j} \leq \frac{2}{3^m} \sum_{\ell=0}^{\infty} 3^{-\ell} = \frac{3}{3^m} \qquad \forall\, a \in \Omega_m^{(n)}. \tag{3.4}$$

3.2. The Cantor set in the plane

The key properties of the Cantor-type set defined in the complex plane in [18] will be preserved in the construction of the next subsection (modulo the required adaptations to the disk). First, since $C_n(R^2;\mathbb{R})$ is a disjoint union of 2^n intervals, the set $C_n(R;\mathbb{C})$ from (1.5) is a disjoint union of 2^n annuli. Furthermore, the condition $|z|^2 \in C_n(R^2;\mathbb{R})$ is chosen since it ensures that all of the 2^n annuli have the same Lebesgue measure.

3.3. The Cantor set in the disk

Our goal is to define the n-th iterate of the Cantor-type set in the disk such that it is a union of 2^n disjoint annuli with the same hyperbolic measure, where we saw in Equation (2.5) that

$$\mu_{\mathbb{D}}(D(0,r)) = \frac{r^2}{1-r^2} \quad \text{for} \quad D(0,r) = \{w \in \mathbb{C} : |w| < r\} \text{ and } r \in [0,1).$$

This suggests defining the n-th Cantor set as the set of all $w \in \mathbb{D}$ such that $\varphi(|w|) \in C_n(R;\mathbb{R})$, where

$$\varphi : \quad [0,1) \to [0,\infty), \quad r \mapsto \varphi(r) = \frac{r^2}{1-r^2}.$$

More explicitly, as in (2.3), we define

$$C_n(R;\mathbb{D}) := \left\{ w \in \mathbb{D} \ : \ \frac{|w|^2}{1-|w|^2} \in C_n(R;\mathbb{R}) \right\}.$$

We next show that this construction indeed yields a set $C_n(R;\mathbb{D})$ that behaves similarly to the set $C_n(R;\mathbb{C})$ from [18] discussed in Sections 3.2 and 1.2. For $a \in \Omega^{(n)}$, let us write

$$D_a^{(n)}(0,R) := \left\{ w \in \mathbb{D} \ : \ \varphi(|w|) \in R \cdot I_a^{(n)} \right\} \quad \text{with} \quad I_a^{(n)} = [b_a,\ 3^{-n} + b_a] \text{ as in (3.2).} \tag{3.5}$$

Proposition 3.1. *The n-th disk Cantor set $C_n(R;\mathbb{D})$ is a disjoint union of 2^n annuli:*

$$C_n(R;\mathbb{D}) = \biguplus_{a \in \Omega^{(n)}} D_a^{(n)}(0,R) \qquad \textit{with} \qquad D_a^{(n)}(0,R) \ \textit{as in (3.5)}.$$

Each annulus has hyperbolic measure $\mu_{\mathbb{D}}(D_a^{(n)}(0,R)) = \frac{R}{3^n}$. Therefore,

$$\mu_{\mathbb{D}}\left(C_n(R;\mathbb{D})\right) = \left(\frac{2}{3}\right)^n \cdot R.$$

Proof. Recall the definition (2.4) of the hyperbolic measure $\mu_{\mathbb{D}}$. Next, given an arbitrary measurable function $F : [0,\infty) \to [0,\infty]$, introduce polar coordinates, set $s = r^2$ and then $t = \frac{s}{1-s}$ to yield

$$\begin{aligned} \int_{\mathbb{D}} F\left(\frac{|z|^2}{1-|z|^2}\right) d\mu_{\mathbb{D}}(z) &= 2\int_0^1 F\left(\frac{r^2}{1-r^2}\right) \frac{r}{(1-r^2)^2}\, dr \\ &= \int_0^1 F\left(\frac{s}{1-s}\right)(1-s)^{-2}\, ds = \int_0^\infty F(t)\, dt. \end{aligned}$$

