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A Balian–Low type theorem for Gabor Riesz sequences of arbitrary density

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Abstract

Gabor systems are used in fields ranging from audio processing to digital communication. Such a Gabor system (g, Λ) consists of all time-frequency shifts $\pi(\lambda)g$ of a window function $g \in L^2(\mathbb{R})$ along a lattice $\Lambda \subset \mathbb{R}^2$. We focus on Gabor systems that are also Riesz sequences, meaning that one can stably reconstruct the coefficients $c = (c_\lambda)_{\lambda\in\Lambda}$ from the function $\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g$. In digital communication, a function of this form is used to transmit the digital sequence c. It is desirable for g to be well localized in time and frequency, since the transmitted signal will then be almost compactly supported in time and frequency if the sequence c has finite support. In this paper, we study what additional structural properties the signal space $\mathcal{G}(g, \Lambda)$, i.e., the span of the Gabor system, satisfies in addition to being a closed subspace of $L^2(\mathbb{R})$. The most well-known result in this direction—the Balian–Low theorem—states that if g is well localized in time and frequency and if (g, Λ) is a Riesz sequence, then $\mathcal{G}(g, \Lambda)$ is necessarily a *proper* subspace of $L^2(\mathbb{R})$. We prove a generalization of this result related to the invariance of $\mathcal{G}(g, \Lambda)$ under time-frequency shifts. Precisely, we show that if (g, Λ) is a Riesz sequence with g being well localized in time and frequency (precisely, g should belong to the so-called Feichtinger algebra), then $\pi(\mu)\mathcal{G}(g, \Lambda) \subset \mathcal{G}(g, \Lambda)$ holds if and only if $\mu \in \Lambda$. For lattices of *rational density*, this was already known, with the proof based on Zak transform techniques. These methods do not generalize to arbitrary lattices, however. Instead, our proof for lattices of irrational density relies on combining methods

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from time-frequency analysis with properties of a special C^*-algebra, the so-called irrational rotation algebra.

Keywords Balian-Low theorem · Gabor systems · Time-frequency shifts · Riesz sequences · Feichtinger algebra · Irrational rotation algebra

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1 Introduction

The central idea of digital signal processing is to represent continuous signals in terms of discrete sequences of coefficients. This is usually achieved by taking the inner products of the signal with respect to some well-structured system of functions. In the area of applied harmonic analysis, many different such systems have been proposed, with wavelet and Gabor systems being the most popular among them. The most important property of such a system is the ability to stably represent arbitrary signals; in this case, the system is called a *frame* [8].

Formally, a Gabor system (g, Λ) consists of all time-frequency shifts $\pi(\lambda)g$ of the window function $g \in L^2(\mathbb{R})$ along a lattice $\Lambda \subset \mathbb{R}^2$, where

$$[\pi(x, \omega)g](y) = e^{2\pi i \omega y} \cdot g(y - x);$$

see [16]. When working with such a Gabor frame, the window function g should have a good time-frequency localization, so that the frame coefficients faithfully reflect the time-frequency behavior of the analyzed function. The *Feichtinger algebra* $S_0(\mathbb{R}^d)$ [13, 21] is a particularly popular window class. Among other advantages, choosing a window from S_0 ensures that the canonical dual window also belongs to the Feichtinger algebra [17], so that for example the membership of a function $f \in L^2(\mathbb{R}^d)$ in the modulation space $M^{p,q}(\mathbb{R}^d)$ can be characterized in terms of the decay properties of its frame coefficients. One crucial obstruction, however, is that a Gabor system with window belonging to $S_0(\mathbb{R}^d)$ can not form an orthonormal basis—in fact not even a Riesz basis—for $L^2(\mathbb{R}^d)$. We call this phenomenon the S_0 *Balian-Low theorem (or the Feichtinger algebra Balian–Low theorem)*; it is a consequence of the Amalgam Balian–Low theorem [2, Theorem 3.2]. The same no-go type result holds for the case where g belongs to the space $\mathbb{H}^1(\mathbb{R}^d)$ consisting of functions in the L^2-Sobolev space $H^1(\mathbb{R}^d)$ whose Fourier transform also belongs to $H^1(\mathbb{R}^d)$. This is the classical Balian-Low theorem; see [16, Theorem 8.4.5] for the case of orthonormal bases, and [10, Theorem 2.3] for the general case.

Yet, even though a Gabor system with $g \in S_0(\mathbb{R}^d)$ cannot form a Riesz basis for all of $L^2(\mathbb{R}^d)$, it might still be a *Riesz sequence*, that is, a Riesz basis for its closed linear span $\mathcal{G}(g, \Lambda)$, at least if $\mathcal{G}(g, \Lambda)$ is a *proper* subspace of $L^2(\mathbb{R}^d)$. In this case, one might wonder about further properties—in addition to being a proper subspace of $L^2(\mathbb{R}^d)$—that the Gabor space $\mathcal{G}(g, \Lambda)$ has to have. One important property in time-frequency analysis is the invariance of $\mathcal{G}(g, \Lambda)$ under time-frequency shifts $T_a M_b$. For lattices Λ *of rational density* and for dimension $d = 1$, it was observed in [4] that if (g, Λ) is a Riesz sequence and if $g \in S_0(\mathbb{R})$, then the set of parameters $(a, b) \in \mathbb{R}^2$ such that $\mathcal{G}(g, \Lambda)$ is invariant under the time-frequency shift $T_a M_b$ is exactly equal to Λ. A multi-dimensional variant of this was derived in [5].

These results generalize the S_0 Balian-Low theorem to subspaces of $L^2(\mathbb{R})$. Indeed, to derive the S_0 Balian-Low theorem from the above result, note that if $\mathcal{G}(g, \Lambda) = L^2(\mathbb{R})$ then

$\mathcal{G}(g, \Lambda)$ is invariant under *all* time-frequency shifts, even under those with $(a, b) \notin \Lambda$; hence, g cannot belong to S_0. A corresponding generalization of the *classical* Balian-Low theorem was proved in [6]; a quantitative version can be found in [7].

We emphasize that in all articles [4–7] it is assumed that the generating lattice Λ has rational density. This restriction is needed in order to utilize the Zak transform which is used extensively in [4–7]. It is thus natural to ask whether the results in [4] and [6] still hold for lattices with *irrational* density.

In a sense, this question has analogies with the research concerning the regularity of the canonical dual window of a Gabor frame. In 1997 it was shown (see [14, Theorem 3.4]) that if $g \in S_0(\mathbb{R}^d)$ generates a Gabor frame for $L^2(\mathbb{R}^d)$ over a lattice *of rational density*, then the canonical dual window also belongs to $S_0(\mathbb{R}^d)$. It was conjectured in the same article that this property continues to hold for general lattices. Six years later, this conjecture was confirmed by Gröchenig and Leinert [17] by using C^*-algebra methods.

Here, we likewise extend the result in [4] from lattices of rational density to arbitrary lattices:

Theorem 1 *If $g \in S_0(\mathbb{R})$ and $\Lambda \subset \mathbb{R}^2$ is a lattice such that the Gabor system (g, Λ) is a Riesz basis for its closed linear span $\mathcal{G}(g, \Lambda)$, then the time-frequency shifts $T_a M_b$ that leave $\mathcal{G}(g, \Lambda)$ invariant satisfy $(a, b) \in \Lambda$.*

In other words, the following conditions cannot hold simultaneously: (i) $g \in S_0(\mathbb{R})$, (ii) (g, Λ) is a Riesz sequence, (iii) $\mathcal{G}(g, \Lambda)$ is invariant under $T_a M_b$ with some $(a, b) \notin \Lambda$. One can easily find examples where only two of these conditions hold simultaneously. For instance, conditions (i) and (ii) hold for the Gaussian $g(x) = e^{-\pi x^2}$ and $\Lambda = \mathbb{Z} \times 2\mathbb{Z}$; conditions (i) and (iii) hold for $g(x) = (1 - 4|x|)\, \mathbf{1}_{[-\frac{1}{4}, \frac{1}{4}]}(x)$ and $\Lambda = \mathbb{Z} \times 2\mathbb{Z}$; conditions (ii) and (iii) hold for all the cases in Example 6 below. Here, $\mathbf{1}_S$ denotes the characteristic function of $S \subset \mathbb{R}$.

As indicated above, the Zak transform is a powerful tool for analyzing Gabor systems generated by lattices with rational density; yet, it is not of much use in the case of irrational density lattices. Consequently, the methods used in the present paper differ substantially from those in [4–7]. Instead of applying the Zak transform and thus dealing with functions on $\mathbb{R}^2$, we work directly with the given objects and exploit the rich theory of time-frequency analysis. Along the way, we obtain several new statements related to time-frequency shift invariance that are interesting in their own right.

The proof of Theorem 1 consists of several steps. First, for $g \in S_0(\mathbb{R})$ and only assuming that (g, Λ) is a *frame sequence*—that is, a frame for its closed linear span— we prove the following dichotomy:

Either (g, Λ) spans all of $L^2(\mathbb{R})$, or the set of $(a, b) \in \mathbb{R}^2$ for which $T_a M_b$ leaves $\mathcal{G}(g, \Lambda)$ invariant is a lattice containing Λ as a sublattice; (D)

see Theorem 8. This result significantly reduces the range of parameters (a, b) that we need to consider. Next, we give a characterization for the invariance of $\mathcal{G}(g, \Lambda)$ under a time-frequency shift $T_a M_b$ with $(a, b) \notin \Lambda$ in terms of the adjoint system of (g, Λ); see Theorem 11. This characterization holds for general $g \in L^2(\mathbb{R})$, not only for $g \in S_0(\mathbb{R})$. Combining this characterization with a deep existing result about traces of projections in the so-called *irrational rotation algebra* (see [26, 27]), we arrive at the conclusion of Theorem 1.

With Theorem 1 established for g in the Feichtinger algebra, it is natural to ask whether the same statement holds in the setting of the *classical* Balian-Low theorem, that is, when g

has finite uncertainty product $\left(\int x^2|g(x)|^2\,dx\right)\cdot\left(\int \omega^2|\widehat{g}(\omega)|^2\,d\omega\right) < \infty$, a condition which we simply write as $g \in \mathbb{H}^1$. Unfortunately, we were not able to prove a full-fledged version of Theorem 1 for $g \in \mathbb{H}^1$; the best we could do is to show that the dichotomy (D) described above for $g \in S_0$ still holds for $g \in \mathbb{H}^1$.

The outline of the paper is as follows: After recalling the necessary background on Janssen's representation, time-frequency shift invariance, symplectic operators, and the two spaces $S_0(\mathbb{R})$ and $\mathbb{H}^1$ in Sect. 2, the paper proper starts in Sect. 3, where we prove the dichotomy (D) described above, for g lying in a broad function space which contains both $S_0(\mathbb{R})$ and $\mathbb{H}^1$. Next, in Sect. 4 we show that one can reduce to the case of a separable lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, with an additional time-frequency shift of the form $T_{\alpha/\nu}$ for some $\nu \in \mathbb{N}_{\geq 2}$. For this setting, we then derive a characterization in terms of the adjoint Gabor system. Throughout Sect. 4, the generating function g is only assumed to be in $L^2(\mathbb{R})$. The paper culminates in Sect. 5, where we prove Theorem 1 using an auxiliary result which relies on irrational rotation algebras. Appendix A contains a proof of the auxiliary result including a short treatise on irrational rotation algebras.

2 Preliminaries

For $a, b \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$ we define the operators of translation by a and modulation by b as

$$T_a f(x) := f(x-a) \quad \text{and} \quad M_b f(x) := e^{2\pi i b x} f(x),$$

respectively. Both T_a and M_b are unitary operators on $L^2(\mathbb{R})$ and hence so is the *time-frequency shift*

$$\pi(a,b) := T_a M_b = e^{-2\pi i a b} M_b T_a.$$

A (full rank) *lattice* $\Lambda \subset \mathbb{R}^2$ is any set of the form $\Lambda = A\mathbb{Z}^2$ with an invertible matrix $A \in \mathbb{R}^{2\times 2}$. The *density* of Λ is defined by $d(\Lambda) = |\det A|^{-1}$. Note that $A\mathbb{Z}^2 = \mathbb{Z}^2$ if and only if $A \in \mathbb{Z}^{2\times 2}$ and $\det A = \pm 1$. This will be used heavily in the proof of Proposition 5 below.

A lattice Λ is called *separable* if A can be chosen to be diagonal, i.e., $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ with $\alpha, \beta > 0$. The next lemma shows that every lattice can be transformed into a separable one by means of a symplectic matrix; this will be used frequently.

