Research Article

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Stable tail dependence functions – some basic properties

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Abstract: We prove some important properties of the extremal coefficients of a stable tail dependence function ("STDF") and characterise logistic and some related STDFs. The well known sufficient conditions for composebility of logistic STDFs are shown to be also necessary.

Keywords: multivariate extreme value distribution, stable tail dependence function, extremal coefficient, logistic, negative logistic, nested logistic, fully d-alternating, Archimedean property

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1 Introduction

A multivariate extreme value (MEV) distribution (in a standardised form) is given by a distribution function ("d.f.") F on $\mathbb{R}_+^d$ with the decisive property

$$(F(tx))^t = F(x) \quad \forall x \in \mathbb{R}_+^d, \quad \forall t > 0,$$

and with standard one-dimensional Fréchet margins, defined by the d.f. $\exp\left(-\frac{1}{u}\right)$ for $u > 0$. The d.f. F is in a one-to-one correspondence with its associated *stable tail dependence function* ("STDF"), defined by

$$\ell(x) := -\log F\left(\frac{1}{x}\right), \quad x \in \mathbb{R}_+^d,$$

where $\frac{1}{x} := \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots\right)$, and these STDFs allow an intrinsic characterisation: $\ell : \mathbb{R}_+^d \to \mathbb{R}$ is a STDF iff ℓ is homogeneous ($\ell(tx) = t \cdot \ell(x) \quad \forall t, \forall x$), normalised ($\ell(e_i) = 1 \quad \forall$unit vector e_i), and "fully d-alternating" (to be explained later on), cf. [6], Theorem 6.

The marginals of F are given by

$$F_\alpha(x_\alpha) := F\left(x_\alpha, \infty_{\alpha^c}\right) = \exp\left[-\ell\left(\frac{1}{x_\alpha}, \mathbf{0}_{\alpha^c}\right)\right]$$

for $\varnothing \neq \alpha \subseteq \{1, \ldots, d\}$, $x_\alpha \in \mathbb{R}_+^\alpha$. F_α is again a MEV distribution with STDF $\ell_\alpha := \ell|\mathbb{R}_+^\alpha$. If $X = (X_1, \ldots, X_d)$ has the d.f. F, the subvector $X_\alpha := (X_i, i \in \alpha)$ has d.f. F_α.

Two main subjects will be treated in this article. The first one is about the so-called *extremal coefficients* of a (d-variate) STDF ℓ, defined by

$$\ell(\alpha) := \ell(\mathbf{1}_\alpha), \quad \alpha \subseteq [d]$$

(slightly abusing notation). Although ℓ is plainly not determined by its restriction to $\{0, 1\}^d$, these coefficients contain important information, especially with respect to the independence of subvectors (Theorem 3).

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The other main theme addressed is about logistic, negative logistic and nested logistic STDFs. A certain functional equation (Theorem 5) turns out to be the key for several characterisations of "Archimedean type." The well known sufficient conditions for "composebility" of logistic STDFs are shown to be necessary as well (Theorem 9) – meaning that the composed function is again a STDF.

Except Theorem 1, all the other theorems in this article are new to the best of our knowledge. A recommendable treatment of STDFs is presented in chapter 8 of [1].

Notations:

$\mathbb{R}_+ := [0, \infty[$, $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, $\overline{\mathbb{R}} := [-\infty, \infty]$,

$\frac{1}{x} := \left(\frac{1}{x_1}, \frac{1}{x_2}, \dots\right)$ with $\frac{1}{0} := \infty$, $\frac{1}{\infty} := 0$

$[d] := \{1, \dots, d\}$, $-\alpha := \alpha^c = [d]\backslash\alpha$ for $\alpha \subseteq [d]$, $\mathbf{1}_d := (1, \dots, 1) \in \mathbb{N}^d$, $e_1, \dots, e_d$ are the usual unit vectors in $\mathbb{R}^d$, $\mathbf{1}_\alpha := \sum_{i\in\alpha} e_i$

$(f \times g)(x, y) := (f(x), g(y))$ for mappings f, g

$M_+(X)$ is the set of Radon measures on a locally compact space X

d.f. = distribution function.

2 Fully d-alternating functions

To define this notion, which is of particular importance in this article, we introduce a special notation for multivariate real-valued functions. Let $A_1, \dots, A_d$ be non-empty sets, $A := A_1 \times \dots \times A_d$, and $f : A \to \mathbb{R}$. First, for $x \in A$ and $\varnothing \neq u \subseteq [d]$, we put $x_u := (x_i)_{i\in u}$, $-u := [d]\backslash u$, and so for $x, z \in A$

$$(z_u, x_{-u}) := \begin{cases} z_i, & i \in u \\ x_i, & i \in -u, \end{cases}$$

i.e., another element of A, being x for $u = \varnothing$ and z for $u = [d]$. Also, $A_u := \prod_{i\in u} A_i$ for $u \neq \varnothing$. We then define

$$D_z^x f := \sum_{u\subseteq[d]} (-1)^{|u|} f(z_u, x_{-u}) = f(x) \mp \dots + (-1)^d f(z).$$

Note that $D_z^x f = (-1)^d D_x^z f$. For $\varnothing \neq u \subsetneq [d]$ and $y_{-u} \in A_{-u}$, we define a "partial version" of f with fixed values in the variables $i \in -u$ by

$$f(\cdot, y_{-u})(x_u) := f(x_u, y_{-u}), \quad x_u \in A_u.$$

(For $u = [d]$, this would be f, and for $u = \varnothing$, the constant $f(y_{[d]})$.)