Since $C_n(R;\mathbb{R}) = \biguplus_{a\in\Omega^{(n)}} R\cdot I_a^{(n)}$, where the intervals $I_a^{(n)} = [b_a, 3^{-n}+b_a] \subset [0,\infty)$ have length $|I_a^{(n)}| = 3^{-n}$, then

$$C_n(R;\mathbb{D}) = \biguplus_{a\in\Omega^{(n)}} D_a^{(n)}(0,R) = \biguplus_{a\in\Omega^{(n)}} \left\{w\in\mathbb{D} : \varphi^{-1}(R\cdot b_a) \le |w| \le \varphi^{-1}(R\cdot(3^{-n}+b_a))\right\}$$

is a disjoint union of 2^n annuli, each of which has $\mu_{\mathbb{D}}$-measure

$$\mu_{\mathbb{D}}\left(D_a^{(n)}(0,R)\right) = \int_{\mathbb{D}} 1_{R\cdot I_a^{(n)}}\left(|z|^2/(1-|z|^2)\right) d\mu_{\mathbb{D}}(z)$$
$$= \int_0^\infty 1_{R\cdot I_a^{(n)}}(t)\, dt = \frac{R}{3^n}. \quad \square$$

4. Localization to the Cantor set in the disk

In this section, we study the spectral properties of the time-scale localization operators

$$P^{(\alpha)}_{C_n(R;\mathbb{D})} \quad \text{for} \quad R\in(0,\infty),\ n\in\mathbb{N},\ \text{and}\ \alpha\in(0,\infty),$$

as defined in (2.2).

4.1. Eigenvalues of the Cantor-type localization operator

Proposition 4.1. *Let $\alpha, R\in(0,\infty)$ and $n\in\mathbb{N}$. Then the eigenvalues of the operator $P^{(\alpha)}_{C_n(R;\mathbb{D})}$ introduced in Equation (2.2) are given by*

$$\lambda_k = \lambda_k^{(\alpha)}\left(C_n(R;\mathbb{D})\right) = \int_0^\infty g_k(t;\alpha)\, 1_{C_n(R;\mathbb{R})}(t)\, dt \quad \textit{for} \quad k\in\mathbb{N}_0,$$

where $g_k(\cdot\,;\alpha) : (0,\infty)\to(0,\infty)$ is given by

$$g_k(t;\alpha) = [B(k+1,2\alpha)]^{-1}\cdot\left(\frac{t}{1+t}\right)^k\cdot(1+t)^{-(1+2\alpha)}. \tag{4.1}$$

Proof. Observe that the orthogonality of the basis functions $e_k^\alpha(z) = \sqrt{\gamma_k} z^k$ in the reproducing kernel expansion (2.1) holds for any radial measure $\mu(|z|)dA(z)$, with $\mu : [0,\infty)\to[0,\infty)$ satisfying $\int_0^1 \mu(r)dr < \infty$. This follows from the following calculation, for $k\neq\ell$:

$$\int_{\mathbb{D}} z^k\,\overline{z^\ell}\,\mu(|z|)dA(z) = \frac{1}{\pi}\int_0^1\int_0^{2\pi}(re^{i\theta})^k(re^{-i\theta})^\ell\mu(|re^{i\theta}|)rd\theta dr$$
$$= \frac{1}{\pi}\int_0^1 r^{k+\ell+1}\mu(r)\int_0^{2\pi} e^{i\theta(n-m)}d\theta dr = 0.$$

Thus, we can set

$$\mu(|z|) = 2\alpha\cdot(1-|z|^2)^{2\alpha-1}\cdot 1_{C_n(R;\mathbb{D})}(z) = 2\alpha\cdot(1-|z|^2)^{2\alpha-1}1_{C_n(R;\mathbb{R})}\left(\frac{|z|^2}{1-|z|^2}\right)$$

and combine (2.1) with the orthogonality of the $e_k^\alpha(z)$ with respect to $\mu(|z|)dA(z)$ to obtain[2]

$$\left[P^{(\alpha)}_{C_n(R;\mathbb{D})}(e_k^\alpha)\right](w) = \int_{\mathbb{D}} e_k^\alpha(z)\overline{\mathcal{K}^\alpha_{\mathbb{D}}(z,w)}\,\mu(|z|)dA(z) = \left[\int_{\mathbb{D}} |e_k^\alpha(z)|^2\,\mu(|z|)dA(z)\right] e_k^\alpha(w).$$

