European Journal of Operational Research 292 (2021) 155–171

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Production, Manufacturing, Transportation and Logistics

Shelf space dimensioning and product allocation in retail stores

Alexander Hübner[a,*], Tobias Düsterhöft[b], Manuel Ostermeier[a]

[a] Technical University of Munich, Supply and Value Chain Management, Am Essigberg 3, Straubing 94315, Germany
[b] Catholic University Eichstätt-Ingolstadt, Operations Management, Auf der Schanz 49, Ingolstadt 85049, Germany

ARTICLE INFO

Article history:
Received 4 March 2020
Accepted 26 October 2020
Available online 3 November 2020

Keywords:
Inventory
Retailing
Shelf space planning
Planogram
Integer problem

ABSTRACT

Retail shelves are adjustable by varying the number of shelf boards as well as the height and depth of each shelf board. Shelf planners adjust the boards accordingly at regular intervals when they create the shelf plans and allocate products. Current shelf planning models assume given shelf configurations and allocate only products. However, the dimensioning of a shelf segment and product allocation are interdependent. For instance, the height of one segment may be reduced if only small products are allocated or products cannot be stacked. This paper proposes the first integrated approach for shelf segment dimensioning and product allocation. It jointly determines the number of facings for each product, the shelf quantity and the size and number of shelf segments. We also identify and consider several restrictions for the shelf structure (e.g., technical options), allocation rules (e.g., maximum inventory reach) and allocation- and shelf-layout-dependent demand. We formulate the decision problem at hand which is an Integer Non-linear Program and apply a solution algorithm based on the application of bounds that are obtained by transferring constraints to a preprocessing stage. Doing so, we can reformulate the problem as Binary Integer Program, provide an exact approach and generate practical applicable and optimal solutions in a time-efficient manner. We show that integrating shelf dimensioning into product allocation results in up to 5% higher profits than benchmarks available in literature. By means of a case study we show how planning can be improved, and that the retailer's profit margin can be improved by up to 7%.

1. Introduction

Retailers pay close attention to shelf space as it is one of their most important and scarcest resources. In current retail practice, however, shelf planning is often based on planner's experience, gut feeling and manual trial-and-error approaches with limited IT support (Bianchi-Aguiar, Hübner, Carravilla, & Oliveira, 2021; Kök, Fisher, & Vaidyanathan, 2015). In this respect, retailing will benefit from an approach that is data-driven and targets the optimization of total profitability as well as allowing time-efficient planning processes via appropriate tool support (Griswold, 2007; Mou, Robb, & DeHoratius, 2018). Retailers must decide, for each product category, how much shelf space they assign to each product within a given total store space (Ghoniem, Flamand, & Haouari, 2016; Ostermeier, Düsterhöft, & Hübner, 2020). The space assignment of individual products implies different effects. If a product receives more space, it is more likely that customers will decide to purchase this product (Chandon, Hutchinson, Bradlow, & Young, 2009; Eisend, 2014). Furthermore, more space results in higher shelf quantity and hence potentially fewer replenishment actions are required. However, this also implies that less shelf space is left for the remaining products and as such, the product allocation problem is a mulitple Knapsack problem.

When defining the space for each product, retailers need to simultaneously consider the options with shelf dimensioning. A shelf rack consists of different shelf segments. A shelf segment is defined by the height, depth and vertical level of the shelf board. Fig. 1 illustrates a rack with four shelf segments, where each segment has a different depth and height of the shelf board, and is located on different vertical levels. In practice, the lower segments are usually deeper and higher than the upper segments for optical reasons.