Lemma 2 *Let $A \in \mathbb{R}^{2\times 2}$ be a non-singular matrix. Then there exists a matrix $C \in \mathbb{R}^{2\times 2}$ with* $\det C = 1$ *such that CA is diagonal, i.e., $CA\mathbb{Z}^2$ is separable.*

Proof Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and note $\Delta := ad - bc \neq 0$. If $a \neq 0$, choose $C = \left(\begin{smallmatrix} 1+bc/\Delta & -ab/\Delta \\ -c/a & 1 \end{smallmatrix}\right)$. A simple calculation shows that $CA = \operatorname{diag}(a, \Delta/a)$ and $\det C = 1$. In the case $a = 0$ we have $b \neq 0 \neq c$ as A is non-singular. Then $C := \left(\begin{smallmatrix} -d/b & 1 \\ -1 & 0 \end{smallmatrix}\right)$ satisfies $\det C = 1$ and $CA = \operatorname{diag}(c, -b)$. □

For a subset $M \subset L^2(\mathbb{R})$, we denote its closure by $\overline{M}$. For $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$, we set

$$(g,\Lambda) := \left\{\pi(\lambda)g : \lambda \in \Lambda\right\} \qquad \text{and} \qquad \mathcal{G}(g,\Lambda) := \overline{\operatorname{span}}(g,\Lambda) \subset L^2(\mathbb{R}).$$

We use the normalization $\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x \omega}\, dx$ for the Fourier transform of $f \in L^1(\mathbb{R})$. It is well-known that $\mathcal{F}$ extends to a unitary map $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called

- a *Bessel sequence* in $\mathcal{H}$ (with a Bessel bound B) if there is a constant $B > 0$ such that
$$\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \le B\, \|f\|^2 \quad \text{for all} \quad f \in \mathcal{H};$$
- a *frame* for $\mathcal{H}$ (with frame bounds A and B) if there exist constants $0 < A \le B < \infty$ such that
$$A\, \|f\|^2 \le \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \le B\, \|f\|^2 \quad \text{for all} \quad f \in \mathcal{H};$$
- a *Riesz sequence* in $\mathcal{H}$ (with Riesz bounds A and B) if there are constants $0 < A \le B < \infty$ such that
$$A\, \|c\|_{\ell^2}^2 \le \Big\| \sum_{n\in\mathbb{Z}} c_n\, f_n \Big\|^2 \le B\, \|c\|_{\ell^2}^2 \quad \text{for all} \quad c = \{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z});$$
- a *Riesz basis* for $\mathcal{H}$ (with Riesz bounds A and B) if it is a complete Riesz sequence in $\mathcal{H}$ (with Riesz bounds A and B).

It is known [8, Theorem 5.4.1] that a Riesz basis for $\mathcal{H}$ is a frame for $\mathcal{H}$, in which case the frame bounds coincide with the Riesz bounds. In fact, a sequence in $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if and only if it is an exact frame for $\mathcal{H}$; see [8, Theorem 7.1.1]. Here, a frame is called *exact* if it ceases to be a frame when an arbitrary element is removed.

2.1 Bessel vectors and Janssen's representation

Let $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ be a separable lattice with $\alpha, \beta > 0$. The *adjoint lattice* of Λ is defined as $\Lambda^\circ = \frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}$. We say that $g \in L^2(\mathbb{R})$ is a *Bessel vector* for Λ if the system (g, Λ) is a Bessel system in $L^2(\mathbb{R})$, meaning that the *analysis operator* $C_{\Lambda,g}$ corresponding to (g, Λ) is bounded as an operator from $L^2(\mathbb{R})$ to $\ell^2(\mathbb{Z}^2)$. It is defined by
$$C_{\Lambda,g} f = \big(\langle f, T_{m\alpha} M_{n\beta} g\rangle\big)_{m,n\in\mathbb{Z}}, \qquad f \in L^2(\mathbb{R}).$$

We denote the set of Bessel vectors for Λ by $\mathcal{B}_\Lambda$. This is a linear subspace of $L^2(\mathbb{R})$ which is dense because it contains the Schwartz space $\mathcal{S}(\mathbb{R})$; see [16, Corollary 6.2.3]. It is well known that $\mathcal{B}_\Lambda = \mathcal{B}_{\Lambda^\circ}$ (see [28, Theorem 2.2(a)]) and that
$$\sum_{m,n\in\mathbb{Z}} \langle f, T_{m\alpha} M_{n\beta} g\rangle \big\langle T_{m\alpha} M_{n\beta} h, u\big\rangle = \frac{1}{\alpha\beta} \sum_{k,\ell\in\mathbb{Z}} \big\langle h, T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}} g\big\rangle \big\langle T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}} f, u\big\rangle \tag{1}$$

whenever at least three of $f, g, h, u \in L^2(\mathbb{R})$ are Bessel vectors for Λ; this follows from [22, Proposition 2.4]. Formula (1) yields the so-called *Janssen representation* of the cross frame operator $S_{\Lambda,g,h} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ associated to Bessel vectors $g, h \in \mathcal{B}_\Lambda$. This operator is defined by
$$S_{\Lambda,g,h} f := \sum_{m,n\in\mathbb{Z}} \langle f, T_{m\alpha} M_{n\beta} g\rangle \cdot T_{m\alpha} M_{n\beta} h, \qquad f \in L^2(\mathbb{R}). \tag{2}$$

Equation (1) implies that
$$S_{\Lambda,g,h} f = \frac{1}{\alpha\beta} \sum_{k,\ell\in\mathbb{Z}} \big\langle h, T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}} g\big\rangle \cdot T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}} f \qquad \text{if} \quad f, g, h \in \mathcal{B}_\Lambda. \tag{3}$$

The series in Equations (2) and (3) both converge unconditionally in $L^2(\mathbb{R})$.

2.2 Time-frequency shift invariance

For a closed linear subspace $\mathcal{G} \subset L^2(\mathbb{R})$, we denote by $\mathfrak{I}(\mathcal{G})$ the set of all pairs $(a, b) \in \mathbb{R}^2$ such that $\mathcal{G}$ is invariant under the time-frequency shift $\pi(a, b)$; that is,

$$\mathfrak{I}(\mathcal{G}) := \{z \in \mathbb{R}^2 : \pi(z)\mathcal{G} \subset \mathcal{G}\}.$$

If $\mathcal{G} = \mathcal{G}(g, \Lambda)$ for some $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$, then clearly $\Lambda \subset \mathfrak{I}(\mathcal{G})$. Any time-frequency shift $\pi(z)$ with $z \in \mathfrak{I}(\mathcal{G}) \setminus \Lambda$ will be called an *additional time-frequency shift* for $\mathcal{G}(g, \Lambda)$. For Gabor spaces $\mathcal{G} = \mathcal{G}(g, \Lambda)$, the set $\mathfrak{I}(\mathcal{G})$ has some additional structure:

Lemma 3 ([3, Proposition A.1]) *Let $g \in L^2(\mathbb{R})$, let $\Lambda \subset \mathbb{R}^2$ be a lattice, and define $\mathcal{G} := \mathcal{G}(g, \Lambda)$. If $z \in \mathbb{R}^2$, then $z \in \mathfrak{I}(\mathcal{G})$ if and only if $\pi(z)g \in \mathcal{G}$. Moreover, $\mathfrak{I}(\mathcal{G})$ is a closed additive subgroup of $\mathbb{R}^2$.*

Lemma 3 shows that $z \in \mathfrak{I}(\mathcal{G})$ implies $-z \in \mathfrak{I}(\mathcal{G})$, i.e., $\pi(z)\mathcal{G} \subset \mathcal{G}$ and $\pi(z)^{-1}\mathcal{G} \subset \mathcal{G}$. Hence, we have $\pi(z)\mathcal{G} = \mathcal{G}$ whenever $z \in \mathfrak{I}(\mathcal{G})$.

The next lemma characterizes the case when $\mathcal{G}$ is invariant under *all* time-frequency shifts.

Lemma 4 *For a nonempty closed linear subspace $\mathcal{G} \subset L^2(\mathbb{R})$, $\mathcal{G} \neq \{0\}$, we have $\mathfrak{I}(\mathcal{G}) = \mathbb{R}^2$ if and only if $\mathcal{G} = L^2(\mathbb{R})$.*

Proof Clearly, if $\mathcal{G} = L^2(\mathbb{R})$, then $\mathfrak{I}(\mathcal{G}) = \mathbb{R}^2$. To prove the converse, assume that $\mathfrak{I}(\mathcal{G}) = \mathbb{R}^2$ and let $f \in \mathcal{G}^\perp$ and $g \in \mathcal{G} \setminus \{0\}$. Then $\langle f, \pi(z)g\rangle = 0$ for all $z \in \mathbb{R}^2$, so that the short-time Fourier transform $V_g f$ of f with window g satisfies $V_g f \equiv 0$. By [16, Corollary 3.2.2] and since $g \neq 0$, this implies $f = 0$. We have thus shown $\mathcal{G}^\perp = \{0\}$, whence $\mathcal{G} = L^2(\mathbb{R})$, since $\mathcal{G}$ is a closed subspace of $L^2(\mathbb{R})$. □

2.3 Symplectic operators

It is often useful to reduce a statement involving a non-separable lattice to one that involves a separable lattice, since separable lattices are usually easier to handle. For this reduction, we will use so-called *symplectic operators* (see [16, Sect. 9.4]). Since we are working in dimension $d = 1$, a matrix $B \in \mathbb{R}^{2\times 2}$ is *symplectic* if and only if $\det B = 1$; see [16, Lemma 9.4.1]. For any such matrix B, it is shown in [16, Eq. (9.39)] that there exists a unitary operator $U_B : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that

$$U_B \rho(z) = \rho(Bz) U_B, \quad z \in \mathbb{R}^2, \tag{4}$$

where (as in [16, Page 185 and Eq. (9.25)])

$$\rho(a, b) := e^{\pi i a b} \cdot \pi(a, b).$$

In the sequel, we fix for each $B \in \mathbb{R}^{2\times 2}$ with $\det B = 1$ one choice of the operator U_B, and for functions $g \in L^2(\mathbb{R})$, closed subspaces $\mathcal{G} \subset L^2(\mathbb{R})$, and sets $\Lambda \subset \mathbb{R}^2$ we write

$$g_B := U_B\, g, \qquad \mathcal{G}_B := U_B\, \mathcal{G}, \qquad \text{and} \qquad \Lambda_B := B\Lambda.$$

As shown in [16, Page 197], given $B, C \in \mathbb{R}^{2\times 2}$ with $\det B = \det C = 1$, we have $U_B U_C = \theta_{B,C}\, U_{BC}$ for some $\theta_{B,C} \in \mathbb{C}$ with $|\theta_{B,C}| = 1$.

Note that (4) implies

$$\pi(z)g \in \mathcal{G} \iff \pi(Bz)g_B \in \mathcal{G}_B, \qquad z \in \mathbb{R}^2. \tag{5}$$

Therefore, (g, Λ) is a frame (Riesz basis, resp.) for its closed linear span $\mathcal{G}$ if and only if (g_B, Λ_B) is a frame (Riesz basis, resp.) for its closed linear span $\mathcal{G}_B$. Thanks to Lemma 3, the equivalence (5) also implies that

$$\mathfrak{I}(\mathcal{G}_B) = B\,\mathfrak{I}(\mathcal{G}). \tag{6}$$

2.4 The Feichtinger algebra

We denote by $S_0(\mathbb{R})$ the *Feichtinger algebra*, which is the space of functions $f \in L^2(\mathbb{R})$ such that $\langle f, \pi(\cdot)\varphi\rangle \in L^1(\mathbb{R}^2)$ for some (and hence every) Schwartz function $\varphi \neq 0$. It is known that a function $f \in L^2(\mathbb{R})$ belongs to $S_0(\mathbb{R})$ if and only if $\langle f, \pi(\cdot) f\rangle \in L^1(\mathbb{R}^2)$. See [16, Proposition 12.1.2] for several characterizations of $S_0(\mathbb{R})$ and their proofs.

Recall that $S_0(\mathbb{R})$ is invariant under each operator U_B (cf. [16, Proposition 12.1.3]), so that $g \in S_0(\mathbb{R})$ always implies $g_B \in S_0(\mathbb{R})$ for $B \in \mathbb{R}^{2\times 2}$ with $\det B = 1$. Also, each $g \in S_0(\mathbb{R})$ is a Bessel vector for any (separable) lattice (see e.g. [16, Propositions 6.2.2 and 12.1.4]). Since for $g, h \in S_0(\mathbb{R})$ and any $\alpha, \beta > 0$ the sequence $\big(\langle h, T_{m\alpha} M_{n\beta} g\rangle\big)_{m,n\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z}^2)$ (see [16, Corollary 12.1.12]), it follows from (3) and from the density of $\mathcal{B}_\Lambda$ in $L^2(\mathbb{R})$ that

$$S_{\Lambda,g,h} = \frac{1}{\alpha\beta} \sum_{k,\ell\in\mathbb{Z}} \big\langle h, T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}} g\big\rangle \cdot T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}} \quad \text{with} \quad \Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}, \tag{7}$$

where the series converges absolutely in operator norm.