Thus, e_k^α is an eigenfunction of $P^{(\alpha)}_{C_n(R;\mathbb{D})}$ with eigenvalue

$$\begin{aligned}\lambda_k^{(\alpha)}(C_n(R;\mathbb{D})) &= \int_{\mathbb{D}} |e_k^\alpha(z)|^2\,\mu(|z|)dA(z) \\ &= 2\alpha\gamma_k \int_{\mathbb{D}} |z|^{2k}\cdot(1-|z|^2)^{2\alpha-1}\cdot \mathbb{1}_{C_n(R;\mathbb{R})}\left(\frac{|z|^2}{1-|z|^2}\right) dA(z) \\ &= 4\alpha\gamma_k \int_0^1 \mathbb{1}_{C_n(R;\mathbb{R})}\left(\frac{r^2}{1-r^2}\right)\cdot r^{2k+1}\cdot(1-r^2)^{2\alpha-1}\,dr.\end{aligned}$$

Using the substitutions $s = r^2$ and $t = \frac{s}{1-s} = \frac{1}{1-s} - 1$, we finally see

$$\begin{aligned}\lambda_k^{(\alpha)}(C_n(R;\mathbb{D})) &= 2\alpha\gamma_k \int_0^1 \mathbb{1}_{C_n(R;\mathbb{R})}\left(\frac{s}{1-s}\right)\cdot s^k\cdot(1-s)^{2\alpha-1}\,ds \\ &= 2\alpha\gamma_k \int_0^\infty \mathbb{1}_{C_n(R;\mathbb{R})}(t)\cdot\left(\frac{t}{1+t}\right)^k\cdot(1+t)^{-(2\alpha+1)}\,dt \\ &= \int_0^\infty \mathbb{1}_{C_n(R;\mathbb{R})}(t)\cdot g_k(t;\alpha)\,dt. \quad \square\end{aligned}$$

Remark 4.2. We observe that the function $g_k(\cdot\,;\alpha)$ is the density function of the *Beta prime distribution* $\beta'(k+1,2\alpha)$ with form parameters $k+1$ and 2α; see [17, Equation (25.79)]. In particular, this implies

$$\int_0^\infty g_k(t;\alpha)\,dt = 1 \tag{4.2}$$

and the following probabilistic interpretation of the eigenvalues

$$\lambda_k^{(\alpha)}(C_n(R;\mathbb{D})) = \mathbb{E}\left(\mathbb{1}_{C_n(R;\mathbb{R})}(X)\right) \quad \text{if } X \sim \beta'(k+1,2\alpha).$$

Recalling the definition of $g_k(\cdot\,;\alpha)$ and applying [11, Section 3.197.8] (with $\mu = 1$, $\nu = k+1$, $\alpha = 1$, $\lambda = -(k+1+2\alpha)$ and $u = y$) shows that the cumulative distribution function of $\beta'(k+1,2\alpha)$ is given by

$$F_{k,\alpha}(y) = \int_0^y g_k(t;\alpha)\,dt = \frac{y^{1+k}}{(1+k)\cdot B(1+k,2\alpha)}\cdot {}_2F_1(k+1+2\alpha,k+1;k+2;-y),$$

[2] Here, we use that the series for $\mathcal{K}^\alpha_{\mathbb{D}}(\cdot,w)$ given in (2.1) is a power series which is convergent on $\mathbb{D}$ and thus converges locally on $\mathbb{D}$. Since the measure μ has compact support in $\mathbb{D}$ (since $C_n(R;\mathbb{D}) \subset \mathbb{D}$ is compact), this allows to interchange the series with the integral.