Shelf segment dimensioning is the decision on the height and depth dimension, while the length (width) is given by the rack

* Corresponding author at: Technical University of Munich, Supply and Value Chain Management, Am Essigberg 3, 94315 Straubing, Germany.
E-mail addresses: alexander.huebner@tum.de (A. Hübner), tobias.duesterhoeft@ku.de (T. Düsterhöft), manuel.ostermeier@tum.de (M. Ostermeier).

https://doi.org/10.1016/j.ejor.2020.10.030
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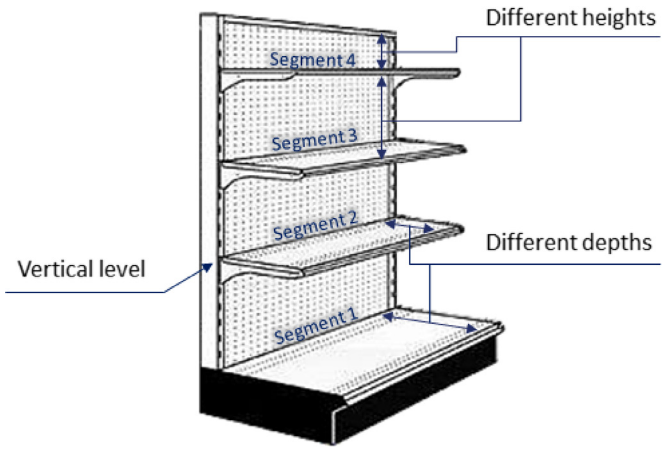

Fig. 1. Example shelf rack with shelf segments with different vertical levels, height and depth.

width. Different shelf segment heights can be selected by a shelf planner that impact the stacking opportunities, as illustrated in Fig. 2. It shows how two products could be allocated to differently sized segments. In option 1, items can only be positioned next to each other due to the low height. Option 2 allows the stacking of only one item that then requires less horizontal space. Option 3 shows the highest segment, where both items are stackable.

Also the height of a segment defines the possibility of allocating an item to a shelf segment (e.g., tall items that do not fit into small segments). Shelf segments with a larger height reduce the total number of shelf levels possible on a shelf rack. Furthermore, the deeper a shelf segment, the more units of a product can be stored. The different options for the number, height and depth of shelf segments raise the question of how segment dimensions should be defined, and how each shelf rack should look. Furthermore, it becomes clear that product allocation heavily depends on the dimensions of the chosen shelf segment. These two aspects are interdependent and hence planned jointly in retail practice.

However, the interrelationship of shelf dimensions and the corresponding options for product allocation has not yet been addressed in literature (see e.g., Bianchi-Aguiar et al. (2021)). We present the first comprehensive model and solution approach for product allocation and shelf segment dimensioning. This means we determine the optimal shelf presentation of each product as well as the related total shelf quantity depending on the height and depth dimensions of the chosen shelf segment. Further, we simultaneously determine the optimal number of shelf segments and the corresponding dimensions for each shelf segment. By doing so, we also include related demand effects that include space-elasticity and vertical and horizontal positioning effects. This increases the combinatorial complexity and requires an efficient and effective solution approach for the NP-hard Knapsack problem.

The remainder of this paper is organized as follows. In Section 2 we introduce the conceptual structure of the novel problem and review related literature. The mathematical model and solution approach are presented in Sections 3 and 4. The numerical

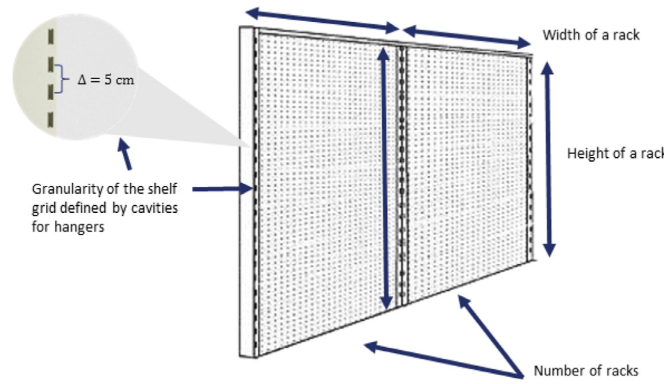

Fig. 3. Example: Empty shelf rack as a starting point for shelf segment dimensioning.

results shown in Section 5 prove the efficiency of our implementation. The approach is tested and solved with real data from cooperation with a large European retailer. Finally, Section 6 concludes the paper and denotes further areas of future research.