2.5 The space $\mathbb{H}^1$

Let $H^1(\mathbb{R})$ denote the space of all functions f in $L^2(\mathbb{R})$ for which the weak derivative f' exists and belongs to $L^2(\mathbb{R})$. In other words, $H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$ is an L^2-Sobolev-space. It is well known (see [23, Theorem 7.16]) that each $f \in H^1(\mathbb{R})$ has a representative that is absolutely continuous on $\mathbb{R}$ and whose classical derivative exists and coincides with the weak derivative f' almost everywhere.

By $\mathbb{H}^1$ we denote the space of all functions $f \in H^1(\mathbb{R})$ whose Fourier transform $\widehat{f}$ also belongs to $H^1(\mathbb{R})$. Equivalently, a function $f \in L^2(\mathbb{R})$ is in $\mathbb{H}^1$ if and only if $f', Xf \in L^2(\mathbb{R})$, where Xf denotes the function $\mathbb{R} \to \mathbb{C}, x \mapsto xf(x)$. The space $\mathbb{H}^1$ also coincides with the modulation space $M^2_m(\mathbb{R})$ with the weight $m(x,\omega) = 1 + \sqrt{x^2+\omega^2}$; see [18, Corollary 2.3].

As shown in [6, Proof of Theorem 1.4], the space $\mathbb{H}^1$ is invariant under symplectic operators, meaning that $U_B g \in \mathbb{H}^1$ if $g \in \mathbb{H}^1$ and $B \in \mathbb{R}^{2\times 2}$ with $\det B = 1$.

It is easily seen that both $S_0(\mathbb{R})$ and $\mathbb{H}^1$ contain the Schwartz space $\mathcal{S}(\mathbb{R})$. To help the readers' understanding, we provide some examples of functions in $L^2(\mathbb{R})\backslash(S_0(\mathbb{R}) \cup \mathbb{H}^1)$, $(S_0(\mathbb{R}) \cap \mathbb{H}^1)\backslash\mathcal{S}(\mathbb{R})$, $S_0(\mathbb{R})\backslash\mathbb{H}^1$, and $\mathbb{H}^1\backslash S_0(\mathbb{R})$.

Examples. The characteristic function $f := \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$ of $[-\frac{1}{2},\frac{1}{2}]$ belongs to neither $S_0(\mathbb{R})$ nor $\mathbb{H}^1$, since $S_0(\mathbb{R}) \cup \mathbb{H}^1 \subset C(\mathbb{R})$ (see Lemma 7 below). The weak derivative of f is given by $f'(x) = \delta(x+\frac{1}{2}) - \delta(x-\frac{1}{2}) \notin L^2(\mathbb{R})$, where δ denotes the Dirac delta. Note that $\frac{1}{2\pi i}\widehat{f'}(\omega) = \omega\,\widehat{f}(\omega) = \frac{1}{\pi}\sin(\pi\omega) \notin L^2(\mathbb{R})$.

The hat function $h(x) := (1-|x|)\,\mathbf{1}_{[-1,1]}(x) = (f * f)(x)$ is continuous but not smooth, hence, does not belong to $\mathcal{S}(\mathbb{R})$. However, $h \in S_0(\mathbb{R}) \cap \mathbb{H}^1$. Indeed, one can directly check $\langle h, \pi(\cdot)h\rangle \in L^1(\mathbb{R}^2)$ to show that $h \in S_0(\mathbb{R})$ (see e.g., [12, Chapter 16]). Also, we have

$h \in \mathbb{H}^1$ since both $h' = \mathbf{1}_{[-1,0)} - \mathbf{1}_{[0,1]}$ (defined a.e.) and $Xh(x) = x\,(1-|x|)\,\mathbf{1}_{[-1,1]}(x)$ are square-integrable. Note that $\frac{1}{2\pi i}\widehat{h'}(\omega) = \omega\,\widehat{h}(\omega) = \omega\,(\widehat{f}(\omega))^2 = \frac{1}{\pi^2\omega}\sin^2(\pi\omega) \in L^2(\mathbb{R})$.

It is shown in [2, Example 3.3] that $g(x) = \sum_{n=1}^{\infty} n^{-3/2}\, h(2(x-n))$ is not in $\mathbb{H}^1$. However, using [15, Theorem 3.2.13] and the fact that $h \in S_0(\mathbb{R})$, we have $g \in S_0(\mathbb{R})$, hence, $g \in S_0(\mathbb{R})\backslash\mathbb{H}^1$. On the other hand, a function in $\mathbb{H}^1\backslash S_0(\mathbb{R})$ is constructed in [2, Example 3.4].

3 Time-frequency shift invariance: a closer look

In this section, we first establish a certain trichotomy concerning the set of invariant time-frequency shifts. We then show that one of the three cases of the trichotomy is excluded if the generator function g is "sufficiently nice".

The next proposition establishes the trichotomy: the invariance set $\mathfrak{I}(\mathcal{G})$ either fills the whole space $\mathbb{R}^2$, or it consists of equispaced lines that are aligned with the lattice, or it is a refinement of Λ (and in particular a lattice itself). Note that this holds regardless of the regularity of the generator g or the (frame) properties of the Gabor system (g, Λ).

Proposition 5 *Let H be a closed additive subgroup of $\mathbb{R}^2$ and suppose that $H \supset \Lambda$ for a non-degenerate lattice $\Lambda \subset \mathbb{R}^2$. Then there exist $\lambda_1, \lambda_2 \in \Lambda$ satisfying $\Lambda = \mathbb{Z}\cdot\lambda_1 + \mathbb{Z}\cdot\lambda_2$ and $m, n \in \mathbb{N}_{\geq 1}$ such that exactly one of the following conditions holds:*

(1) $H = \mathbb{R}^2$.
(2) $H = \mathbb{R}\cdot\lambda_1 + \mathbb{Z}\cdot\frac{\lambda_2}{n}$.
(3) $H = \mathbb{Z}\cdot\frac{\lambda_1}{m} + \mathbb{Z}\cdot\frac{\lambda_2}{n}$.

In particular, if $\Lambda \subset \mathbb{R}^2$ is a lattice and $g \in L^2(\mathbb{R})$, then one of the above cases holds for $H = \mathfrak{I}(\mathcal{G}(g, \Lambda))$.

Proof By [19, Theorem 9.11], there are $\alpha, \beta \in \mathbb{N}_0$ and linearly independent vectors $x_1, \ldots, x_\alpha, y_1, \ldots, y_\beta \in \mathbb{R}^2$ (hence, $\alpha + \beta \leq 2$) such that

$$H = \mathbb{R}\,x_1 + \cdots + \mathbb{R}\,x_\alpha + \mathbb{Z}\,y_1 + \cdots + \mathbb{Z}\,y_\beta.$$

Since H contains the non-degenerate lattice Λ (and thus two linearly independent vectors), we must have $\alpha + \beta = 2$. Hence, there are three cases:

(i) $(\alpha, \beta) = (2, 0)$ and hence $H = \mathbb{R}^2$,
(ii) $(\alpha, \beta) = (1, 1)$, so that $H = \mathbb{R}v + \mathbb{Z}w$ with linearly independent $v, w \in \mathbb{R}^2$,
(iii) $(\alpha, \beta) = (0, 2)$, so that H is a (non-degenerate) lattice.

Clearly, in Case (i), Condition (1) holds.

Let us consider the case (ii): $H = \mathbb{R}\cdot v + \mathbb{Z}\cdot w$. Let $\Lambda = \mathbb{Z}\cdot\mu + \mathbb{Z}\cdot\lambda$ be an arbitrary representation of Λ. Since $\Lambda \subset H$, there exist $m, n \in \mathbb{Z}$ and $s, t \in \mathbb{R}$ such that

$$[\mu, \lambda] = [sv + mw,\ tv + nw] = [v, w]\begin{bmatrix} s & t \\ m & n \end{bmatrix}.$$

Note that μ, λ are linearly independent, and hence $sn - tm \neq 0$, so that $d = (sn - tm)^{-1}$ is well-defined. Furthermore, we see

$$[v, w] = d\cdot[\mu, \lambda]\begin{bmatrix} n & -t \\ -m & s \end{bmatrix},$$

which shows that $v = d(n\mu - m\lambda)$ and thus $\mathbb{R}\cdot v = \mathbb{R}\cdot\lambda_1$ with some $\lambda_1 \in \Lambda$. By rescaling λ_1, we can ensure that $\frac{1}{k}\lambda_1 \notin \Lambda$ for each $k \in \mathbb{Z}\setminus\{-1,0,1\}$. Note because of $\mathbb{R}\cdot v = \mathbb{R}\cdot\lambda_1$ that $H = \mathbb{R}\cdot\lambda_1 + \mathbb{Z}\cdot w$.

Now, there exists $\lambda_2 \in \Lambda$ such that $\Lambda = \mathbb{Z}\cdot\lambda_1 + \mathbb{Z}\cdot\lambda_2$. Indeed, writing $\Lambda = A\mathbb{Z}^2$ with $A = [a_1, a_2] \in \mathbb{R}^{2\times 2}$ invertible, there exist $i, j \in \mathbb{Z}$ such that $\lambda_1 = ia_1 + ja_2$. The numbers i, j are necessarily coprime, since $\frac{1}{k}\lambda_1 \notin \Lambda$ for $k \in \mathbb{Z}\setminus\{-1,0,1\}$. Hence, by Bézout's lemma there exist $k, \ell \in \mathbb{Z}$ such that $i\ell - jk = 1$. Set $\lambda_2 = ka_1 + \ell a_2$. Then $[\lambda_1, \lambda_2]\mathbb{Z}^2 = A\left[\begin{smallmatrix} i & k \\ j & \ell \end{smallmatrix}\right]\mathbb{Z}^2 = A\mathbb{Z}^2 = \Lambda$. Since $\lambda_2 \in \Lambda \subset H$, there exist $\sigma \in \mathbb{R}$ and $\nu \in \mathbb{Z}$ such that $\lambda_2 = \sigma\lambda_1 + \nu w$. Then $\nu \neq 0$ and so $H = \mathbb{R}\cdot\lambda_1 + \mathbb{Z}\cdot(\frac{\lambda_2}{\nu} - \frac{\sigma}{\nu}\lambda_1) = \mathbb{R}\cdot\lambda_1 + \mathbb{Z}\cdot\frac{\lambda_2}{|\nu|}$. Hence, Condition (2) of the statement of the proposition holds.

Assume now that Case (iii) holds: H is a lattice, i.e., $H = \mathbb{Z}\cdot v + \mathbb{Z}\cdot w$ with linearly independent vectors $v, w \in \mathbb{R}^2$. Write $\Lambda = \mathbb{Z}\cdot\mu + \mathbb{Z}\cdot\lambda$ with linearly independent $\lambda, \mu \in \mathbb{R}^2$. Then, because of $\Lambda \subset H$, there exist $a, b, c, d \in \mathbb{Z}$ such that

$$[\mu, \lambda] = [av + cw, bv + dw] = [v, w]\begin{bmatrix} a & b \\ c & d \end{bmatrix} =: [v, w]\cdot A, \tag{8}$$

with $A \in \mathbb{Z}^{2\times 2}$. Let $A = MDN^{-1}$ be the Smith canonical form of A (see, for instance [24, Theorem 26.2] or [20, Theorem 3.8]), where $M, N, D \in \mathbb{Z}^{2\times 2}$ with $\det M = \det N = 1$ and D is a diagonal matrix. Note that A (and hence D) is invertible; this follows from (8) since μ and λ are linearly independent. Moreover, note that $[\mu, \lambda]ND^{-1} = [v, w]M$.

Define $\lambda_1, \lambda_2 \in \Lambda$ via $[\lambda_1, \lambda_2] := [\mu, \lambda]N$. Then

$$\Lambda = [\mu, \lambda]\mathbb{Z}^2 = [\mu, \lambda]N\mathbb{Z}^2 = [\lambda_1, \lambda_2]\mathbb{Z}^2 = \mathbb{Z}\cdot\lambda_1 + \mathbb{Z}\cdot\lambda_2.$$

Further, writing $D = \mathrm{diag}(m, n)$ with $m, n \in \mathbb{Z}\setminus\{0\}$, we see

$$H = [v, w]\mathbb{Z}^2 = [v, w]M\mathbb{Z}^2 = [\mu, \lambda]ND^{-1}\mathbb{Z}^2 = [\lambda_1, \lambda_2]D^{-1}\mathbb{Z}^2 = \mathbb{Z}\cdot\tfrac{\lambda_1}{|m|} + \mathbb{Z}\cdot\tfrac{\lambda_2}{|n|}.$$

This completes the proof of the proposition, since the conditions (1)–(3) are clearly mutually exclusive. □

We now provide some examples for the three cases in Proposition 5. In particular, we show that all three cases can occur for $H = \mathfrak{I}(\mathcal{G}(g, \Lambda))$ with $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$ such that (g, Λ) is a Riesz sequence.