in terms of the ordinary hypergeometric function ${}_2F_1(a,b,c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{j!(c)_j} z^j$. When $n = 0$, then $1_{C_0(R;\mathbb{R})} = 1_{[0,R]}$ and thus

$$\lambda_k^{(\alpha)}(C_0(R;\mathbb{D})) = \frac{R^{1+k}}{(1+k)\cdot B(1+k,2\alpha)} \cdot {}_2F_1(k+1+2\alpha, 1+k; k+2; -R).$$

4.2. Upper bounding the eigenvalues

The upper bounds for the eigenvalues depend on the following pointwise estimate for the density functions $g_k(\cdot\,;\alpha)$. The details of the proof are in Section 5.

Lemma 4.3. *For each $\alpha \in (0,\infty)$, there is a constant $C = C(\alpha) > 0$ satisfying*

$$g_k(x;\alpha) \le C \cdot (1+x)^{-1} \qquad \forall\, x \in (0,\infty) \text{ and } k \in \mathbb{N}_0.$$

In this section, we prove the following upper bound for the eigenvalues $\lambda_k^{(\alpha)}(C_n(R;\mathbb{D}))$:

Proposition 4.4. *For each $\alpha \in (0,\infty)$, there is a constant $C = C(\alpha) > 0$ satisfying*

$$\lambda_k^{(\alpha)}(C_n(R;\mathbb{D})) \le C \begin{cases} (2/3)^n \cdot R, & \text{if } 0 < R \le 1, \\ (2/3)^n \cdot R^{1-\frac{\ln 2}{\ln 3}}, & \text{if } 1 \le R \le 3^n, \\ 1, & \text{if } R \ge 3^n \end{cases}$$

for all $k \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $R \in (0,\infty)$.

Proof. First of all, recall from Equation (4.2) that each $g_k(\cdot\,;\alpha)$ is a probability density function on $(0,\infty)$, so that

$$\lambda_k^{(\alpha)}(C_n(R;\mathbb{D})) = \int_{C_n(R;\mathbb{R})} g_k(y;\alpha)\,dy \le \int_0^\infty g_k(y;\alpha)\,dy = 1.$$

This establishes the desired estimate in case of $R \ge 3^n$. Next, recall that the Lebesgue measure $\lambda(C_n(R;\mathbb{R}))$ of the Cantor set $C_n(R;\mathbb{R})$ is $\lambda(C_n(R;\mathbb{R})) = (2/3)^n \cdot R$. Furthermore, Lemma 4.3 yields a constant $C = C(\alpha) > 0$ satisfying $g_k(x;\alpha) \le C \cdot (1+x)^{-1} \le C$ for all $x \in (0,\infty)$ and $k \in \mathbb{N}_0$. Therefore,

$$\lambda_k^{(\alpha)}(C_n(R;\mathbb{D})) = \int_{C_n(R;\mathbb{R})} g_k(y;\alpha)\,dy \le C \cdot \lambda(C_n(R;\mathbb{R})) = C \cdot (2/3)^n \cdot R.$$

This proves the desired estimate for the case that $0 < R \le 1$. Finally, let us consider the case $1 \le R \le 3^n$. Due to this assumption on R, there is $t \in \{0,\ldots,n-1\}$ such that

$$3^t \le R \le 3^{t+1} \quad \text{and hence} \quad \frac{\ln R}{\ln 3} - 1 \le t \le \frac{\ln R}{\ln 3}.$$

Using the representation (3.1) of the Cantor set $C_n(R;\mathbb{R})$ and $\Omega^{(n)} \setminus \{0\} = \biguplus_{m=1}^n \Omega_m^{(n)}$, we see

$$\lambda_k^{(\alpha)}(C_n(R;\mathbb{D})) = \int_0^{R/3^n} g_k(y;\alpha)\,dy + \sum_{m=1}^n \sum_{a \in \Omega_m^{(n)}} \int_{R\cdot b_a}^{R\cdot(b_a+3^{-n})} g_k(y;\alpha)\,dy.$$