2. Problem statement and related literature

This section defines the conceptual background of our planning problem. It builds the foundation for scoping the planning problem, analyzing related literature, identifying the research gap and defining the contribution of this paper. A detailed understanding of the actual scope of the planning problem is required to model the dependencies and restrictions.

2.1. Setting and related planning problems

To maximize the profit of a category, the shelf planner must decide how to place a given set of products (i.e., the assortment of a category) on a limited area of shelf space containing several shelf segments of certain dimensions (Hübner, 2017). This is done in regular intervals and is therefore part of a tactical planning problem. It is usually updated after major assortment changes (e.g., after regular negotiations with suppliers) or when category sizes are adjusted (e.g., when additional categories are added to the store) (Hübner & Kuhn, 2012). The two main decisions in this process are the setup of shelf racks (i.e., *shelf segment dimensioning*) and the product placement on this racks (i.e., *product allocation*).

Shelf segment dimensioning. The shelf segment dimensioning considers the definition of the number, size and height of rectangle areas given an overall space allowance. A shelf rack with given total width and total height (see Fig. 3) needs to be subdivided into different shelf segments (i.e., rectangle areas) to enable the most profitable allocation of products.

The segments are depicted by the physical depth and height as well as the vertical level of shelf boards within the rack. The level means the vertical position of the segment (e.g., on eye level at 1.50 meter), whereas the height represents the vertical size of the segment where items can be placed (e.g., 30 centimeter). The ver-

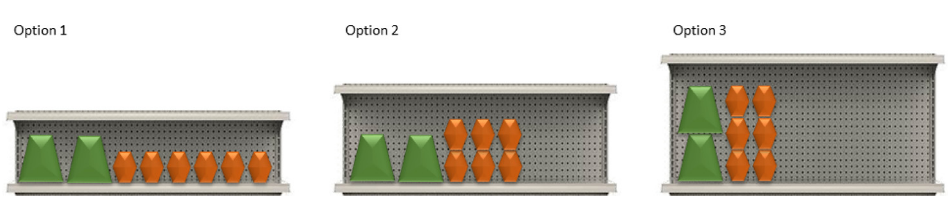

Fig. 2. Different heights of a shelf segment allow different options for product allocation.

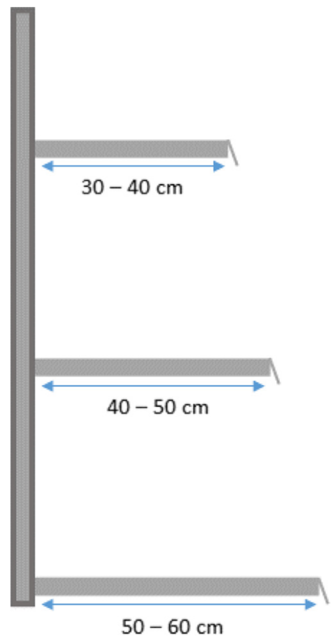

Fig. 4. Example: Different segment depths possible on a shelf rack, side-facing view.

tical level of a shelf segment is bound to the shelf grid and the available cavities for hangers (see Fig. 3). The shelf grid represents the physical possibility of placing vertical hangers on a shelf rack, e.g., every 5 centimeter. The segment width is determined by the width of a rack. Based on this information, retail planners determine the dimensions of each shelf segment, which includes the *(1) vertical level*, *(2) height* and *(3) depth*, as well as the *(4) total number of shelf segments*.

To define the (1) vertical level and (2) height of a shelf segment, further restrictions need to be considered. A shelf segment may have a specific *minimum height* representing the distance between two shelf boards. This usually depends on marketing and layout guidelines. Additionally, a certain grabbing gap must be considered such that customers can still remove products from this segment. The *maximum height* of a segment is the upper limit for the vertical distance between two segments. It depends on the product category (e.g., large dog food bag vs. small candy) and the options for stacking items (e.g., canned food vs. bagged food), so as to avoid shaky stacks. Furthermore, (3) the depth of each segment needs to be set. Retailers use steps between different segments as shown in Fig. 4 to make all products on a shelf better visible to customers (see also Düsterhöft, Hübner, & Schaal, 2020). In retail practice usually up to three different depth sizes are applied. The choice depends on optical reasons and inventory requirements. Less deep segments allow lower inventory and thus affect availability and replenishment frequency. It is a common rule that each upper segment needs to be equal or smaller than the next lower segment, both in terms of depth and height. (4) The total number of segments of each rack is derived by the selection of the height and vertical levels of the segments.