Example 6

Case (1) : Let $g := \mathbf{1}_{[0,1]}$. Then $(g, \mathbb{Z}\times\mathbb{Z})$ is an orthonormal basis for $\mathcal{G} = \mathcal{G}(g, \mathbb{Z}\times\mathbb{Z}) = L^2(\mathbb{R})$ and thus $\mathfrak{I}(\mathcal{G}) = \mathbb{R}^2$ by Lemma 4.

Case (2) : We will show that for every lattice $\Lambda \subset \mathbb{R}^2$ with density smaller than 1, there exists a function $g \in L^2(\mathbb{R})$ such that (g, Λ) is a Riesz sequence and Case (2) holds for $H = \mathfrak{I}(\mathcal{G}(g, \Lambda))$. Due to Eq. (6) and Lemma 2, it suffices to consider separable lattices $\Lambda = \alpha\mathbb{Z}\times\beta\mathbb{Z}$ with $\alpha, \beta > 0$, $\alpha\beta > 1$. For simplicity of presentation, we will set $\alpha = 1$. For $m \in \mathbb{Z}$, define $E_m := m + [0, \frac{1}{\beta}]$, and let $g := \sqrt{\beta}\cdot\mathbf{1}_{E_0}$. Then

$$\begin{aligned} T_m M_{n\beta} g(x) &= \sqrt{\beta}\cdot e^{2\pi i n\beta(x-m)}\cdot\mathbf{1}_{[0,\frac{1}{\beta}]}(x-m) \\ &= \sqrt{\beta}\, e^{-2\pi i m n\beta}\cdot e^{2\pi i n\beta x}\cdot\mathbf{1}_{E_m}(x). \end{aligned}$$

Hence, for each $m \in \mathbb{Z}$ the system $(T_m M_{n\beta} g)_{n\in\mathbb{Z}}$ is an orthonormal basis for the subspace $L^2(E_m)$ of $L^2(\mathbb{R})$. Note that since $\frac{1}{\beta} < 1$, we have $[0, \frac{1}{\beta}] \subsetneq [0, 1]$ and therefore $E := \bigcup_{m\in\mathbb{Z}} E_m \subsetneq \mathbb{R}$. The system (g, Λ) is thus an orthonormal basis for $\mathcal{G} = \mathcal{G}(g, \Lambda) = L^2(E) \subsetneq L^2(\mathbb{R})$. It is easy to see that $T_x g \in \mathcal{G}$ if and only if $x \in \mathbb{Z}$. Moreover, we have $M_\omega g = \sqrt{\beta} \cdot e^{2\pi i \omega(\cdot)} \cdot \mathbf{1}_{E_0} \in L^2(E_0) \subset \mathcal{G}$ for all $\omega \in \mathbb{R}$. Therefore, Lemma 3 implies that $\mathfrak{I}(\mathcal{G}) = \mathbb{Z} \times \mathbb{R}$.

Case (3) : Let $g := \frac{1}{2}\,\mathbf{1}_{[-\frac{5}{6},-\frac{2}{3}]} + 2\,\mathbf{1}_{[0,\frac{1}{6}]} + 2\,\mathbf{1}_{[\frac{1}{3},\frac{1}{2}]} + \mathbf{1}_{[\frac{1}{2},1]}$. Then $(g, \mathbb{Z} \times 3\mathbb{Z})$ is a Riesz basis for $\mathcal{G} = \mathcal{G}(g, \mathbb{Z} \times 3\mathbb{Z}) \subsetneq L^2(\mathbb{R})$ and $\mathfrak{I}(\mathcal{G}) = \frac{1}{2}\mathbb{Z} \times 3\mathbb{Z}$ (see [4, Example 1] for more details).

Note that all the functions g in Example 6 are well localized in time but not in frequency. In the remainder of this section, we show that Case (2) in Proposition 5 *cannot* occur if (g, Λ) is a frame sequence with a sufficiently nice window g. In this case, the trichotomy from Proposition 5 becomes a dichotomy. By g being "sufficiently nice" we mean that $g \in \mathbb{W}(C, \ell^2)$, where

$$\mathbb{W}(C, \ell^2) := \left\{ f \in L^2(\mathbb{R}) : U_B f \in W(C, \ell^2) \text{ for all } B \in \mathbb{R}^{2\times 2} \text{ with } \det B = 1 \right\}.$$

Here, $W(C, \ell^2)$ is the so-called *Wiener Amalgam space* consisting of all *continuous* functions $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\|f\|_{W(C,\ell^2)} := \left(\sum_{k\in\mathbb{Z}} \sup_{x\in[k-1,k+1]} |f(x)|^2 \right)^{1/2} < \infty.$$

Recall from Sect. 2.3 that if $g \in \mathbb{W}(C, \ell^2)$ and if $B, C \in \mathbb{R}^{2\times 2}$ satisfy $\det B = \det C = 1$, then $U_C U_B g = \theta_{B,C}\, U_{CB}\, g \in W(C, \ell^2)$ for a suitable $\theta_{B,C} \in \mathbb{C}$. This shows that $U_B g \in \mathbb{W}(C, \ell^2)$ whenever $g \in \mathbb{W}(C, \ell^2)$ and $\det B = 1$.

The following lemma shows that the function classes considered in this paper (namely, $S_0(\mathbb{R})$ and $\mathbb{H}^1$) are contained in $\mathbb{W}(C, \ell^2)$.

Lemma 7 *We have $S_0(\mathbb{R}) \subset \mathbb{W}(C, \ell^2)$ and $\mathbb{H}^1 \subset \mathbb{W}(C, \ell^2)$.*

Proof If $g \in S_0(\mathbb{R})$, then [16, Proposition 12.1.3] shows that $U_B g \in S_0(\mathbb{R})$ for each $B \in \mathbb{R}^{2\times 2}$ with $\det B = 1$. Similarly, if $g \in \mathbb{H}^1$, then [6, Proof of Theorem 1.4] shows that $U_B g \in \mathbb{H}^1$ for each $B \in \mathbb{R}^{2\times 2}$ with $\det B = 1$. Therefore, it suffices to show that $S_0(\mathbb{R}) \subset W(C, \ell^2)$ and $\mathbb{H}^1 \subset W(C, \ell^2)$.

First, if $f \in S_0(\mathbb{R})$, then [16, Proposition 12.1.4] shows that $\widehat{f} \in L^1(\mathbb{R})$. By Fourier inversion, this implies that f has a continuous representative. Furthermore, by [16, Proposition 12.1.4] we have $f \in W(L^\infty, \ell^1)$. Combined with the embedding $\ell^1(\mathbb{Z}) \hookrightarrow \ell^2(\mathbb{Z})$, this easily implies $f \in W(L^\infty, \ell^2)$ and thus $f \in W(C, \ell^2)$.

Next, if $f \in \mathbb{H}^1 \subset H^1 = W^{1,2}(\mathbb{R})$, then [23, Theorem 7.16] shows (after changing f on a null-set) that f is absolutely continuous, and hence continuous, and satisfies $f(x) - f(y) = \int_y^x f'(t)\, dt$ for all $y < x$, where $f' \in L^2(\mathbb{R})$ is the weak derivative of f. Now, note that if $n \in \mathbb{Z}$ and $x, y \in [n-1, n+1]$, then

$$\begin{aligned} |f(x)| &\le |f(y)| + \int_{\min\{x,y\}}^{\max\{x,y\}} |f'(t)|\, dt \le |f(y)| + \int_{n-1}^{n+1} |f'(t)|\, dt \\ &\le |f(y)| + \sqrt{2} \left(\int_{n-1}^{n+1} |f'(t)|^2\, dt \right)^{1/2}, \end{aligned}$$

and hence $|f(x)|^2 \le 2|f(y)|^2 + 4\int_{n-1}^{n+1} |f'(t)|^2 dt$. Integration over $y \in [n-1, n+1]$ gives

$$2|f(x)|^2 \le 2\int_{n-1}^{n+1} |f(y)|^2\, dy + 8\int_{n-1}^{n+1} |f'(t)|^2\, dt,$$

for all $x \in [n-1, n+1]$, which finally implies

$$\|f\|_{W(C,\ell^2)}^2 \leq \sum_{n\in\mathbb{Z}} \left(\int_{n-1}^{n+1} |f(y)|^2 \, dy + 4 \int_{n-1}^{n+1} |f'(t)|^2 \, dt \right) \lesssim \|f\|_{L^2}^2 + \|f'\|_{L^2}^2 < \infty,$$

and hence $f \in W(C, \ell^2)$. □

We now show that Case (2) in Proposition 5 cannot occur if (g, Λ) is a frame sequence with generator $g \in \mathbb{W}(C, \ell^2) \setminus \{0\}$.

Theorem 8 *Let $g \in \mathbb{W}(C, \ell^2)\backslash\{0\}$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice such that (g, Λ) is a frame for $\mathcal{G} = \mathcal{G}(g, \Lambda)$. Then either $\mathfrak{I}(\mathcal{G}) = \mathbb{R}^2$ or there exist $\lambda_1, \lambda_2 \in \Lambda$ and $m, n \in \mathbb{N}_{\geq 1}$ such that*

$$\Lambda = \mathbb{Z} \cdot \lambda_1 + \mathbb{Z} \cdot \lambda_2 \quad \text{ and } \quad \mathfrak{I}(\mathcal{G}) = \mathbb{Z} \cdot \tfrac{\lambda_1}{m} + \mathbb{Z} \cdot \tfrac{\lambda_2}{n}. \tag{9}$$

Proof The claim is equivalent to the statement that either Case (1) or (3) in Proposition 5 holds for $H = \mathfrak{I}(\mathcal{G})$. Since Proposition 5 shows the trichotomy into Cases (1)–(3), it is enough to show that Case (2) in Proposition 5 cannot hold for H. Therefore, we assume towards a contradiction that Case (2) holds, i.e., there are $\lambda_1, \lambda_2 \in \Lambda$ and $n \in \mathbb{N}_{\geq 1}$ such that $\Lambda = \mathbb{Z} \cdot \lambda_1 + \mathbb{Z} \cdot \lambda_2$ and $\mathfrak{I}(\mathcal{G}) = \mathbb{Z} \cdot \frac{\lambda_1}{n} + \mathbb{R} \cdot \lambda_2$.

Step 1. We first derive a contradiction for the case $\lambda_1 = (\alpha, 0)^\top$ and $\lambda_2 = (0, \beta)^\top$ with some $\alpha, \beta > 0$. Then $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, and $\{0\} \times \mathbb{R} \subset \mathfrak{I}(\mathcal{G})$. For $f \in \mathcal{G}$ we thus have $M_\omega f \in \mathcal{G}$ for all $\omega \in \mathbb{R}$. By [29, Theorem 9.17] (applied to the translation invariant space $\mathcal{F}^{-1}\mathcal{G}$, with $\mathcal{F}$ denoting the Fourier transform), there exists a Borel measurable set $E \subset \mathbb{R}$ such that $\mathcal{G} = L^2(E)$, where we consider $L^2(E)$ as a closed subspace of $L^2(\mathbb{R})$, in the sense that $L^2(E) = \{f \in L^2(\mathbb{R}) \colon f = 0 \text{ a.e. on } \mathbb{R}\backslash E\}$.

Our goal is to show that $E = \mathbb{R}$, up to null-sets. This will then imply $\mathcal{G} = L^2(E) = L^2(\mathbb{R})$ and hence $\mathfrak{I}(\mathcal{G}) = \mathbb{R}^2$, providing the desired contradiction. Towards proving $E = \mathbb{R}$, let us consider for given $f \in L^2(\mathbb{R})$ the continuous function $\Gamma_f : \mathbb{R} \to \mathbb{R}$ defined by

$$\Gamma_f(\omega) := \langle S M_\omega f, M_\omega f \rangle, \quad \omega \in \mathbb{R},$$

where $S : L^2(\mathbb{R}) \to \mathcal{G}$ denotes the frame operator of (g, Λ). By [16, Proposition 7.1.1], the operator S has the *Walnut representation*

$$\langle Sf, h \rangle = \beta^{-1} \sum_{n\in\mathbb{Z}} \langle G_n \cdot T_{\frac{n}{\beta}} f, h \rangle \qquad \forall\, f, h \in L^\infty(\mathbb{R}) \text{ with compact support,}$$

where only finitely many terms of the sum do not vanish, and where

$$G_n(x) := \sum_{m\in\mathbb{Z}} g(x - m\alpha) \cdot \overline{g(x - \tfrac{n}{\beta} - m\alpha)}, \qquad x \in \mathbb{R}, n \in \mathbb{Z}.$$

The fact that $g \in W(C, \ell^2)$ easily implies that the series defining G_n converges locally uniformly, and that the G_n are continuous functions. Since G_n is also α-periodic, this means that each G_n is bounded.