First, note as a consequence of $g_k(x;\alpha) \le C \cdot (1+x)^{-1} \le C$ that $\int_0^{R/3^n} g_k(y;\alpha)\,dy \le C\frac{R}{3^n}$. Next, note that if $a \in \Omega_m^{(n)}$, then (3.4) implies $b_a \ge 2/3^m \ge 3^{-m}$ and $R \cdot b_a \ge 3^{t-m}$. This implies for $y \in [R \cdot b_a, R \cdot (b_a + 3^{-n})]$ that

$$(1+y)^{-1} \le (1+3^{t-m})^{-1} \le 3^{-(t-m)_+} \quad \text{where} \quad x_+ := \max\{0,x\},$$

and hence

$$\int_{R\cdot b_a}^{R\cdot(b_a+3^{-n})} g_k(y;\alpha)\,dy \le C \int_{R\cdot b_a}^{R\cdot(b_a+3^{-n})} (1+y)^{-1}\,dy \le C \cdot \frac{R}{3^n} \cdot 3^{-(t-m)_+}.$$

Recalling, from (3.3) that $\#\Omega_m^{(n)} = 2^{n-m}$, we conclude

$$\sum_{m=1}^{n} \sum_{a \in \Omega_m^{(n)}} \int_{R\cdot b_a}^{R\cdot(b_a+3^{-n})} g_k(y;\alpha)\,dy \le C \cdot (2/3)^n \cdot R \cdot 2^{-t} \sum_{m=1}^{n} 2^{t-m} 3^{-(t-m)_+}.$$

Introducing the new summation index $\ell = t - m$, we see that

$$\sum_{m=1}^{n} 2^{t-m} 3^{-(t-m)_+} \le \sum_{\ell \in \mathbb{Z}} 2^{\ell} 3^{-\ell_+} = \sum_{\ell=-\infty}^{-1} 2^{\ell} + \sum_{\ell=0}^{\infty} (2/3)^{\ell} = 1 + \frac{1}{1-\frac{2}{3}} = 4.$$

Furthermore, since $t \ge \frac{\ln R}{\ln 3} - 1$, we have $2^{-t} \le 2 \cdot 2^{-\ln R/\ln 3} = 2 \cdot R^{-\ln 2/\ln 3}$. By combining the estimates that we collected, we finally conclude that

$$\lambda_k^{(\alpha)}(C_n(R;\mathbb{D})) \le C\frac{R}{3^n} + 8C \cdot (2/3)^n \cdot R^{1-\frac{\ln 2}{\ln 3}}.$$

It remains to observe that $R/3^n \le (2/3)^n \cdot R^{1-\frac{\ln 2}{\ln 3}}$, which easily follows from the condition $R \le 3^n$. $\square$

4.3. Lower bounding the first eigenvalue

In this subsection we prove that the eigenvalue $\lambda_0^{(\alpha)}(C_n(R;\mathbb{D}))$ of $P_{C_n(R;\mathbb{D})}^{(\alpha)}$ fulfills a lower bound which matches the upper bound from Proposition 4.4.

Proposition 4.5. *For each $\alpha \in (0,\infty)$, there is a constant $C = C(\alpha) > 0$ satisfying*

$$\lambda_0^{(\alpha)}(C_n(R;\mathbb{D})) \ge C \begin{cases} (2/3)^n \cdot R, & \text{if } 0 < R \le 1, \\ (2/3)^n \cdot R^{1-\frac{\ln 2}{\ln 3}}, & \text{if } 1 \le R \le 3^n, \\ 1, & \text{if } R \ge 3^n \end{cases}$$

for all $n \in \mathbb{N}$ and $R \in (0,\infty)$.