Product allocation. The allocation of products to multi-dimensional shelf segments represents a multi-dimensional knapsack problem as items of differing value (i.e., profit contribution) need to be allocated to a limited space, and due to this limitation, not all units may be allocated. As multiple segments are available it represents additionally a multiple knapsack problem. In practice, this means defining for each product the quantity allocated to each shelf segment and the corresponding shelf racks. The product quantity is indicated by facings. A facing is the first visible unit of an item in the front row of a shelf. In this sense, retailers define the number of units per product in the front row of a shelf that are visible to the customer (Corstjens & Doyle, 1981; Hansen & Heinsbroek, 1979). Items can be placed next to each other (horizontal facings) and some items may be stacked (vertical facings) (see Fig. 5). The stacking options depend on the height of the segment, the item height and stackability of the item (Zelst, van Donselaar, van Woensel, Broekmeulen, & Fransoo, 2009). The total number of facings is the number of vertical facings times the number of horizontal facings. Behind each facing, items can be lined up one behind the other depending on the segment and item depth. Stacking and lining up items allow the maximum possible vertical and horizontal space to be fully utilized.

Retailers apply minimum and maximum requirements for inventory and facings. *Minimum inventory* limits can ensure a certain service level (e.g, by using safety stocks) or comply with optical guidelines for shelf layout even if a product is very slow moving and would only require little shelf space (see e.g., Baldauf, Eng-Larsson, & Isaksson (2019)). Upper limits on the other hand are necessary to limit the *maximum inventory* reach, especially for perishables. Similarly, a *minimum number of facings* can be applied to ensure a certain shelf representation (e.g., for newly listed products with low current demand) or to fulfill supplier targets (e.g., contractual agreements for shelf shares; see e.g., Martinez-de Albeniz & Roels (2011)). A *maximum number of facings* sets an upper bound to limit the shelf share for certain products. Finally, some items are restricted to certain shelf segments. For example, heavy and bulky items are placed at the lower levels and small, light items are on the upper levels.

Summary. Retail planners face three decisions that need to be determined simultaneously. On the shelf segment dimensioning side they need to decide on (1) the height and (2) depth of each segment. (1) also determines the vertical level of each shelf segment and ultimately the total number of shelf segments, whereas (1) and (2) together determine the total available shelf space. On the product allocation side, they (3) need to assign products to shelf segments and the respective number of facings and units behind one facing, which also implies the total shelf inventory for this product as items are stacked and strung according to the dimensions of the chosen shelf segment. The described decisions are interrelated and need to be considered simultaneously. By way of example, product allocation depends on the number and dimensions of available shelf segments on the one hand, and the dimensions of the shelf segments depend on the specific attributes of allocated items (e.g., item sizes, number of horizontal and vertical facings) on the other hand.

2.2. Impact of simultaneous consideration

The simultaneous decision of these related planning problems also affects further parameters in the stores. More precisely, both the demand for products and the replenishment costs are affected by the given decisions. We detail the corresponding effects in the following and discuss their relevance for the presented setting.