Now, since multiplication with G_n commutes with the modulation M_ω, using the identity $T_{n/\beta} M_\omega = e^{-2\pi i \frac{n}{\beta}\omega} M_\omega T_{n/\beta}$, we get

$$\Gamma_f(\omega) = \beta^{-1} \sum_{n\in\mathbb{Z}} e^{-2\pi i \frac{n}{\beta}\omega} \langle G_n \cdot T_{\frac{n}{\beta}} f, f \rangle \qquad \forall\, f \in L_c^\infty(\mathbb{R}), \tag{10}$$

where $L_c^\infty(\mathbb{R})$ denotes the space of functions $f \in L^\infty(\mathbb{R})$ with compact support, and where there are only finitely many $n \in \mathbb{Z}$ (depending only on f, but not on the choice of ω) for which $\langle G_n \cdot T_{\frac{n}{\beta}} f, f\rangle \neq 0$.

Since (g, Λ) is a frame for $\mathcal{G}$ and since $M_\omega f \in \mathcal{G}$ for all $\omega \in \mathbb{R}$ and all $f \in \mathcal{G}$, there exists $A > 0$ such that $\Gamma_f(\omega) = \langle S M_\omega f, M_\omega f\rangle \geq A\|f\|_{L^2}^2$ for all $f \in \mathcal{G}$. Let us write $L_c^\infty(E)$ for the set of all compactly supported $f \in L^\infty(\mathbb{R})$ which satisfy $f = 0$ on $\mathbb{R}\setminus E$, and note that $L_c^\infty(E) \subset L^2(E) = \mathcal{G}$. For $f \in L_c^\infty(E)$, integrate the estimate $\Gamma_f(\omega) \geq A\|f\|_{L^2}^2$ over $[0, \beta]$ and apply Equation (10) to see

$$\beta A \|f\|_{L^2}^2 \leq \beta^{-1} \sum_{n\in\mathbb{Z}} \langle G_n \cdot T_{\frac{n}{\beta}} f, f\rangle \int_0^\beta e^{-2\pi i \frac{n}{\beta}\omega}\, d\omega = \langle G_0 f, f\rangle = \langle hf, f\rangle,$$

where $h := G_0 = \sum_{m\in\mathbb{Z}} |T_{m\alpha} g|^2$. We have thus shown

$$\int_E \big(h(x) - \beta A\big) \cdot |f(x)|^2\, dx \geq 0 \qquad \forall f \in L_c^\infty(E).$$

Using standard arguments, this implies that $h(x) \geq \beta A$ for almost all $x \in E$.

Since $T_{m\alpha} g \in \mathcal{G} = L^2(E)$ and thus $T_{m\alpha} g(x) = 0$ for almost all $x \in \mathbb{R}\setminus E$ and arbitrary $m \in \mathbb{Z}$, it follows that $h(x) = 0$ for almost all $x \in \mathbb{R}\setminus E$. Recall from above that $h(x) \geq \beta A$ for almost all $x \in E$; thus, $h(x) \in \{0\} \cup [\beta A, \infty)$ almost everywhere. Also recall from above that $h = G_0$ is continuous. Hence, the *open* set $h^{-1}((0, \beta A))$ has measure zero and is thus empty; therefore, we see $h(x) \in \{0\} \cup [\beta A, \infty)$ for *all* $x \in \mathbb{R}$. By the intermediate value theorem, this implies that $h(x) \geq \beta A$ for all $x \in \mathbb{R}$ (since $h \geq |g|^2$ and $g \not\equiv 0$) and thus, indeed, $E = \mathbb{R}$ (up to null-sets), since $h(x) = 0$ a.e. on $\mathbb{R}\setminus E$.

Step 2. Let Λ be a general lattice. Recall that $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ and $\mathfrak{I}(\mathcal{G}) = \mathbb{Z}\frac{\lambda_1}{n} + \mathbb{R}\lambda_2$. By Lemma 2 there exists $B \in \mathbb{R}^{2\times 2}$ with $\det B = 1$ such that $B[\lambda_1, \lambda_2] = \operatorname{diag}(\alpha, \beta)$ for certain $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. We thus obtain $\Lambda_B = B\Lambda = B[\lambda_1, \lambda_2]\mathbb{Z}^2 = |\alpha|\mathbb{Z} \times |\beta|\mathbb{Z}$ and

$$\begin{aligned} \mathfrak{I}(\mathcal{G}_B) &= B\mathfrak{I}(\mathcal{G}) = B[\lambda_1, \lambda_2]\operatorname{diag}(\tfrac{1}{n}, 1)(\mathbb{Z} \times \mathbb{R}) \\ &= \operatorname{diag}(\tfrac{|\alpha|}{n}, |\beta|)(\mathbb{Z} \times \mathbb{R}) = \tfrac{|\alpha|}{n}\mathbb{Z} \times \mathbb{R}; \end{aligned}$$

see (6). In particular, $\{0\} \times \mathbb{R} \subset \mathfrak{I}(\mathcal{G}_B)$. Hence, since $g_B = U_B g \in \mathbb{W}(C, \ell^2)$ and (g_B, Λ_B) is a frame for $\mathcal{G}_B = U_B\mathcal{G} \subsetneq L^2(\mathbb{R})$ (cf. Sect. 2.3), we are in the situation of Step 1, which we proved to be impossible. □

By combining Theorem 8 and Lemma 7, we obtain the following corollary.

Corollary 9 *Let $g \in S_0(\mathbb{R}) \setminus \{0\}$ or $g \in \mathbb{H}^1 \setminus \{0\}$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice such that (g, Λ) is a Riesz basis for $\mathcal{G} := \mathcal{G}(g, \Lambda)$. Then $\mathfrak{I}(\mathcal{G})$ is a refinement of Λ as in (9).*

Proof The Balian-Low theorem [10, Theorem 2.3] and the Amalgam Balian-Low theorem [2, Theorem 3.2], show that $\mathcal{G} \neq L^2(\mathbb{R})$. Therefore, Lemma 4 implies $\mathfrak{I}(\mathcal{G}) \neq \mathbb{R}^2$. The rest follows from Lemma 7 and Theorem 8. □

4 Time-frequency shift invariance: duality

Let us consider a Gabor Riesz sequence (g, Λ) with $g \in \mathbb{W}(C, \ell^2)$ as in the previous section, and assume that $\mathcal{G} := \mathcal{G}(g, \Lambda) \subsetneq L^2(\mathbb{R})$, but that there exists an additional time-frequency shift, meaning $\mathfrak{I}(\mathcal{G}) \neq \Lambda$. In view of Theorem 1 it is our goal to show that this is impossible, at least if $g \in S_0$. To make the situation more accessible, we first reduce to the case where $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ is separable, and where the additional time-frequency shift is of the form $(\frac{\alpha}{\nu}, 0)^\top$ for some $\nu \in \mathbb{N}_{\geq 2}$, meaning that $T_{\alpha/\nu}\, g \in \mathcal{G}$. After that, we provide a characterization of this simplified condition in terms of the adjoint Gabor system. It is this characterization that we will use to prove our main result, Theorem 1, in the next section.

Lemma 10 *Let $g \in \mathbb{W}(C, \ell^2)\backslash\{0\}$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice such that (g, Λ) is a frame for $\mathcal{G} := \mathcal{G}(g, \Lambda)$. If $\mathcal{G} \neq L^2(\mathbb{R})$ and $\mathfrak{I}(\mathcal{G}) \neq \Lambda$, there exist a matrix $B \in \mathbb{R}^{2\times 2}$ with* $\det B = 1$ *and $\alpha, \beta > 0$ such that $\Lambda_B = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ and $(\frac{\alpha}{\nu}, 0)^\top \in \mathfrak{I}(\mathcal{G}_B)$ for some $\nu \in \mathbb{N}$, $\nu \geq 2$ (i.e., $T_{\frac{\alpha}{\nu}}\, g_B \in \mathcal{G}_B$).*

Proof Theorem 8 and Lemma 4 show $\Lambda = [\lambda_1, \lambda_2]\,\mathbb{Z}^2$ and $\mathfrak{I}(\mathcal{G}) = [\frac{\lambda_1}{m}, \frac{\lambda_2}{n}]\,\mathbb{Z}^2$ for suitable vectors $\lambda_1, \lambda_2 \in \mathbb{R}^2$ and $m, n \in \mathbb{N}\backslash\{0\}$. We may safely assume that $m \neq 1$. Indeed, since $\mathfrak{I}(\mathcal{G}) \neq \Lambda$, we have $(m, n) \neq (1, 1)$. If $m = 1$, then with $J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ also $\Lambda = [\lambda_1, \lambda_2]J\mathbb{Z}^2 = [-\lambda_2, \lambda_1]\mathbb{Z}^2$ and $\mathfrak{I}(\mathcal{G}) = [\frac{-\lambda_2}{n}, \frac{\lambda_1}{m}]\mathbb{Z}^2$.

Now, by Lemma 2 there exists a matrix $B \in \mathbb{R}^{2\times 2}$ with $\det B = 1$ such that $B[\lambda_1, \lambda_2] = \operatorname{diag}(\alpha, \beta)$, where $\alpha, \beta \in \mathbb{R}\backslash\{0\}$. Hence, $\Lambda_B = B\Lambda = |\alpha|\mathbb{Z} \times |\beta|\mathbb{Z}$ and

$$\mathfrak{I}(\mathcal{G}_B) = B\mathfrak{I}(\mathcal{G}) = B[\lambda_1, \lambda_2]\operatorname{diag}(\tfrac{1}{m}, \tfrac{1}{n})\mathbb{Z}^2 = \tfrac{|\alpha|}{m}\mathbb{Z} \times \tfrac{|\beta|}{n}\mathbb{Z};$$

see (6). In particular, $(\frac{|\alpha|}{m}, 0)^\top \in \mathfrak{I}(\mathcal{G}_B)$ and $m \geq 2$. □

In what follows, fix $g \in L^2(\mathbb{R})$, $\alpha, \beta > 0$, $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, and $\nu \in \mathbb{N}_{\geq 2}$, and assume that (g, Λ) is a frame for $\mathcal{G} = \mathcal{G}(g, \Lambda)$. The adjoint system

$$\mathcal{F} := \left(g, \tfrac{1}{\beta}\mathbb{Z} \times \tfrac{1}{\alpha}\mathbb{Z}\right) = \left\{T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}}\, g : k, \ell \in \mathbb{Z}\right\}$$

is then a frame for its closed linear span $\mathcal{K}$ by [28, Theorem 2.2 (c)]. Note that $\mathcal{K} = L^2(\mathbb{R})$ if and only if (g, Λ) is a Riesz sequence (cf. [28, Theorem 2.2 (e)] or [16, Theorem 7.4.3]).

It is natural to ask what an additional time-frequency shift invariance of the form $T_{\frac{\alpha}{\nu}}\, g \in \mathcal{G}$ means for the adjoint system $\mathcal{F}$. To answer this question, we set

$$\mathcal{F}_s := \left\{T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} M_{\frac{s}{\alpha}}\, g : k, \ell \in \mathbb{Z}\right\}, \qquad s = 0, \ldots, \nu - 1.$$

Again by [28, Theorem 2.2 (c)], $\mathcal{F}_0$ is a frame sequence if and only if the system $(g, \frac{\alpha}{\nu}\mathbb{Z} \times \beta\mathbb{Z})$ is a frame sequence. In this case, each $\mathcal{F}_s$ is a frame sequence because $M_{\frac{s}{\alpha}}\mathcal{F}_0$ is, and multiplying the vectors of a frame sequence by unimodular constants results in a frame sequence. For $s \in \{0, \ldots, \nu - 1\}$, we set $\mathcal{L}_s := \overline{\operatorname{span}}\, \mathcal{F}_s = M_{\frac{s}{\alpha}}\mathcal{L}_0$. Note that

$$\mathcal{K} = \mathcal{L}_0 + \cdots + \mathcal{L}_{\nu-1}.$$

Indeed, the inclusion "$\supset$" is trivial. Conversely, since $\mathcal{F}$ is a frame sequence, each $f \in \mathcal{K}$ satisfies $f = \sum_{k,\ell\in\mathbb{Z}} c_{k,\ell}\, T_{k/\beta} M_{\ell/\alpha}\, g$ with a suitable sequence $c = (c_{k,\ell})_{k,\ell\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$. Since $\mathcal{F}$ is a Bessel sequence, the function

$$f_s := \sum_{k,\ell\in\mathbb{Z}} c_{k,\ell\nu+s} T_{\frac{k}{\beta}} M_{\frac{\ell\nu+s}{\alpha}}\, g \quad \in \mathcal{L}_s$$

is well-defined for $s \in \{0, \ldots, \nu - 1\}$, and $f = f_0 + \cdots + f_{\nu-1} \in \mathcal{L}_0 + \cdots + \mathcal{L}_{\nu-1}$.