There is a two-step procedure to determine $D_z^x f$ which will be needed later on:

Lemma 1. *Let $f : A \to \mathbb{R}$, $\varnothing \neq v \subsetneq [d]$, $x, z \in A$, and define $g : A_{-v} \to \mathbb{R}$ by*

$$g(y_{-v}) := D_{z_v}^{x_v} f(\cdot, y_{-v}), \quad y_{-v} \in A_{-v}.$$

Then

$$D_z^x f = D_{z_{-v}}^{x_{-v}} g.$$

Proof.

$$\begin{aligned} D_{z_{-v}}^{x_{-v}} g &= \sum_{w\subseteq -v} (-1)^{|w|} g(z_w, x_{(-v)\backslash w}) \\ &= \sum_{w\subseteq -v} (-1)^{|w|} \sum_{u\subseteq v} (-1)^{|u|} f(z_u, x_{v\backslash u}, z_w, x_{(-v)\backslash w}) \\ &\quad (\text{noting } u \cap w = \varnothing \text{ here, putting } \alpha := u \cup w) \\ &= \sum_{\alpha\subseteq[d]} (-1)^{|\alpha|} f(z_\alpha, x_{-\alpha}) \\ &= D_z^x f. \end{aligned}$$

□

Definition. Let $A_1,\dots,A_d \subseteq \overline{\mathbb{R}}$ be non-empty, $A := A_1\times\cdots\times A_d$. Then $f : A \to \mathbb{R}$ is *fully d-alternating* (in symbols "$\mathbf{1}_d$-$\updownarrow$") iff $D_z^x f \le 0$ for $x \le z$ (both in A), and if also $D_{z_v}^{x_v} f(\cdot, y_{-v}) \le 0$ for each $\varnothing \ne v \subsetneq [d]$ and $x_v \le z_v$ (both in A_v), and for all $y_{-v} \in A_{-v}$.

This property is specific for co-survival functions, i.e., $f(x) := P(X \not\geq x)$, e.g., for X uniform on $[0,1]^d$

$$f(x) = \sum x_i - \sum_{i<j} x_i x_j + \sum_{i<j<k} x_i x_j x_k \mp \cdots ,$$

but it is of special importance also for some infinite measures, as we will see shortly.

Remark 1. There is a more general notion of $\mathbf{n}$-alternating ("$\mathbf{n}$-$\updownarrow$") functions, with $\mathbf{n} \in \mathbb{N}^d$, cf. [7], describing monotonicity conditions of higher orders, not needed in this work.

For $A_1 = \cdots = A_d = \mathbb{R}_+$, i.e., $A = \mathbb{R}_+^d$, a function $\ell : A \to \mathbb{R}$ is a STDF iff ℓ is homogeneous (i.e., $\ell(tx) = t \cdot \ell(x) \quad \forall t \ge 0, \quad \forall x \in \mathbb{R}_+^d$), $\mathbf{1}_d$-$\updownarrow$ and normalised, i.e., $\ell(e_i) = 1$ for each unit vector. Disregarding normalisation, we consider

$$K := \{f : \mathbb{R}_+^d \to \mathbb{R}_+ | f \text{ is } \mathbf{1}_d\text{-}\updownarrow \text{ and homogeneous}, \ f(\mathbf{1}_d) = 1\}.$$

This set, obviously compact and convex, was shown in ref. [6] to be a so-called Bauer simplex (i.e., a compact convex subset of some locally convex Hausdorff space, for which the extreme boundary is closed, and for which the integral representation given by the Krein-Milman theorem is unique), with extreme boundary

$$\mathrm{ex}(K) = \left\{ x \mapsto \max_{i\le d}(x_i w_i) | w \in C_d \right\},$$

where $C_d := \{w \in [0,1]^d | \max_{i\le d} w_i = 1\}$, and for each homogeneous $\mathbf{1}_d$-$\updownarrow$ function f ($\not\equiv 0$), we have the unique integral representation

$$f(x) = f(\mathbf{1}_d) \cdot \int \max_{i\le d}(x_i w_i) \mathrm{d}\nu(w)$$

with ν a probability measure on C_d.

The function f is then the so-called *co-survival function* of a homogeneous Radon measure μ on the locally compact space $Z_d := [0,\infty]^d \setminus \{\infty_d\}$, i.e.,

$$f(x) = \mu([x,\infty]^c) =: \check{\mu}(x)$$

(which is finite by the definition of a Radon measure).

3 Properties depending on the extremal coefficients

Let ℓ be a d-variate STDF. Its restriction $\ell|\{0,1\}^d$ gives the so-called *extremal coefficients* $\ell(\alpha) := \ell(\mathbf{1}_\alpha)$ for $\alpha \subseteq [d]$ (hence, $\ell([d]) = \ell(\mathbf{1}_d)$). From the integral representation,

$$\ell(x) = \ell(\mathbf{1}_d) \cdot \int_{C_d} \max_{i\le d}(x_i w_i) \mathrm{d}\nu(w),$$

we obtain

$$\ell(\alpha) = \ell(\mathbf{1}_d) \cdot \int \max_{i\in\alpha} w_i \mathrm{d}\nu(w), \quad \varnothing \ne \alpha \subseteq [d].$$

Clearly, the restriction of ℓ to $\{0,1\}^d$ does in general not determine ℓ, with the following exception, known since a long time, and due to Takahashi [9,10]. Note that

$$\max_{i\le d} x_i \le \ell(x) \le \sum_{i=1}^{d} x_i$$

holds for any d-variate STDF ℓ, and therefore,

$$1 \le \ell(\mathbf{1}_d) \le d.$$

Theorem 1. *Let ℓ be a d-variate STDF. Then,*

(*i*) $\ell(\mathbf{1}_d) = 1 \Leftrightarrow \ell(x) = \max_{i\le d} x_i \quad \forall x$

(*ii*) $\ell(\mathbf{1}_d) = d \Leftrightarrow \ell(x) = \sum_{i=1}^{d} x_i \quad \forall x.$

Proof.