Impact of shelf segment dimensioning and product allocation on demand. Shelving decisions and product allocations affect customer demand. As only 30% of all purchasing decisions are fixed before entering the store (GFK, 2009), the possibility of influencing customers in their choices is significant through instore manipulations (Chandon et al., 2009). The potential demand sources are: (1) Demand dependent on *product allocation of an item*, (2) demand dependent on *product allocation across items*, (3) demand dependent on *chosen rack and segment*, and (4) demand dependent on *position within a segment*. We further refer to the reviews and consumer studies of Dréze, Hoch, and Purk (1994), Chandon et al. (2009),

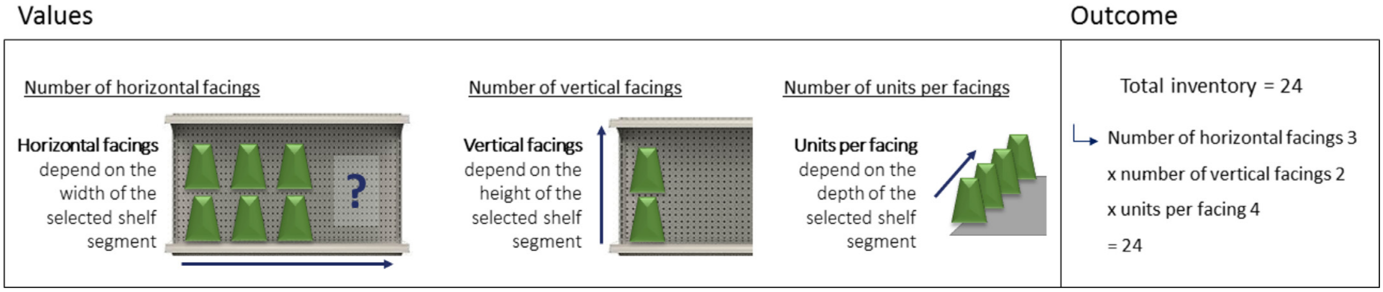

Fig. 5. Options and related outcome of product allocation.

Eisend (2014) and Bianchi-Aguiar, Silva, Guimaraes, Carravilla, and Oliveira (2017) that summarize related demand effects from an empirical point of view.

(1) Item demand depends on the visible quantity on the shelf (Hansen & Heinsbroek, 1979). The higher the visibility of an item, the higher its demand. The visibility of an item increases with the number of facings assigned to that item. Empirical studies examine these so-called “space-elasticity effects” (see e.g., Cox (1964), Frank and Massy (1970), Curhan (1972), Anderson (1979) Hansen and Heinsbroek (1979), Dréze et al. (1994), Desment and Renaudin (1998)). Chandon et al. (2009) show that number of facings is the most important instore factor affecting customer demand. Using a meta-analysis across empirical studies, Eisend (2014) quantified the average space-elasticity factor as 17%, which implies a demand increase of 17% each time the number of facings is doubled.

(2) Product allocation may also impact the demand across items. First of all, cross-space elasticity describes the impact on the demand of items when the space assigned to one item is changed (Corstjens & Doyle, 1981; Desment & Renaudin, 1998; Dréze et al., 1994). However, Schaal and Hübner (2018) show that the impact of this demand source on product allocations and retail profit is very limited. This also holds true if elasticities are significantly higher than the existing empirical values. Secondly, substitution effects describe the demand transfer from non-available to available items. A differentiation needs to be made between permanently non-available items that are out-of-assortment, and temporarily non-available items that are out-of-stock (see e.g., Fitzsimons (2000), Campo, Gijsbrechts, and Nisol (2000), van Woensel, van Donselaar, Broekmeulen, and Fransoo (2007), Xin, Messinger, and Li (2009)). Decisions in shelf planning are usually based on a given assortment, so that out-of-assortment substitution is not within the scope (Irion, Lu, Al-Khayyal, & Tsao, 2012; Kök & Fisher, 2007). Furthermore, retailers try to avoid out-of-stock situations by immediately refilling empty shelves with inventory from the backroom and applying safety stocks and minimum representation quantities (Urban, 1998).

(3) Shelf-segment-dependent demand. With the approach of considering different segments for product allocation, it is understandable that different segments may have different influences on the demand of a product. First, the segments are located on different vertical levels within the shelf. Following Dréze et al. (1994) and Underhill (1999), this means that some segments lie within a specific zone running approximately from eye- to knee-level, where products are more likely to be seen by customers than outside this zone. Chandon et al. (2009) points out that a top-level shelf is superior to a bottom-level shelf in terms of both attention and sales. Additionally, segments can be differentiated in horizontal order across racks. This touches on the question of whether a specific area of the total shelf, e.g., the horizontal center or the beginning of an aisle, is more attractive regarding sales (see e.g., Ghoniem et al. (2016)).