In the sequel, the symbol $\boxplus$ denotes the *direct* (not necessarily orthogonal) sum of subspaces, whereas $\oplus$ is used to denote an orthogonal sum.

The next theorem provides several equivalent conditions for the additional time-frequency invariance $T_{\frac{\alpha}{\nu}} g \in \mathcal{G}$ in terms of properties of the adjoint system $\mathcal{F}$. The—for our purposes—most useful statement from Theorem 11 is that an additional time-frequency shift of the form $T_{\frac{\alpha}{\nu}} g \in \mathcal{G}$ implies that the scaled frame operator $(\alpha\beta)^{-1} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g}$ is a projection.

Theorem 11 *Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$, and assume that $(g, \alpha\mathbb{Z}\times\beta\mathbb{Z})$ is a frame sequence with canonical dual window $\gamma \in \mathcal{G}$, where $\mathcal{G} = \mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$. Let $\nu \in \mathbb{N}_{\geq 2}$, and define the systems $\mathcal{F}_s$ and the spaces $\mathcal{K}$, $\mathcal{L}_s$ as above, and let $S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g}$ be the operator defined as in Equation* (2).

Then the following are equivalent:

(i) $T_{\frac{\alpha}{\nu}} g \in \mathcal{G}$.
(ii) $(\alpha\beta)^{-1} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g} M_{\frac{s}{\alpha}} g = \delta_{s,0} \cdot g$ *for* $s = 0, \dots, \nu - 1$.
(iii) $\mathcal{K} = \mathcal{L}_0 \boxplus \cdots \boxplus \mathcal{L}_{\nu-1}$.
(iv) $\gamma \perp \mathcal{L}_s$ *for* $s = 1, \dots, \nu - 1$, *that is,* $\langle \gamma, T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}} g \rangle = 0$ *for all* $k \in \mathbb{Z}$ *and all* $\ell \in \mathbb{Z}\setminus\nu\mathbb{Z}$.

If one of (i)–(iv) *holds, then for each $s = 0, \dots, \nu - 1$ the system $\mathcal{F}_s$ is a frame for $\mathcal{L}_s$ and the operator $P_s := (\alpha\beta)^{-1} M_{\frac{s}{\alpha}} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g} M_{-\frac{s}{\alpha}}$ is the* (*possibly non-orthogonal*) *projection onto $\mathcal{L}_s$ with respect to the decomposition $L^2(\mathbb{R}) = (\mathcal{L}_0 \boxplus \cdots \boxplus \mathcal{L}_{\nu-1}) \oplus \mathcal{K}^\perp$.*

Remark 12 The statement (ii) is equivalent to having $P_0 g = g$ and $P_s g = 0$ for $s = 1, \dots, \nu - 1$.

Example 13 In both of the following two examples we let $g = \mathbf{1}_{[0,1]}$, $\beta = 1$, and $\nu = 2$.

(a) Let $\alpha = 2$ (i.e., $\Lambda = 2\mathbb{Z} \times \mathbb{Z}$). Then $\mathcal{G}$ consists of those functions in $L^2(\mathbb{R})$ with support in $\bigcup_{k\in\mathbb{Z}}[2k, 2k + 1]$. Hence, $g = T_1 g \notin \mathcal{G}$, so that (i) is violated. Moreover, $\mathcal{L}_0 = \mathcal{L}_1 = L^2(\mathbb{R})$, resulting in $\mathcal{K} = L^2(\mathbb{R}) = \mathcal{L}_0 + \mathcal{L}_1$, which is *not* a direct sum.
(b) Let $\alpha = 1$ (i.e., $\Lambda = \mathbb{Z} \times \mathbb{Z}$). In this case, $\mathcal{G} = L^2(\mathbb{R})$ so that (i) is trivially satisfied. Furthermore, $\mathcal{L}_0$ is the set of functions $f \in L^2(\mathbb{R})$ whose restrictions to the intervals $[2k, 2k + 1]$, $k \in \mathbb{Z}$, only have non-zero Fourier coefficients with even index. The space $\mathcal{L}_1$ is described similarly with "even" replaced by "odd". Hence $\mathcal{K} = L^2(\mathbb{R}) = \mathcal{L}_0 \oplus \mathcal{L}_1$.

Proof of Theorem 11 First, note that $\mathcal{F}$ is a frame sequence by Ron-Shen duality (see [28, Theorem 2.2(c)]). We will frequently use the following fact (see [28, Theorem 2.3]):

$$(\alpha\beta)^{-1}\gamma \text{ is the canonical dual window of } \mathcal{F} = \left\{T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}} g : k, \ell \in \mathbb{Z}\right\}; \tag{11}$$

in particular, $\gamma \in \overline{\operatorname{span}}\, \mathcal{F} = \mathcal{K}$.

For the rest of the proof we set $P := (\alpha\beta)^{-1} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g}$. It is well known (see for instance [16, Eq. (5.25)]) that

$$P\, T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} = T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} P \qquad \text{for all } k, \ell \in \mathbb{Z}. \tag{12}$$

Moreover, Eq. (3) applied to the lattice $\frac{1}{\beta}\mathbb{Z} \times \frac{\nu}{\alpha}\mathbb{Z}$ shows for $f \in \mathcal{B}_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z}}$ that

$$Pf = \sum_{m,n\in\mathbb{Z}} c_{m,n} \cdot T_{\frac{m\alpha}{\nu}} M_{n\beta} f \qquad \text{with} \qquad c_{m,n} = \tfrac{1}{\nu}\langle g, T_{\frac{m\alpha}{\nu}} M_{n\beta}\gamma\rangle. \tag{13}$$

Let us denote the orthogonal projection onto the subspace $\mathcal{K} = \overline{\operatorname{span}}\,\mathcal{F}$ by $P_{\mathcal{K}}$. Note that Equation (11) implies

$$S_{\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z},(\alpha\beta)^{-1}\gamma,g}|_{\mathcal{K}} = \mathrm{id}_{\mathcal{K}} \quad \text{and} \quad S_{\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z},(\alpha\beta)^{-1}\gamma,g}|_{\mathcal{K}^\perp} \equiv 0,$$

where $\mathrm{id}_{\mathcal{K}}$ denotes the identity operator on $\mathcal{K}$, so that $P_{\mathcal{K}} = S_{\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z},(\alpha\beta)^{-1}\gamma,g}$ (see [8, Proposition 5.2.3]). Similarly, the orthogonal projection $P_{\mathcal{G}}$ onto $\mathcal{G}$ satisfies $P_{\mathcal{G}} = S_{\alpha\mathbb{Z}\times\beta\mathbb{Z},\gamma,g}$. Next, using Equation (13) and the elementary identity $\sum_{s=0}^{\nu-1} e^{2\pi i\frac{ms}{\nu}} = \nu\cdot\mathbf{1}_{\nu\mathbb{Z}}(m)$, we obtain

$$\begin{aligned}
\sum_{s=0}^{\nu-1} M_{\frac{s}{\alpha}} P M_{-\frac{s}{\alpha}} f &= \sum_{s=0}^{\nu-1} \sum_{m,n\in\mathbb{Z}} c_{m,n}\cdot M_{\frac{s}{\alpha}} T_{\frac{m\alpha}{\nu}} M_{n\beta} M_{-\frac{s}{\alpha}} f \\
&= \sum_{m,n\in\mathbb{Z}} c_{m,n}\left(\sum_{s=0}^{\nu-1} e^{2\pi i\frac{ms}{\nu}}\right) T_{\frac{m\alpha}{\nu}} M_{n\beta} f \\
&= \nu \sum_{m,n\in\mathbb{Z}} c_{\nu m,n} T_{m\alpha} M_{n\beta} f \\
&= \sum_{m,n\in\mathbb{Z}} \langle g, T_{m\alpha} M_{n\beta}\gamma\rangle \cdot T_{m\alpha} M_{n\beta} f \\
(\text{Equation (3)}) &= (\alpha\beta)^{-1} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z},\gamma,g} f = S_{\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z},(\alpha\beta)^{-1}\gamma,g} f = P_{\mathcal{K}} f,
\end{aligned} \tag{14}$$

for all $f \in \mathcal{B}_{\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z}}$ and hence for all $f \in L^2(\mathbb{R})$ by density. Here, we used that if $f \in \mathcal{B}_{\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z}}$, then $M_{-\frac{s}{\alpha}} f \in \mathcal{B}_{\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z}} \subset \mathcal{B}_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z}}$. Next, for $s = 0,\dots,\nu-1$, we see by another application of Eq. (13) that

$$\begin{aligned}
M_{-\frac{s}{\alpha}} P M_{\frac{s}{\alpha}} g &= \frac{1}{\nu} \sum_{m,n\in\mathbb{Z}} \langle g, T_{\frac{m\alpha}{\nu}} M_{n\beta}\gamma\rangle \cdot M_{-\frac{s}{\alpha}} T_{\frac{m\alpha}{\nu}} M_{n\beta} M_{\frac{s}{\alpha}} g \\
&= \frac{1}{\nu} \sum_{m,n\in\mathbb{Z}} \sum_{r=0}^{\nu-1} \left\langle g, T_{\frac{\nu m-r}{\nu}\alpha} M_{n\beta}\gamma\right\rangle \cdot M_{-\frac{s}{\alpha}} T_{\frac{\nu m-r}{\nu}\alpha} M_{n\beta} M_{\frac{s}{\alpha}} g \\
&= \frac{1}{\nu} \sum_{r=0}^{\nu-1} e^{2\pi i\frac{sr}{\nu}} \cdot T_{-\frac{r\alpha}{\nu}} \sum_{m,n\in\mathbb{Z}} \left\langle T_{\frac{r\alpha}{\nu}} g, T_{m\alpha} M_{n\beta}\gamma\right\rangle \cdot T_{m\alpha} M_{n\beta}\, g \\
&= \frac{1}{\nu} \sum_{r=0}^{\nu-1} e^{2\pi i\frac{sr}{\nu}} \cdot T_{-\frac{r\alpha}{\nu}} P_{\mathcal{G}} T_{\frac{r\alpha}{\nu}} g,
\end{aligned} \tag{15}$$

where we used that $P_{\mathcal{G}} = S_{\alpha\mathbb{Z}\times\beta\mathbb{Z},\gamma,g}$. Equation (15) shows that the vectors

$$v = \left(M_{-\frac{s}{\alpha}} P M_{\frac{s}{\alpha}} g\right)_{s=0}^{\nu-1} \quad \text{and} \quad u = \left(T_{-\frac{r\alpha}{\nu}} P_{\mathcal{G}} T_{\frac{r\alpha}{\nu}} g\right)_{r=0}^{\nu-1}$$

in $\left(L^2(\mathbb{R})\right)^\nu$ satisfy $F_\nu u = \sqrt{\nu}\cdot v$, where F_ν is the ν-dimensional discrete Fourier matrix $F_\nu = \nu^{-1/2}\left(\omega^{sr}\right)_{s,r=0}^{\nu-1}$ with $\omega = e^{2\pi i/\nu}$.

With this preparation, we now prove the equivalence of (i)–(iv).

(i)⇔(ii): If $T_{\frac{\alpha}{\nu}} g \in \mathcal{G}$, then Lemma 3 shows that $T_{\frac{r\alpha}{\nu}} g \in \mathcal{G}$ for all $r \in \mathbb{Z}$, so that $T_{-\frac{r\alpha}{\nu}} P_{\mathcal{G}} T_{\frac{r\alpha}{\nu}} g = g$ for all $r \in \mathbb{Z}$. Since $\frac{1}{\nu}\sum_{r=0}^{\nu-1} e^{2\pi i\frac{sr}{\nu}} = \delta_{s,0}$ for $s \in \{0,\dots,\nu-1\}$, the statement (ii) then follows from (15). Conversely, if (ii) holds, then $v = (g, 0, \dots, 0)^\top$, which implies that $u = \sqrt{\nu}\cdot F_\nu^* v = (g, g, \dots, g)^\top$. In particular, $T_{-\frac{\alpha}{\nu}} P_{\mathcal{G}} T_{\frac{\alpha}{\nu}} g = g$, i.e., $T_{\frac{\alpha}{\nu}} g \in \mathcal{G}$.