Proof. In case of $R \ge 3^n$, we have $C_n(R;\mathbb{R}) \supset [0, R/3^n] \supset [0,1]$, and hence

$$\lambda_0^{(\alpha)}(C_n(R;\mathbb{D})) \ge \int_0^1 g_0(y;\alpha)\,dy =: C_1(\alpha) > 0,$$

since $g_0(\cdot\,;\alpha)$ is a positive continuous function. Likewise, if $0 < R \le 1$, then $C_n(R;\mathbb{R}) \subset [0,1]$. Since the continuous, positive function $g_0(\cdot\,;\alpha)$ is lower bounded on the compact set $[0,1]$ (say, $g_0(x;\alpha) \ge C_2$ for $x \in [0,1]$ with $C_2 = C_2(\alpha) > 0$), we thus see

$$\lambda_0^{(\alpha)}(C_n(R;\mathbb{D})) = \int\limits_{C_n(R;\mathbb{R})} g_0(y;\alpha)\,dy \ge C_2 \int\limits_{C_n(R;\mathbb{R})} 1\,dy = C_2 \cdot (2/3)^n \cdot R,$$

proving the desired bound for the case $0 < R \le 1$. Finally, consider the case $1 \le R \le 3^n$. For brevity, define $C_3 := [B(1,2\alpha)]^{-1}$. Choose $t \in \{0,\dots,n-1\}$ such that $3^t \le R \le 3^{t+1}$, whence $\frac{\ln R}{\ln 3} - 1 \le t \le \frac{\ln R}{\ln 3}$. Using the representation (3.1) of the Cantor set, and observing that $\Omega_{t+1}^{(n)} \subset \Omega^{(n)}$ since $t+1 \in \{1,\dots,n\}$, it follows that

$$\lambda_0^{(\alpha)}(C_n(R;\mathbb{D})) \ge C_3 \sum_{a\in\Omega_{t+1}^{(n)}} \int\limits_{R\cdot b_a}^{R\cdot(b_a+3^{-n})} (1+y)^{-(1+2\alpha)}\,dy.$$

Now, note that if $a \in \Omega_{t+1}^{(n)}$ and $y \le R\cdot(b_a + 3^{-n})$, then Equation (3.4) shows that

$$y \le R\cdot(b_a + 3^{-n}) \le R\cdot(3\cdot 3^{-(t+1)} + 3^{-n}) \le 4R\cdot 3^{-(t+1)} \le 4,$$

whence $(1+y)^{-(1+2\alpha)} \ge 5^{-(1+2\alpha)} =: C_4$. Next, recall from Equation (3.3) that $\#\Omega_{t+1}^{(n)} = 2^{n-t-1}$. Overall, we thus see as desired that

$$\begin{aligned}\lambda_0^{(\alpha)}(C_n(R;\mathbb{D})) &\ge C_3C_4\cdot 2^{n-t-1}\frac{R}{3^n} = \frac{C_3C_4}{2}\cdot(2/3)^n\cdot R\cdot 2^{-t}\\ &\ge \frac{C_3C_4}{2}\cdot(2/3)^n\cdot 2^{-\frac{\ln R}{\ln 3}}\cdot R = \frac{C_3C_4}{2}\cdot(2/3)^n\cdot R^{1-\frac{\ln 2}{\ln 3}}. \quad\square\end{aligned}$$

4.4. Proof of Theorem 2.1

Since $P_{C_n(R;\mathbb{D})}^{(\alpha)} : L^2(dA_\alpha) \to L^2(dA_\alpha)$ is self-adjoint, with $P_{C_n(R;\mathbb{D})}^{(\alpha)} = 0$ on the orthogonal complement of $\operatorname{span}\{e_k^\alpha : k \in \mathbb{N}_0\}$, we see by combining Propositions 4.1, 4.4 and 4.5 that

$$C_1\begin{cases}(\frac{2}{3})^n\cdot R, & \text{if } 0<R\le 1\\ (\frac{2}{3})^n\cdot R^{1-\frac{\ln 2}{\ln 3}}, & \text{if } 1\le R\le 3^n\\ 1, & \text{if } R\ge 3^n\end{cases} \le \left\|P_{C_n(R;\mathbb{D})}^{(\alpha)}\right\|_{op} \le C_2\begin{cases}(\frac{2}{3})^n\cdot R, & \text{if } 0<R\le 1,\\ (\frac{2}{3})^n\cdot R^{1-\frac{\ln 2}{\ln 3}}, & \text{if } 1\le R\le 3^n,\\ 1, & \text{if } R\ge 3^n.\end{cases} \tag{4.3}$$