(i) If $\ell(\mathbf{1}_d) = 1$, then $\ell(e_i) = \int w_i \mathrm{d}\nu(w) = 1$; hence, $\nu(\{w \in C_d | w_i = 1\}) = 1 \quad \forall i \le d$, i.e., $\nu(\{\mathbf{1}_d\}) = 1$ and $\ell(x) = \max_{i\le d} x_i$.

(ii) If $\ell(\mathbf{1}_d) = d$, then $\int w_i \mathrm{d}\nu(w) = \frac{1}{d}$ for each i; hence,

$$\int \sum_{i=1}^{d} w_i \mathrm{d}\nu(w) = 1 = \int \max_{i\le d} w_i \mathrm{d}\nu(w),$$

i.e., $\sum_{i=1}^{d} w_i = \max_{i\le d} w_i = 1$ ν-a.s., or $\nu(\{e_1, \dots, e_d\}) = 1$. From

$$\frac{1}{d} = \int w_i \mathrm{d}\nu(w) = \nu(\{e_i\}) \quad \forall i \le d,$$

we deduce $\nu = \frac{1}{d}\sum_{i=1}^{d} \varepsilon_{e_i}$, or $\ell(x) = \sum_{i=1}^{d} x_i$. □

Let $X = (X_1, \dots, X_d)$ have the MEV-distribution associated with the STDF ℓ, i.e., with d.f. $F(x) = \exp\left[-\ell\left(\frac{1}{x}\right)\right]$, for $x \in]0,\infty]^d$. For $\varnothing \ne \alpha \subseteq [d]$, the subvector $X_\alpha := (X_i, i \in \alpha)$ then has the d.f.

$$F_\alpha(x_\alpha) = F(x_\alpha, \infty_{-\alpha}) = \exp[-\ell(x_\alpha^{-1}, \mathbf{0}_{-\alpha})],$$

including $F_{\{i\}}(x_i) = \exp\left(-\frac{1}{x_i}\right)$, $i = 1, \dots, d$.

Condition (i) in Theorem 1 means that a.s. $X_1 = X_2 = \dots = X_d$, and (ii) is equivalent with $X_1, \dots, X_d$ being iid (standard Fréchet). The independence of two subvectors of X also depends only on the extremal coefficients, as we now shall see.

Theorem 2. *For disjoint (non-empty) subsets $\alpha, \beta \subseteq [d]$, the following properties are equivalent:*

(*i*) $\ell(\alpha) + \ell(\beta) = \ell(\alpha \cup \beta)$

(*ii*) $\ell(x_\alpha, \mathbf{0}_{-\alpha}) + \ell(x_\beta, \mathbf{0}_{-\beta}) = \ell(x_{\alpha\cup\beta}, \mathbf{0}_{-(\alpha\cup\beta)}) \quad \forall x$

(*iii*) *X_α and X_β are independent.*

Proof. In view of the connection between ℓ and F, only (i) ⇒ (ii) has to be shown. Without restriction $\alpha \cup \beta = [d]$, i.e., $\beta = -\alpha$. So, let us assume (i), then from

$$\ell(\alpha) = \ell(\mathbf{1}_d)\int \max_{i\in\alpha} w_i \mathrm{d}\nu(w)$$
$$\ell(\beta) = \ell(\mathbf{1}_d)\int \max_{i\in\beta} w_i \mathrm{d}\nu(w),$$

we obtain

$$\int \left(\max_{i\in\alpha} w_i + \max_{i\in\beta} w_i\right) \mathrm{d}\nu(w) = \int \max_{i\le d} w_i \mathrm{d}\nu(w) = 1.$$

Let $f(w) := \max_{i\in\alpha} w_i$, $g(w) := \max_{i\in\beta} w_i$, $w \in C_d$. Then $0 \le f \le 1$, $0 \le g \le 1$, $f \vee g = 1$, $\int (f+g)\mathrm{d}\nu = \int f \vee g \mathrm{d}\nu$. Since $f \vee g + f \wedge g = f + g$, we obtain $\int f \wedge g \mathrm{d}\nu = 0$, or $\nu(\{f > 0\} \cap \{g > 0\}) = 0$, and $C_d = \{f = 1\} \cup \{g = 1\}$. It follows

$$\ell(x) = \ell(\mathbf{1}_d) \int \max_{i \le d}(x_i w_i)\mathrm{d}\nu(w) = \ell(\mathbf{1}_d) \cdot \left(\int\limits_{\{f=1\}} \ldots + \int\limits_{\{g=1\}} \ldots \right) = \ell(x_\alpha, \mathbf{0}_{-\alpha}) + \ell(x_\beta, \mathbf{0}_{-\beta}). \qquad \square$$

Before we extend Theorem 2 to more than two subvectors, we need the following.