(ii)⇒(iii): Since $Pg = g$, it is a consequence of (12) that $P|_{\mathcal{L}_0} = \mathrm{id}_{\mathcal{L}_0}$. Furthermore, for $s \in \{1, \ldots, \nu-1\}$ and $k, \ell \in \mathbb{Z}$, Eq. (12) implies

$$P T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} M_{\frac{s}{\alpha}} g = T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} P M_{\frac{s}{\alpha}} g = 0,$$

which shows $P|_{\mathcal{L}_s} = 0$. Using these observations and noting that $\mathcal{L}_r = M_{r/\alpha}\mathcal{L}_0$, we see for $r, s \in \{0, \ldots, \nu-1\}$ that $P_r|_{\mathcal{L}_r} = M_{r/\alpha} P M_{-r/\alpha}|_{\mathcal{L}_r} = \mathrm{id}_{\mathcal{L}_r}$ and furthermore $P_r|_{\mathcal{L}_s} = M_{r/\alpha} P M_{-r/\alpha}|_{\mathcal{L}_s} = 0$ for $s \neq r$. Therefore, the sum $\mathcal{K} = \mathcal{L}_0 \boxplus \cdots \boxplus \mathcal{L}_{\nu-1}$ is direct, and $P_s|_{\mathcal{K}} = M_{s/\alpha} P M_{-s/\alpha}|_{\mathcal{K}}$ is the projection onto $\mathcal{L}_s$ with respect to this decomposition. Finally, since $\gamma \in \mathcal{K}$ and since $\mathcal{K}$ is invariant under $T_{k/\beta} M_{\ell/\alpha}$, it follows by definition of the operator $P_s = (\alpha\beta)^{-1} M_{\frac{s}{\alpha}} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g} M_{-\frac{s}{\alpha}}$ that $P_s|_{\mathcal{K}^\perp} = 0$. Hence, P_s is the projection onto $\mathcal{L}_s$ with respect to the decomposition $L^2(\mathbb{R}) = (\mathcal{L}_0 \boxplus \cdots \boxplus \mathcal{L}_{\nu-1}) \oplus \mathcal{K}^\perp$.

Finally, we show that $\mathcal{F}_s$ is a frame for $\mathcal{L}_s$, where it clearly suffices to show this for $s = 0$. Since $\mathcal{F}_0$ is a Bessel sequence, [8, Corollary 5.5.2] shows that we only need to prove that the synthesis operator

$$D: \quad \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R}), \qquad (c_{k,\ell})_{k,\ell\in\mathbb{Z}} \mapsto \sum_{k,\ell\in\mathbb{Z}} c_{k,\ell}\, T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} g$$

has closed range $\operatorname{ran} D = \mathcal{L}_0$. Directly from the definition of $\mathcal{L}_0 = \overline{\operatorname{span}}\, \mathcal{F}_0$, we see $\operatorname{ran} D \subset \mathcal{L}_0$. Conversely, if $f \in \mathcal{L}_0$, then

$$f = Pf = (\alpha\beta)^{-1} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g} f = (\alpha\beta)^{-1} Dc \in \operatorname{ran} D$$

for the sequence $c = (c_{k,\ell})_{k,\ell\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$ given by $c_{k,\ell} = \langle f, T_{k/\beta} M_{\ell\nu/\alpha}\gamma\rangle$.

(iii)⇒(ii): Since $P = (\alpha\beta)^{-1} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g}$, we see directly from the definition of $S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g}$ that $\operatorname{ran} P \subset \mathcal{L}_0$. Hence, $Pg - g \in \mathcal{L}_0$. On the other hand, again as a consequence of $\operatorname{ran} P \subset \mathcal{L}_0$ we see that $M_{s/\alpha} P M_{-s/\alpha}\, g \in \mathcal{L}_s$, so that Equation (14) implies

$$\mathcal{L}_0 \ni P\,g - g = P\,g - P_{\mathcal{K}}\,g = -\sum_{s=1}^{\nu-1} M_{\frac{s}{\alpha}} P M_{-\frac{s}{\alpha}}\, g \in \mathcal{L}_1 + \cdots + \mathcal{L}_{\nu-1}, \tag{16}$$

and thus $Pg = g$ since the sum $\mathcal{L}_0 + \cdots + \mathcal{L}_{\nu-1}$ is direct. Similarly, for any $s \in \{1, \ldots, \nu-1\}$ we get because of $\operatorname{ran} P \subset \mathcal{L}_0$ that $M_{s/\alpha} P M_{-s/\alpha}\, g \in \mathcal{L}_s$; but this implies as in Equation (16) that

$$\mathcal{L}_s \ni M_{\frac{s}{\alpha}} P M_{-\frac{s}{\alpha}}\, g = P_{\mathcal{K}} g - \sum_{r\neq s} M_{\frac{r}{\alpha}} P M_{-\frac{r}{\alpha}}\, g$$
$$\in \mathcal{L}_0 + \operatorname{span}\{\mathcal{L}_r : r \neq s\} = \operatorname{span}\{\mathcal{L}_r : r \neq s\}.$$

Again, since $\mathcal{L}_0 + \cdots + \mathcal{L}_{\nu-1}$ is a direct sum, this implies $P M_{-\frac{s}{\alpha}}\, g = 0$ for $s = 1, \ldots, \nu-1$. Since P commutes with $M_{\pm\nu/\alpha}$ (see Equation (12)), we have $P M_{(\nu-s)/\alpha}\, g = 0$ and therefore $P M_{s/\alpha}\, g = 0$ for $s = 1, \ldots, \nu-1$.

(i)⇒(iv): Note that $(\gamma, \alpha\mathbb{Z}\times\beta\mathbb{Z})$ is a frame sequence and furthermore that $\mathcal{G} = \mathcal{G}(\gamma, \alpha\mathbb{Z}\times\beta\mathbb{Z})$. Moreover, Lemma 3 shows that $T_{\frac{\alpha}{\nu}} g \in \mathcal{G}$ if and only if $\mathcal{G}$ is invariant under $T_{\frac{\alpha}{\nu}}$, if and only if $T_{\frac{\alpha}{\nu}}\gamma \in \mathcal{G}$. Let us consider the setting above with g and γ interchanged: Define $\mathcal{F}^* := \left\{T_{k/\beta} M_{\ell/\alpha}\gamma : k, \ell \in \mathbb{Z}\right\}$ and

$$\mathcal{F}_s^* := \left\{T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} M_{\frac{s}{\alpha}}\gamma : k, \ell \in \mathbb{Z}\right\}, \qquad s = 0, \ldots, \nu-1.$$

Then, by using the implication "(i)⇒(iii)" in this setting, we obtain that $\mathcal{K}^* = \mathcal{L}_0^* \boxplus \cdots \boxplus \mathcal{L}_{\nu-1}^*$, where $\mathcal{L}_s^* := \overline{\operatorname{span}}\,\mathcal{F}_s^*$ and $\mathcal{K}^* := \overline{\operatorname{span}}\,\mathcal{F}^*$, and that $(\alpha\beta)^{-1} M_{\frac{s}{\alpha}} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},g,\gamma} M_{-\frac{s}{\alpha}}$ is the projection onto $\mathcal{L}_s^*$ with respect to the decomposition $L^2(\mathbb{R}) = (\mathcal{L}_0^* \boxplus \cdots \boxplus \mathcal{L}_{\nu-1}^*) \oplus (\mathcal{K}^*)^\perp$. Note that $\mathcal{K}^* = \mathcal{K}$ by Equation (11) and we have $S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},g,\gamma} = (S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g})^*$ from the definition of cross frame operators in (2). Hence, $M_{s/\alpha} P^* M_{-s/\alpha}$ is the projection onto $\mathcal{L}_s^*$ with respect to the decomposition $L^2(\mathbb{R}) = (\mathcal{L}_0^* \boxplus \cdots \boxplus \mathcal{L}_{\nu-1}^*) \oplus \mathcal{K}^\perp$. In particular, using the general formula $(\ker T)^\perp = \overline{\operatorname{ran} T^*}$ for a bounded operator $T : \mathcal{H} \to \mathcal{H}$ (see [9, Remarks after Theorem II.2.19]) and the elementary identity $(A + B)^\perp = A^\perp \cap B^\perp$ for subspaces $A, B \subset \mathcal{H}$, we get

$$\mathcal{L}_0^* = \overline{\operatorname{ran} P^*} = (\ker P)^\perp = \big(\mathcal{L}_1 \boxplus \cdots \boxplus \mathcal{L}_{\nu-1}\big)^\perp \cap \mathcal{K}.$$

This implies $\mathcal{L}_0^* \perp \mathcal{L}_s$ for $s = 1, \ldots, \nu - 1$, which is equivalent to (iv).

(iv)⇒(ii): For $s \in \{1, \ldots, \nu - 1\}$, we have

$$P M_{\frac{s}{\alpha}} g = \frac{1}{\alpha\beta} \sum_{k,\ell\in\mathbb{Z}} \Big\langle M_{\frac{s}{\alpha}} g, T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} \gamma \Big\rangle T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} g = 0.$$

Thanks to Eq. (14), this implies $g = P_{\mathcal{K}} g = P g$. Overall, we have thus shown $P M_{s/\alpha}\, g = \delta_{s,0}\, g$ for all $s \in \{0, \ldots, \nu - 1\}$. □

Note that with $P := (\alpha\beta)^{-1} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,g}$, the condition $Pf = f$ for $f \in \mathcal{L}_0$ means that $(\alpha\beta)^{-1}\gamma$ is a dual window for the frame sequence $\mathcal{F}_0 = (g, \frac{1}{\beta}\mathbb{Z} \times \frac{\nu}{\alpha}\mathbb{Z})$. However, it is possible that $\gamma \notin \mathcal{L}_0 = \overline{\operatorname{span}}\,\mathcal{F}_0$.

5 Proof of the main theorem

In this section, we prove our main result, Theorem 1, which we state here once more for the convenience of the reader.

Theorem 1 *If $g \in S_0(\mathbb{R})$ and $\Lambda \subset \mathbb{R}^2$ is a lattice such that the Gabor system (g, Λ) is a Riesz basis for its closed linear span $\mathcal{G}(g, \Lambda)$, then the time-frequency shifts $T_a M_b$ that leave $\mathcal{G}(g, \Lambda)$ invariant satisfy $(a, b) \in \Lambda$.*

A crucial ingredient for the proof of Theorem 1 is the following auxiliary statement which relies on a deep result concerning the structure of the *irrational rotation algebra* $\mathcal{A}_\theta$ (see [11, 26, 27]). We postpone its proof to Appendix A.

Theorem 14 *Let $\mathcal{H} \neq \{0\}$ be a Hilbert space and let $U, V \in \mathcal{B}(\mathcal{H})$ be unitary operators on $\mathcal{H}$ with $UV = e^{2\pi i\theta} VU$ for some $\theta \in \mathbb{R}\setminus\mathbb{Q}$.*

If $a = (a_{k,\ell})_{k,\ell\in\mathbb{Z}} \in \ell^1(\mathbb{Z}^2)$ is such that the operator $P_a := \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell} V^k U^\ell$ satisfies $P_a^2 = P_a$, then $a_{0,0} \in \mathbb{Z} + \theta\mathbb{Z}$.

Proof of Theorem 1 The claim is true if Λ has rational density; see [4, Theorem 1]. Thus, assume that Λ has irrational density $d(\Lambda) \in \mathbb{R}\setminus\mathbb{Q}$. Write $\Lambda = A\mathbb{Z}^2$ with an invertible matrix $A \in \mathbb{R}^{2\times 2}$.

Due to the Amalgam Balian–Low theorem [2, Theorem 3.2], it is not possible that $\mathcal{G} := \mathcal{G}(g, \Lambda) = L^2(\mathbb{R})$. Hence, $\mathcal{G} \neq L^2(\mathbb{R})$. Suppose towards a contradiction that $\mathfrak{I}(\mathcal{G}) \supsetneq \Lambda$. According to Lemma 10 there exist $B \in \mathbb{R}^{2\times 2}$ with $\det B = 1$ and $\alpha, \beta > 0$ such that $\Lambda_B =$

$\alpha\mathbb{Z} \times \beta\mathbb{Z}$ and $T_{\frac{\alpha}{\nu}} g_B \in \mathcal{G}_B$ for some $\nu \in \mathbb{N}_{\geq 2}$. Set $h := g_B$ and $\mathcal{G}_h := \mathcal{G}_B = \mathcal{G}(h, \alpha\mathbb{Z} \times \beta\mathbb{Z})$. Then $h \in S_0(\mathbb{R})$ by [16, Proposition 12.1.3] and $T_{\frac{\alpha}{\nu}} h \in \mathcal{G}_h$. Furthermore, note that since (g, Λ) is a Riesz sequence, so is $(h, \alpha\mathbb{Z} \times \beta\mathbb{Z}) = (g_B, \Lambda_B)$ (cf. Sect. 2.3).