The condition $3^n \asymp R^2$ implies (for $R \to \infty$) that $1 \ll R \asymp 3^{n/2} \ll 3^n$. The same condition also implies that $R^{\frac{\ln 2}{\ln 3}} \asymp 2^{n/2}$, and thus $R^{2\frac{\ln 2}{\ln 3}} \asymp 2^n$. This implies for $R \to \infty$ that

$$(2/3)^n \cdot R^{1-\frac{\ln 2}{\ln 3}} \asymp (2/3)^{n/2} \asymp R^{-1+\frac{\ln 2}{\ln 3}}$$

and the result follows from (4.3). $\square$

5. Proof of Lemma 4.3

Recall that the goal is to prove that there exists a constant $C = C(\alpha) > 0$ satisfying

$$g_k(x;\alpha) \le C\cdot(1+x)^{-1} \qquad \forall\, x\in(0,\infty) \text{ and } k\in\mathbb{N}_0,$$

where the function

$$g_k(x;\alpha) = [B(k+1,2\alpha)]^{-1} \cdot \left(\frac{x}{1+x}\right)^k \cdot (1+x)^{-(1+2\alpha)}$$

was defined in Equation (4.1).

The proof will be given in three steps. All implied constants will either be absolute constants or constants that only depend on α.

Step 1 *(Estimating the Beta function):* By definition, the Beta function can be written in terms of the Gamma function as $B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$ for $x,y>0$. Furthermore, a precise form of Stirling's formula (see [16]) shows that the Gamma function satisfies

$$\Gamma(x) = \sqrt{2\pi/x} \cdot (x/e)^x \cdot e^{\mu(x)} \quad \text{for} \quad x>0 \quad \text{where} \quad 0 \le \mu(x) \le (12\,x)^{-1}$$

and therefore

$$\Gamma(x) \asymp x^{-1/2} \cdot (x/e)^x \qquad \forall\, x \ge 1,$$

which is the only case that we will need. Indeed, using this estimate, we see for $k \ge 1$ that

$$\begin{aligned}
[B(k+1,2\alpha)]^{-1} &= \frac{\Gamma(k+1+2\alpha)}{\Gamma(2\alpha)\,\Gamma(1+k)} \\
&\asymp \frac{\sqrt{1/(k+1+2\alpha)} \cdot \left(\frac{k+1+2\alpha}{e}\right)^{k+1+2\alpha}}{\Gamma(2\alpha) \cdot \sqrt{1/(k+1)} \cdot \left(\frac{k+1}{e}\right)^{k+1}} \\
&\lesssim \sqrt{\frac{k+1}{k+1+2\alpha}} \cdot \left(\frac{k+1+2\alpha}{e}\right)^{2\alpha} \cdot \left(1+\frac{2\alpha}{k+1}\right)^{k+1} \\
&\overset{(*)}{\lesssim} \left(\frac{k+1+2\alpha}{e}\right)^{2\alpha} \lesssim k^{2\alpha} \quad \text{since } k \ge 1.
\end{aligned}$$

Here, the step marked with $(*)$ used the well-known fact that $(1+\frac{x}{k})^k \xrightarrow[k\to\infty]{} e^x$ for all $x \in \mathbb{R}$, and thus $(1+\frac{2\alpha}{k+1})^{k+1} \lesssim 1$, with the implied constant only depending on α.

We remark that the estimate for the Beta function that we derived in this step is probably well-known. We nevertheless decided to give the relatively easy proof since we could not locate a handy reference.