Lemma 2. *Let $I \subseteq \overline{\mathbb{R}}$ be any non-degenerate interval, $f : I^d \to \mathbb{R}$ a $\mathbf{1}_d$$\updownarrow$ function, and $[d] = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$ a partition with non-empty $\alpha_1, \ldots, \alpha_n$. Define $g : I^n \to \mathbb{R}$ by $g(x_1, \ldots, x_n) := f\left(\sum_{j=1}^n x_j \mathbf{1}_{\alpha_j}\right)$. Then g is $\mathbf{1}_n$-$\updownarrow$. If f is homogeneous, so is g.*

Proof. $d = 2$ being trivial, assume $d \ge 3$. It is clearly enough to consider the partition

$$\alpha_1 = \{1\}, \ldots, \alpha_{d-2} = \{d-2\}, \alpha_{d-1} = \{d-1, d\},$$

since the general case then follows easily by iteration.

We have $g(x_1, \ldots, x_{d-1}) = f(x_1, \ldots, x_{d-2}, x_{d-1}, x_{d-1})$. Let $x, z \in I^{d-1}$ with $x \le z$, define $x' := (x_1, \ldots, x_{d-2})$, and $z' := (z_1, \ldots, z_{d-2})$, then by Lemma 1,

$$\begin{aligned}
D_z^x g &= D_{z'}^{x'} g(\cdots, x_{d-1}) - D_{z'}^{x'} g(\cdots, z_{d-1}) \\
&= D_{z'}^{x'} f(\cdots, x_{d-1}, x_{d-1}) - D_{z'}^{x'} f(\cdots, z_{d-1}, z_{d-1}) \\
&= [D_{z'}^{x'} f(\cdots, x_{d-1}, x_{d-1}) - D_{z'}^{x'} f(\cdots, z_{d-1}, x_{d-1})] + [D_{z'}^{x'} f(\cdots, z_{d-1}, x_{d-1}) - D_{z'}^{x'} f(\cdots, z_{d-1}, z_{d-1})] \\
&= D_z^x f(\cdots, \cdot, x_{d-1}) + D_z^x f(\cdots, z_{d-1}, \cdot) \\
&\le 0
\end{aligned}$$

as the sum of two non-positive numbers. $\square$

The following result is a considerable generalisation of Theorem 1 (ii) and Theorem 2.

Theorem 3. *Let ℓ be a d-variate STDF, and let $\alpha_1, \ldots, \alpha_n$ be disjoint non-empty subsets of $[d]$. The random vector $X = (X_1, \ldots, X_d)$ is supposed to have the d.f. $\exp\left[-\ell\left(\frac{1}{x}\right)\right]$, $x \in \mathbb{R}_+^d$. Then the following conditions are equivalent*:

(*i*) $\ell(\bigcup_{j=1}^n \alpha_j) = \sum_{j=1}^n \ell(\alpha_j)$.

(*ii*) $\{X_{\alpha_j} | j = 1, \ldots, n\}$ *are independent.*

(*iii*) $\{X_{\alpha_j} | j = 1, \ldots, n\}$ *are pairwise independent.*

Proof. Without restriction, we assume $\bigcup_{j=1}^n \alpha_j = [d]$.

(**i**) ⇒ (**ii**): We use induction, the case $n = 2$ being true by Theorem 2. Supposing the conclusion for n, we use ℓ's subadditivity to obtain

$$\ell\left(\bigcup_{j=1}^{n+1} \alpha_j\right) \le \ell\left(\bigcup_{j=1}^{n} \alpha_j\right) + \ell(\alpha_{n+1}) \le \sum_{j=1}^n \ell(\alpha_j) + \ell(\alpha_{n+1}) = \ell\left(\bigcup_{j=1}^{n+1} \alpha_j\right),$$

and hence, $\ell(\bigcup_{j=1}^n \alpha_j) = \sum_{j=1}^n \ell(\alpha_j)$, $\{X_{\alpha_j} | j \le n\}$ are independent, and $n = 2$ may be applied to $\bigcup_{j=1}^n \alpha_j$ and α_{n+1}.

(**iii**) ⇒ (**i**): We use again induction. For $n = 2$, there is nothing to prove. We assume validity for some $n \ge 2$ and consider the case $n + 1$. Let

$$\alpha := \bigcup_{j<n} \alpha_j, \quad \beta := \alpha_n, \quad \gamma := \alpha_{n+1}$$

and define f on $\mathbb{R}_+^3$ by $f(a,b,c) := \ell(a\mathbf{1}_\alpha + b\mathbf{1}_\beta + c\mathbf{1}_\gamma)$. By Lemma 2, f is $\mathbf{1}_3$-$\updownarrow$ (not normalised!). Therefore,

$$\begin{aligned}0 \geq D_{\mathbf{1}_3}^{\mathbf{0}_3} f &= f(0) - \sum f(e_i) + \sum_{i<j} f(e_i+e_j) - f(\mathbf{1}_3)\\ &= 0 - \ell(\alpha) - \ell(\beta) - \ell(\gamma) + \ell(\alpha\cup\beta) + \ell(\alpha\cup\gamma) + \ell(\beta\cup\gamma) - \ell(\alpha\cup\beta\cup\gamma)\\ &= -\sum_{j<n}\ell(\alpha_j) - \ell(\alpha_n) - \ell(\alpha_{n+1}) + \sum_{j\leq n}\ell(\alpha_j) + \sum_{j\neq n}\ell(\alpha_j) + (\ell(\alpha_n)+\ell(\alpha_{n+1})) - \ell\left(\bigcup_{j=1}^{n+1}\alpha_j\right),\end{aligned}$$

i.e.,

$$\sum_{j=1}^{n+1}\ell(\alpha_j) \leq \ell\left(\bigcup_{j=1}^{n+1}\alpha_j\right) \leq \sum_{j=1}^{n+1}\ell(\alpha_j). \qquad \square$$

Considering in Theorem 3 the special case $\alpha_j = \{j\}$, $j = 1,\dots,d$, we are back to Theorem 1 (ii), with the additional equivalence to pairwise independence, i.e., $\ell(e_i + e_j) = 2 \quad \forall i \neq j$. One might be tempted to believe that there is a corresponding generalisation of part (i) of Theorem 1 as well. This is not the case.