Let γ be the canonical dual window for $(h, \alpha\mathbb{Z}\times\beta\mathbb{Z})$, so that $\langle \gamma, T_{m\alpha} M_{n\beta} h\rangle = \delta_{m,0}\,\delta_{n,0}$ for $m, n \in \mathbb{Z}$. By Ron-Shen duality (see [16, Theorem 7.4.3]), the adjoint system $(h, \frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z})$ is a frame for $L^2(\mathbb{R})$.

Since $T_{\frac{\alpha}{\nu}} h \in \mathcal{G}_h$, Theorem 11 implies that $P_0 := (\alpha\beta)^{-1} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,h}$ is an idempotent (i.e., $P_0^2 = P_0$). We now wish to apply Theorem 14 to derive a contradiction. To this end, first note that

$$\alpha\beta = \big(d(\Lambda_B)\big)^{-1} = |\det BA| = |\det A| = \big(d(\Lambda)\big)^{-1} \in \mathbb{R} \setminus \mathbb{Q}.$$

Next, set $U := M_\beta$ and $V := T_{\frac{\alpha}{\nu}}$. A direct calculation shows that

$$UV = e^{2\pi i\theta} VU, \qquad \text{where} \qquad \theta := \tfrac{\alpha\beta}{\nu} \in \mathbb{R}\backslash\mathbb{Q}.$$

Note that also $\gamma \in S_0(\mathbb{R})$; see [1, Theorem 7]. Hence, we may use Eq. (7) and obtain

$$P_0 = \frac{1}{\nu} \sum_{m,n\in\mathbb{Z}} \big\langle h, T_{\frac{m\alpha}{\nu}} M_{n\beta}\gamma\big\rangle T_{\frac{m\alpha}{\nu}} M_{n\beta} = \sum_{m,n\in\mathbb{Z}} \frac{1}{\nu}\big\langle h, V^m U^n \gamma\big\rangle V^m U^n, \tag{17}$$

with coefficient sequence $a = (a_{m,n})_{m,n\in\mathbb{Z}} := \big(\frac{1}{\nu}\langle h, V^m U^n \gamma\rangle\big)_{m,n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}^2)$. Therefore, Theorem 14 shows that $\frac{1}{\nu} = a_{0,0} \in \mathbb{Z} + \theta\mathbb{Z}$, say $\frac{1}{\nu} = m + n\theta$ for some $m, n \in \mathbb{Z}$. We must have $n \neq 0$, since otherwise $\frac{1}{\nu} = m \in \mathbb{Z}$, in contradiction to $\nu \geq 2$. Thus, $\theta = \frac{1}{n\nu} - \frac{m}{n} \in \mathbb{Q}$, which is the desired contradiction, since $\theta = \frac{\alpha\beta}{\nu}$ is irrational. □

Remark 15 On a first look, it might appear as if the proof of Theorem 1 would also apply in case of $g \in \mathbb{H}^1$: First, the classical Balian-Low theorem implies that $\mathcal{G} := \mathcal{G}(g, \Lambda) \subsetneq L^2(\mathbb{R})$, so that Lemma 10 allows the reduction to a Gabor Riesz sequence (h, Λ) with $h \in \mathbb{H}^1$, a separable lattice $\Lambda = \alpha\mathbb{Z}\times\beta\mathbb{Z}$, and an additional time-frequency shift of the form $T_{\alpha/\nu}\, h \in \mathcal{G}$. One can then apply Theorem 11 to see that that $L^2(\mathbb{R}) = \mathcal{L}_0 \boxplus \cdots \boxplus \mathcal{L}_{\nu-1}$. In the S_0-case, we then employed Janssen's representation (17) for the projection $P_0 = (\alpha\beta)^{-1} S_{\frac{1}{\beta}\mathbb{Z}\times\frac{\nu}{\alpha}\mathbb{Z},\gamma,h}$, which then led to success in the proof of Theorem 1, thanks to existing results concerning the structure of the irrational rotation algebra. However, in the case $h \in \mathbb{H}^1$ the series in (17) might not converge *in operator norm*, so that one does not know whether P_0 belongs to the irrational rotation algebra. Thus, the proof breaks down at this point.

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Declarations

Conflict of interest There is no conflict of interest nor a competing interest.

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A Appendix—Proof of Theorem 14

As a foreword, it bears to point out that the intent of this appendix is to give a minimal and self-contained 'translation' of a deep result from the theory of C^*-algebras (namely that the trace of a projection operator on an irrational rotation algebra has a very specific form, Theorem 1.2 in [27]) to the standard language of Gabor analysis literature. As such, within this paper, the result will essentially be treated like a 'black box' and we will only show how it applies to our present problem. For more details, we refer the interested reader to [27] for a proof of the result or to [11] for more background.

The proof will make use of some parts of the theory of C^*-algebras, which we recall here for the convenience of the reader, based on [25]. Readers familiar with C^*-algebras will probably want to skip this part—except possibly Lemma 16.

A C^*-algebra is a (complex) Banach algebra $(A, \|\cdot\|)$, additionally equipped with a map $A \to A, x \mapsto x^*$ (called the *involution* on A), satisfying the following properties:

- $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda}\, x^*$, and $(x\,y)^* = y^*\,x^*$ and $(x^*)^* = x$ for all $x, y \in A$ and $\lambda \in \mathbb{C}$;
- $\|x^*\| = \|x\|$ and $\|x^* x\| = \|x\|^2$ for all $x \in A$.

An element $p \in A$ is called an *idempotent* if $p^2 = p$. An idempotent p is called a *projection* if additionally $p = p^*$ holds. A C^*-algebra A is called *unital* if it contains a (necessarily unique) element $1 \in A$ satisfying $1 \neq 0$ and $x\,1 = 1\,x = x$ for all $x \in A$. In a unital C^*-algebra A, an element $x \in A$ is called *unitary* if $x^*x = 1 = xx^*$. If A is a unital C^*-algebra and $a \in A$, then $\sigma(a^*a) \subset [0, \infty)$; see [25, Theorem 2.2.4]. Here, $\sigma(b) = \{\lambda \in \mathbb{C} \colon b - \lambda 1 \text{ not invertible in } A\}$.

Lemma 16 *Any idempotent e in a unital C^*-algebra A is similar to a projection $p \in A$. That is, there exist a projection $p \in A$ and an invertible element $a \in A$ such that $e = a^{-1}pa$.*

Proof We set $b := e^* - e$ and $z := 1 + b^*b$. Note that z is invertible since $\sigma(b^*b) \subset [0, \infty)$. We have

$$ez = e + (e - ee^*)(e^* - e) = ee^*e = e + (e - e^*)(e^*e - e) = ze.$$

Consequently, $ez^{-1} = z^{-1}e$ and, as $z = z^*$, also $e^*z^{-1} = z^{-1}e^*$. Now, define the element $p := ez^{-1}e^*$. We have $p^* = p$. Furthermore, since we just saw that z^{-1} commutes with e and e^* and that $ee^*e = ze$, we also see that $p^2 = z^{-2}(ee^*e)e^* = z^{-1}ee^* = p$. Hence, p is a projection. We further observe that $ep = p$ and $pe = ez^{-1}e^*e = z^{-1}ee^*e = z^{-1}ze = e$. Set $a := 1 - p + e$. Then we see because of

$$(1 \mp p \pm e)(1 \pm p \mp e) = 1 \pm p \mp e \mp p - p + e \pm e + p - e = 1$$

that a is invertible with $a^{-1} = 1 + p - e$. Hence, from $ae = e - pe + e = e$ we obtain

$$aea^{-1} = e(1 + p - e) = e + ep - e = ep = p,$$

which proves the lemma. □

A closed subspace B of a C^*-algebra A is called a *C^*-subalgebra* of A if it is closed under both multiplication and involution. It is clear that B is then itself a C^*-algebra. As usual, given a subset $S \subset A$, there is a smallest (with respect to inclusion) C^*-subalgebra of A containing S. We call it the *C^*-algebra generated by S*, and denote it by $C^*(S)$.

A map $\varphi : A \to B$ between two C^*-algebras A and B is called a $*$-homomorphism if it is linear and satisfies $\varphi(x\,y) = \varphi(x)\,\varphi(y)$ as well as $\varphi(x^*) = [\varphi(x)]^*$ for all $x, y \in A$. A bijective $*$-homomorphism is called a $*$-isomorphism. Any $*$-homomorphism $\varphi : A \to B$ necessarily satisfies $\|\varphi(x)\|_B \leq \|x\|_A$ for all $x \in A$, and is hence continuous; see [25, Theorem 2.1.7].

Proof of Theorem 14 We will make use of the so-called *irrational rotation algebra* $\mathcal{A}_\theta$ with $\theta \in \mathbb{R}\backslash\mathbb{Q}$, as introduced for instance in [11, Chapter VI]. The actual definition of this algebra is not relevant for us; we will only need to know that it satisfies the following properties:

– $\mathcal{A}_\theta$ is a unital C^*-algebra;
– The algebra $\mathcal{A}_\theta$ is *universal* among all unital C^*-algebras generated by unitary elements U, V satisfying $UV = e^{2\pi i\theta}VU$. Thus, defining $\mathcal{A} := C^*(U, V)$ as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ with U, V as in the statement of Theorem 14, there is a $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{A}_\theta$; this follows from [11, Theorem VI.1.4].
– As shown in [11, Corollary VI.1.2 and Proposition VI.1.3], there is a unique *(unital) trace* $\tau : \mathcal{A}_\theta \to \mathbb{C}$. By definition of a trace, this means in particular that τ is linear and continuous, satisfying $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{A}_\theta$.
– For any projection $p \in \mathcal{A}_\theta$, we have $\tau(p) \in \mathbb{Z} + \theta\mathbb{Z}$; see [27, Theorem 1.2]. We remark that this result was originally proven in [26].

Let us define $\tau^\natural := \tau \circ \varphi$, and note that $\tau^\natural : \mathcal{A} \to \mathbb{C}$ is continuous. It is easy to see that $\tau^\natural$ is linear with $\tau^\natural(\mathrm{id}_\mathcal{H}) = 1$ and $\tau^\natural(AB) = \tau^\natural(BA)$ for all $A, B \in \mathcal{A}$; this is called the *cyclicity* of the trace. Next, from the relation $UV = e^{2\pi i\theta}VU$, we immediately get for $k, \ell \in \mathbb{Z}$ that

$$V^kU^\ell = e^{-2\pi i\ell\theta}V^{k-1}U^\ell V = e^{-2\pi ik\theta}UV^kU^{\ell-1}.$$

Thus, noting that $V^kU^\ell \in \mathcal{A}$, we obtain $\tau^\natural(V^kU^\ell) = e^{-2\pi i\ell\theta}\tau^\natural(V^kU^\ell) = e^{-2\pi ik\theta}\tau^\natural(V^kU^\ell)$ by cyclicity. As θ is irrational, this implies $\tau^\natural(V^kU^\ell) = \delta_{\ell,0}\delta_{k,0}$. Next, since we have $\|V^kU^\ell\| = 1$ for all $k, \ell \in \mathbb{Z}$ and since $a \in \ell^1(\mathbb{Z}^2)$, we see that $P_a = \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell}V^kU^\ell \in \mathcal{A}$, with unconditional convergence of the defining series. Hence,

$$\tau^\natural(P_a) = \sum_{k,\ell\in\mathbb{Z}} a_{k,\ell} \cdot \tau^\natural(V^kU^\ell) = a_{0,0}.$$

Since $P_a^2 = P_a$ and since $\varphi : \mathcal{A} \to \mathcal{A}_\theta$ is a $*$-homomorphism, we see that $e := \varphi(P_a) \in \mathcal{A}_\theta$ is an idempotent. By Lemma 16 there exist $b, p \in \mathcal{A}_\theta$ such that b is invertible, p is a projection, and $e = b^{-1}pb$. Thanks to the cyclicity of the trace, we thus see that

$$a_{0,0} = \tau^\natural(P_a) = \tau(e) = \tau(b^{-1}pb) = \tau(p) \in \mathbb{Z} + \theta\mathbb{Z},$$

as claimed. □

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