Step 2 *(Estimating $k^{2\alpha} \cdot y^k$):* Let $y \in (0,1)$ be fixed, and define

$$f: \quad [0,\infty) \to [0,\infty), \quad t \mapsto t^{2\alpha} \cdot y^t.$$

Then f is differentiable on $(0,\infty)$ with derivative

$$f'(t) = 2\alpha\, t^{2\alpha-1} y^t + t^{2\alpha}\, \ln(y)\, y^t = t^{2\alpha-1} \cdot y^t (2\alpha + t\ln(y)).$$

From this, it follows that if we define $t_0 := -\frac{2\alpha}{\ln y} = \frac{2\alpha}{\ln(1/y)} \in (0,\infty)$, then $f'(t) > 0$ for $t \in (0,t_0)$ and $f'(t) < 0$ for $t \in (t_0,\infty)$. Hence, f has a global maximum in $t = t_0$, showing that

$$t^{2\alpha} y^t = f(t) \le f(t_0) = \left(\frac{2\alpha}{\ln(1/y)}\right)^{2\alpha} y^{-\frac{2\alpha}{\ln(y)}} = \left(\frac{2\alpha}{\ln(1/y)}\right)^{2\alpha} e^{-2\alpha} \lesssim [\ln(1/y)]^{-2\alpha}$$

for all $t \in [0,\infty)$ and $y \in (0,1)$.

Step 3 *(Completing the proof):* If we apply the estimate from the preceding step for $t = k$ and $y = \frac{x}{1+x} \in (0,1)$ (where $x \in (0,\infty)$), then we see that

$$k^{2\alpha} \cdot \left(\frac{x}{1+x}\right)^k \lesssim \left[\ln \frac{1+x}{x}\right]^{-2\alpha} = [\ln(1+x^{-1})]^{-2\alpha} \quad \forall\, x \in (0,\infty).$$

Now, for $x \geq 1$ note that $1 + x^{-1} \leq 2$, and hence

$$\ln(1+x^{-1}) = \int_1^{1+x^{-1}} t^{-1}\, dt \geq \frac{1}{2} x^{-1}$$

from which we see—because of $\alpha > 0$—that

$$[\ln(1+x^{-1})]^{-2\alpha} \leq 2^{2\alpha} \cdot x^{2\alpha} \leq 2^{2\alpha} \cdot (1+x)^{2\alpha}.$$

Likewise, if $0 < x \leq 1$ then $\ln(1+x^{-1}) \geq \ln(2)$ and hence

$$[\ln(1+x^{-1})]^{-2\alpha} \leq [\ln(2)]^{-2\alpha} \leq [\ln(2)]^{-2\alpha} \cdot (1+x)^{2\alpha}.$$

All in all, we have thus shown that

$$k^{2\alpha} \cdot \left(\frac{x}{1+x}\right)^k \lesssim (1+x)^{2\alpha} \qquad \forall\, x \in [0,\infty) \text{ and } k \in \mathbb{N}_0.$$

Combining this with the estimate from Step 1, we see for $k \geq 1$ that

$$\begin{aligned} g_k(x;\alpha) &= [B(k+1,2\alpha)]^{-1} \cdot \left(\frac{x}{1+x}\right)^k \cdot (1+x)^{-(1+2\alpha)} \\ &\lesssim k^{2\alpha} \cdot \left(\frac{x}{1+x}\right)^k \cdot (1+x)^{-2\alpha}\,(1+x)^{-1} \lesssim (1+x)^{-1}. \end{aligned}$$

Finally, in case of $k = 0$, we see directly from the definition of the Beta function that

$$B(k+1,2\alpha) = B(1,2\alpha) = \int_0^1 (1-t)^{2\alpha-1}\, dt = \int_0^1 s^{2\alpha-1}\, ds = \frac{1}{2\alpha},$$

and hence

$$g_0(x;\alpha) = 2\alpha \cdot (1+x)^{-(1+2\alpha)} \leq 2\alpha \cdot (1+x)^{-1},$$

since $1 + x \geq 1$ and $1 + 2\alpha \geq 1$. $\square$

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