Theorem 4.

(i) *Let ℓ be a d-variate STDF; $\alpha, \beta \subseteq [d]$ such that $\alpha \cap \beta \neq \varnothing$. If $\ell(\alpha) = \ell(\beta) = 1$, then also $\ell(\alpha\cup\beta) = 1$.*

(ii) *Let f, g be m- (resp. n-)variate STDFs, such that also $\ell(x,y) := f(x) \vee g(y)$ is a STDF. Then $f = \max$, $g = \max$ (and $\ell = \max$).*

Proof.

(i) Let X be a $\mathbb{R}_+^d$-valued random vector with STDF ℓ. Then, if $\ell(\alpha) = \ell(\beta) = 1$, $X_i = X_j$ a.s. $\forall i,j \in \alpha$ and $\forall i,j \in \beta$, and because of $\alpha\cap\beta \neq \varnothing$, $X_i = X_j$ a.s. $\forall i,j \in \alpha\cup\beta$. That is, $\ell(\alpha\cup\beta) = 1$.

(ii) Again let $(X_1, \dots, X_m, Y_1, \dots, Y_n)$ have ℓ as its STDF. Then, $X_i = Y_j$ a.s. $\forall i \in [m]$, $\forall j \in [n]$, i.e., $X_1 = \dots = X_m = Y_1 = \dots = Y_n$ a.s., leading to $\ell = \max$, $f = \max$ and $g = \max$. $\square$

Remark 2. For "overlapping variables", this is different:

$$\ell(x,y,z) := (x+y) \vee (y+z) = x \vee z + y$$

is a STDF, as is also (with $a, b \in [0,1]$)

$$\ell(x,y) := (ax+y) \vee (x+by) = ax + by + [(1-a)x] \vee [(1-b)y].$$

With iid standard Fréchet random variables X, Y, and Z, a stochastic model for these two STDFs would be the random vector (X, Y, X), resp. $((aX) \vee (1-a)Z, (bY) \vee (1-b)Z)$.

Note, however, that

$$f(x,y,z) := (x+y) \vee (y+z) \vee (z+x)$$

is not a STDF: $D_{\mathbf{1}_3}^{\mathbf{0}_3} f = 0 - 3 + 6 - 2 = 1 > 0$.

4 Characterisation of logistic and related STDFs

Perhaps the best-known STDFs are the logistic ones, i.e., the family $\{\ell_p | p \in [1,\infty[\}$, defined by

$$\ell_p(x) := \left(\sum_{i=1}^d x_i^p\right)^{1/p}, \quad x \in \mathbb{R}_+^d.$$

Among all symmetric STDFs they are particular, depending on x in an "additive way," being a function of $\sum_{i=1}^{d} g(x_i)$ for some $g : \mathbb{R}_+ \to \mathbb{R}_+$. We shall see that there are no other STDFs with this property besides the logistic ones.

We begin by solving a functional equation.

Theorem 5. *Let $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be homogeneous, $\neq 0$, and let $g : [0, \infty] \to [0, \infty]$ be a continuous bijection, such that $g(1) = 1$ and*

$$g(\varphi(x, y)) = g(x) + g(y) \quad \forall x, y \in \mathbb{R}_+.$$

Then, $\exists p \in \mathbb{R}\setminus\{0\}$ such that $g(x) = x^p \quad \forall x \in\,]0, \infty[$ (which of course extends uniquely to $[0, \infty]$).

Proof. Obviously $g(]0, \infty[) =]0, \infty[$ and $g(\{0, \infty\}) = \{0, \infty\}$, and g is either (strictly) increasing or decreasing. Since g^{-1} is also continuous, so is φ.

For $a := \varphi(1, 1)$, we have $g(a) = 2$ and $g(ta) = g(t \cdot \varphi(1, 1)) = g(\varphi(t, t)) = 2g(t) \quad \forall t \in \mathbb{R}_+$ (i.e., $g(0) = 2g(0)$, in accordance with $g(0) \in \{0, \infty\}$). The equality $g(ta) = g(t)g(a)$ shows a to belong to

$$G := \{x \in\,]0, \infty[\,|\, g(tx) = g(t)g(x) \quad \forall t \in\,]0, \infty[\},$$

a multiplicative subgroup of $]0, \infty[$ as is easily seen. Hence, $\{a^n | n \in \mathbb{Z}\} \subseteq G$.

For $n \in \mathbb{Z}$,

$$g(\varphi(1, a^n)) = 1 + g(a^n) = 1 + (g(a))^n = 1 + 2^n,$$

and for $t > 0$,

$$g(t \cdot \varphi(1, a^n)) = g(\varphi(t, ta^n)) = g(t) + g(ta^n) = g(t)(1 + 2^n) = g(t)g(\varphi(1, a^n)),$$

i.e., also $\{\varphi(1, a^n) | n \in \mathbb{Z}\} \subseteq G$, where $\varphi(1, a^n) = g^{-1}(1 + 2^n)$, and this converges to $g^{-1}(1) = 1$ for $n \to -\infty$.

This implies G to be dense in $]0, \infty[$: it suffices to show

$$1 < u < v \Rightarrow G \cap\,]u, v[\, \neq \emptyset,$$

and this follows because for any $x \in \left]1, \frac{v}{u}\right[$,

$$\{x^j | j \in \mathbb{N}\} \cap\,]u, v[\, \neq \emptyset$$

(choose k with $x^{k-1} \leq u < x^k$, then $x^k = x \cdot x^{k-1} < \frac{v}{u} \cdot u = v$). If g is increasing, then $g^{-1}(1 + 2^n) \in \left]1, \frac{v}{u}\right[$ for some (negative!) n, and for decreasing g, we may choose instead $[g^{-1}(1 + 2^n)]^{-1}$.

Now G is closed, g being continuous; hence, $G =]0, \infty[$ and $g(xy) = g(x)g(y) \quad \forall x, y \in\,]0, \infty[$. It is well known that this implies $g(x) = x^p$ for some $p \neq 0$. (For $f := \log \circ g \circ \exp : \mathbb{R} \to \mathbb{R}$, we have $f(s + t) = f(s) + f(t) \quad \forall s, t$; this is the standard Cauchy equation, and f being continuous, it has the form $f(s) = c \cdot s$ with $c \in \mathbb{R}$; therefore, $g(x) = x^c$.) From $g(a) = 2 = a^p$, we obtain

$$p = \frac{\log 2}{\log a} \begin{cases} >0 & \text{for } a > 1 \\ <0 & \text{for } a < 1. \end{cases}$$

□

Theorem 6. (Characterisation of logistic STDFs) *Let ℓ be a d-variate STDF of the form*

$$\ell(x) = g^{-1}\left(\sum_{i=1}^{d} g(x_i)\right)$$

for some continuous bijection $g : \mathbb{R}_+ \to \mathbb{R}_+$, $g(1) = 1$ without restriction.

Then, $g(x) = x^p$ for some $p \geq 1$, i.e., $\ell = \ell_p$.

Proof. Obviously g is (strictly) increasing, in particular $g(0) = 0$, and it suffices to consider $d = 2$. By the preceding theorem, $g(x) = x^p$ for some $p \in\,]0, \infty[$, and $g(\ell(1, 1)) = (\ell(1, 1))^p = 2$ implies $\ell(1, 1) > 1$, and

$$p = \frac{\log 2}{\log \ell(1, 1)} \geq 1$$

since $\ell(1, 1) \leq 2$.

□

This result, assuming from the outset (though tacitly) the function g to be differentiable, was shown in an equivalent form for copulas, stating that the only Archimedean extreme value copulas are the logistic (or Gumbel) ones, cf. [3]. We state this as a corollary, being slightly more general while not assuming differentiability:

Corollary 1. *Let C be a d-variate Archimedean copula, i.e.,*

$$C(u) = h^{-1}\left(\sum_{i=1}^{d} h(u_i)\right), \quad u \in]0,1]^d$$

with a decreasing bijection $h :]0,1] \to \mathbb{R}_+$, and assume that C is also "extreme", i.e.,

$$C(u_1^t, \dots, u_d^t) = (C(u))^t \quad \forall t > 0, \quad \forall u \in]0,1]^d.$$

Then, $h(s) = (-\log s)^p$ for some $p \geq 1$.

Proof. We only need to consider $d = 2$. It is easy to see that $C(u) > 0$ for $u \in]0,1]^2$. The corresponding STDF ℓ is given as follows:

$$\ell(x,y) = -\log C(e^{-x}, e^{-y}) = g^{-1}(g(x) + g(y)),$$

where $g(x) := h(e^{-x})$. The preceding theorem implies $g(x) = x^p$ for some $p \geq 1$; hence, $h(s) = (-\log s)^p$. □

Remark 3. In the definition of an Archimedean copula (as in Nelsen's book [4]), it is not assumed that h is unbounded; the case of a decreasing bijection

$$h :]0,1] \to [0,a[$$

for some finite a is also allowed, with h^{-1} extended to $\mathbb{R}_+$ by $h^{-1}(x) := 0 \quad \forall x \geq a$. But then a copula of the form

$$C(u,v) = h^{-1}(h(u) + h(v))$$

cannot be extreme: choose $u, v > 0$ such that $h(u) > \frac{a}{2}$, $h(v) > \frac{a}{2}$, so that $h(u) + h(v) > a$. Then $u^t \to 1$, $v^t \to 1$ for $t \to 0$, $h(u^t) \to 0$, $h(v^t) \to 0$ and

$$h^{-1}(h(u^t) + h(v^t)) \to h^{-1}(0) = 1, \, t \to 0.$$

However,

$$(h^{-1}(h(u) + h(v)))^t = 0 \quad \forall t > 0.$$

Also so-called *negative logistic* (or Galambos) STDFs can be characterised by an "Archimedean property", which however is not obvious at first sight. We remind that any STDF ℓ on $\mathbb{R}_+^d$ is the co-survival function of some homogeneous Radon measure μ on the locally compact space $[0,\infty]^d \setminus \{\infty_d\} =: Z_d$, i.e.,

$$\ell(x) = \mu([x,\infty]^c) =: \check{\mu}(x), \quad x \in \mathbb{R}_+^d.$$

By definition of a Radon measure, $\check{\mu}(x) < \infty \quad \forall x$. The d.f. $\hat{\mu}(x) := \mu([0,x])$ of μ is of course also finite and homogeneous.

The family $\{f_p | p \in]-\infty, 0[\}$ of negative logistic STDFs is defined by

$$f_p(x) := \sum_{\emptyset \neq \alpha \subseteq [d]} (-1)^{|\alpha|+1} \left(\sum_{i \in \alpha} x_i^p\right)^{1/p}, \quad x \in \mathbb{R}_+^d.$$

Theorem 7. *Let ℓ be a d-variate STDF, $\ell = \check{\mu}$, with $\mu \in M_+(Z_d)$, such that the d.f. $\widehat{\mu}$ is "Archimedean," i.e.,*

$$\widehat{\mu}(x) = g^{-1}\left(\sum_{i=1}^{d} g(x_i)\right), \quad x \in \mathbb{R}_+^d,$$

where $g : [0,\infty] \to [0,\infty]$ is a continuous bijection. Then, $\exists p < 0$ such that $g(x) = x^p$, and $\ell = f_p$.

Proof. By assumption

$$\widehat{\mu}(x_1, \ldots, x_{d-1}, \infty) = \lim_{x_d \to \infty} g^{-1}\left(\sum_{i<d} g(x_i) + g(x_d)\right)$$

is finite; hence, $g(\infty) = 0$ and g is decreasing. By iteration, we obtain

$$\widehat{\mu}(x_1, x_2, \infty, \ldots, \infty) = g^{-1}(g(x_1) + g(x_2)),$$

and from Theorem 5, we infer $g(x) = x^p$, where $p < 0$, i.e., $\widehat{\mu}(x) = (\sum_{i \le d} x_i^p)^{1/p} \quad \forall x \in \mathbb{R}_+^d$.

The co-survival function $\check{\mu}$ of μ is easily expressed in terms of $\widehat{\mu}$ (where we use that boundaries of intervals are μ-null sets, μ being homogeneous, see [6], p. 248): with $B_i := \{y \in Z_d | y_i < x_i\}$, we have

$$\check{\mu}(x) = \mu([x,\infty]^c) = \mu\left(\bigcup_{i \le d} B_i\right) = \sum_i \mu(B_i) - \sum_{i<j} \mu(B_i \cap B_j) \pm \cdots,$$

and since $\mu(B_1) = \widehat{\mu}(x_1, \infty, \infty, \ldots)$ etc., $\mu(B_1 \cap B_2) = \widehat{\mu}(x_1, x_2, \infty, \infty, \ldots)$ etc., we arrive at

$$\ell(x) = \check{\mu}(x) = \sum x_i - \sum_{i<j} (x_i^p + x_j^p)^{1/p} \pm \cdots = f_p(x). \qquad \square$$

Remark 4. In the above theorem, we have

$$\widehat{\mu}(x) = \left(\sum_{i=1}^{d} x_i^p\right)^{1/p} =: \widehat{\mu}_p(x).$$

In ref. [5], Theorem 6, it was shown that this function is a "bona fide" d.f. iff $p \in \left[-\infty, \frac{1}{d-1}\right] \cup \left\{\frac{1}{d-2}, \ldots, \frac{1}{2}, 1\right\}$. The question arises if for positive p in this set, there is also a corresponding STDF: the answer is NO: μ_p is then a Radon measure on $\mathbb{R}_+^d$, not on $Z_d = [0,\infty]^d \setminus \{\infty_d\}$, in fact $\check{\mu}_p(x) = \infty$ for each $x \ne 0$ in $\mathbb{R}_+^d$. For $p \to 0$, we obtain as limit $\widehat{\mu}_0(x) = (\prod_{i=1}^d x_i)^{1/d}$, and μ_0 is likewise not a Radon measure on Z_d.

Remark 5. Theorems 6 and 7 add to the many common features between Gumbel (logistic) and Galambos (negative logistic) STDFs resp. copulas, nicely described in ref. [2].

We also want to characterise nested logistic STDFs, but here we are first confronted with the interesting general question of the "composebility" of several STDFs in its simplest (already non-trivial) form: if f, g, h are bivariate STDFs, when is $f \circ (g \times h)$ again a STDF? For logistic STDFs $f = \ell_r$, $g = \ell_p$ and $h = \ell_q$ ($r, p, q \in [1, \infty[$), a sufficient condition is well known: $r \le p$ and $r \le q$. We shall show that this is necessary, too.

Theorem 8. *Let $r, p > 0$ such that*

$$\ell(x, y, z) := \sqrt[r]{\sqrt[p]{x^p + y^p}^{\,r} + z^r}$$

is a STDF. Then, $1 \le r \le p$.

Proof. $r \ge 1$ and $p \ge 1$ is clear. We start with

$$0 \ge D_{(1,1)}^{(0,0)} \ell(\cdot,\cdot,t) = t - 2(1 + t^r)^{1/r} + (2^{r/p} + t^r)^{1/r}$$

or

$$(2^{r/p} + t^r)^{1/r} + t \le 2(1 + t^r)^{1/r}$$

for any $t \ge 0$.

Equivalently,

$$2^{r/p} \le [2(1 + t^r)^{1/r} - t]^r - t^r \quad \forall t > 0.$$

We now make use of the Binomial series

$$(1 + x)^r = 1 + rx + \binom{r}{2}x^2 + \cdots$$

valid for $|x| < 1$ and all $r \in \mathbb{R}$. We obtain

$$\begin{aligned}
[2(1 + t^r)^{1/r} - t]^r - t^r &= t^r[2(1 + t^{-r})^{1/r} - 1]^r - t^r \\
&= t^r\left[2(1 + \frac{1}{r} \cdot t^{-r} + \binom{1/r}{2}t^{-2r} + \cdots) - 1\right]^r - t^r \\
&= t^r\left[1 + \frac{2}{r} \cdot t^{-r} + 2\binom{1/r}{2}t^{-2r} + \cdots \quad\right]^r - t^r \\
&= t^r\left[1 + \frac{2}{r} \cdot t^{-r}(1 + o(1))\right]^r - t^r \\
&= t^r\left[1 + 2t^{-r}(1 + o(1)) + \binom{r}{2}\left(\frac{2(1 + o(1))}{rt^r}\right)^2 + \cdots \quad\right] - t^r \\
&= 2(1 + o(1)) + O(1)/t^r \to 2 \text{ for } t \to \infty.
\end{aligned}$$

As a consequence, $r/p \le 1$, or $r \le p$. □

We arrive at a complete characterisation for composite logistic STDFs.

Theorem 9. *Let $r, p_1, \dots, p_d \in [1, \infty[$ and $\alpha_1 \cup \cdots \cup \alpha_d = [n]$ a partition, $|\alpha_j| \ge 2 \quad \forall j$. Then $\ell_r \circ (\ell_{p_1} \times \cdots \times \ell_{p_d})$ is a STDF if and only if $r \le p_i$ for all i.*

Proof. Sufficiency is well known, see, e.g., ref. [8], p. 256. The other direction follows from the previous theorem by considering (for p_1)

$$\ell_r\big(\ell_{p_1}(xe_i + ye_j), \ell_{p_2}(z \cdot e_k), \ell_{p_3}(0), \dots\big),$$

where $i, j \in \alpha_1$ and $k \in \alpha_2$, which gives $r \le p_1$. □

The "nested" STDFs just considered allow the following "Archimedean" characterisation:

Theorem 10. *Let f, g, h be continuous bijections of $\mathbb{R}_+$, $m \ge 2$, $n \ge 2$. If*

$$\ell(x, y) = f^{-1}\left\{f\left[g^{-1}\left(\sum_{i \le m} g(x_i)\right)\right] + f\left[h^{-1}\left(\sum_{j \le n} h(y_j)\right)\right]\right\}$$

is a STDF on $\mathbb{R}_+^m \times \mathbb{R}_+^n$, then $\ell = \ell_r \circ (\ell_p \times \ell_q)$ for some $r, p, q \ge 1$, such that $r \le p$ and $r \le q$ (ℓ_r bivariate).

Proof. Putting $y = 0$, we have $\ell(x, 0) = g^{-1}(\sum_{i \le m} g(x_i))$; hence, $g(s) = s^p$, where $p \ge 1$. Similarly, $h(s) = s^q$, with $q \ge 1$. For $x = (x_1, 0, \dots, 0)$, $y = (y_1, 0, \dots, 0)$, we obtain $\ell(x_1, 0, \dots, 0, y_1, 0, \dots, 0) = f^{-1}(f(x_1) + f(y_1))$, and so $f(s) = s^r$, where $r \ge 1$. By Theorem 9, finally, $r \le p$ and $r \le q$. □

Corollary 2. *For $m, n \geq 2$ and decreasing bijections $f, g, h :]0, 1] \to \mathbb{R}_+$, if*

$$C(u, v) = f^{-1}\left\{f\left[g^{-1}\left(\sum_{i\leq m} g(u_i)\right)\right] + f\left[h^{-1}\left(\sum_{j\leq n} h(v_j)\right)\right]\right\}$$

is an extreme value copula, then

$$C(u, v) = \left\{\left[\sum_i (-\log u_i)^p\right]^{r/p} + \left[\sum_j (-\log v_j)^q\right]^{r/q}\right\}^{1/r}$$

for some $r, p, q \geq 1$ with $r \leq p$ and $r \leq q$.

This class of copulas of “composite Gumbel type” was already considered in ref. [7], p. 366, as particular examples of so-called *generalised Archimedean copulas*.

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References

[1] Beirlant, J., Goegebour, Y., Segers, J., & Teugels, J. (2004). *Statistics of extremes, theory and applications*. Chichester: Wiley.

[2] Genest, C., & Nešlehová, J. G. (2017). When Gumbel met Galambos. In U. Flores, E. de Amo Artero, F. Durante, & J. Sánchez (Eds.), *Copulas and dependence models with applications* (83–93). New York: Springer.

[3] Genest, C., & Rivest L. P. (1989). A characterisation of Gumbel's family of extreme value distributions. *Statistics & Probability Letters*, *8*, 207–211.

[4] Nelsen, R. B. (2006). *An Introduction to Copulas* (2nd ed.). New York: Springer.

[5] Ressel, P. (2012). Functions operation on multivariate distribution and survival functions – With applications to classical mean-values and to copulas. *Journal of Multivariate Analysis*, *105*(1), 55–67.

[6] Ressel, P. (2013). Homogeneous distributions – and a spectral representation of classical mean values and stable tail dependence functions. *Journal of Multivariate Analysis*, *117*, 246–256.

[7] Ressel, P. (2018). A multivariate version of Williamson's theorem, ℓ^1-symmetric survival functions, and generalised Archimedean copulas. *Dependence Modeling*, *6*, 356–368.

[8] Ressel, P. (2019). Copulas, stable tail dependence functions, and multivariate monotonicity. *Dependence Modeling*, *7*, 247–258.

[9] Takahashi, R. (1987). Some properties of multivariate extreme value distributions and multivariate tail equivalence. *Annals of the Institute of Statistical Mathematics*, *39*, 637–647.

[10] Takahashi, R. (1988). Characterizations of a multi-variate extreme value distribution. *Advances in Applied Probability*, *20*, 235–236.