(4) Item positioning within a segment determines how products are arranged next to each other, how far a product is positioned from the edge of a segment (i.e., the horizontal location), and the way product facings are arranged, e.g., in rectangular shapes or as a family grouping. However, generally these effects are attributed a lower to negligible demand impact (see e.g., Chandon et al. (2009), Geismar, Dawande, Murthi, and Sriskandarajah (2015)). Nevertheless, shelf layout may be subject to some layout restrictions that may require keeping certain products together (e.g., brand grouping), but without changing demand (Bianchi-Aguiar et al., 2017; Bianchi-Aguiar et al., 2016; Pieters, Wedel, & Batra, 2010).

In summary, shelf dimensioning and product allocation impact customer demand in various ways. With respect to the described demand effects, only (1) the product-allocation-dependent and (3) the segment-dependent demand impacts are attributed with a major effect on customer behavior. For effects across products (2) we can state that: cross-space allocation has a neglectable impact due to the low magnitude; cross-product demand is relevant for assortment decisions, but this is out of scope for our planning problem; out-of-stock situations result from poor planning or missing safety stocks and are prevented using minimum representation quantities and safety stocks plus backroom inventory. Finally, there is so far no empirical evidence that positioning items differently within a segment has an impact on demand (4). Consequently, product- and segment-dependent demand effects are assumed to be decision relevant for our problem specification and are therefore considered in our modelling approach.

Impact of shelf segment dimensioning and product allocation on costs. To fulfill customer demand, retailers employ safety stocks (as part of the minimum inventory) and two types of replenishment procedures (Hübner, Kuhn, & Sternbeck, 2013; Hübner & Schaal, 2017a; Kotzab & Teller, 2005; Reiner, Teller, & Kotzab, 2013). First, there is a regular, scheduled replenishment procedure of an entire category. This depends on delivery patterns, which define specific days for store deliveries from the warehouse for each store and each product group (Holzapfel, Hübner, Kuhn, & Sternbeck, 2016; Taube & Minner, 2018). These delivery patterns mainly depend on given network structures and product groups (e.g., fresh products are delivered more often than dry foods). Hübner and Schaal (2017a) identify that these replenishment costs per category are related to the delivery frequency and not to shelf planning. Hence, in our context they can be assumed as not decision relevant. Second, if demand (defined by position of the segment and number of facings) is higher than shelf inventory (defined by the size of the segment and the number of facings), additional and ad hoc replenishment from the backroom is required. This second type is a product-specific procedure (Kotzab & Teller, 2005; Kuhn & Sternbeck, 2013). There are quantity-dependent costs for additional shelf-refilling activities that are required between the regular, scheduled refill processes related to the delivery patterns.

The analysis above on related decisions, demand and cost implications build the foundation for the literature review in the following subsection.

2.3. Related literature

Within this section, we will analyze literature from three different streams for product allocation. There is no dedicated literature for shelf segment dimensioning, as a dimensioning model on its own without product allocation is not reasonable.

Literature related to product allocation. Several models have already been developed to support retailers in the product allocation to shelves. Early approaches addressing this reach back to the 1960s (e.g., Cox (1964)), followed by the development of non-linear decision models (e.g., Hansen and Heinsbroek (1979), Corstjens and Doyle (1981) or Zufryden (1986)) considering space-elastic demand and other demand and cost effects. Based on the conclusive concept of space-elastic demand functions, further developments extended these approaches. Approaches that are more recent include a wide selection of different parameters (e.g., cross-space elasticity, stochastic demand, replenishment costs) or integrate related decisions (e.g., replenishment frequency, assortment decisions). For example, Borin, Farris, and Freeland (1994) integrate an assortment decision and generate a cost function for stock-outs. Urban (1998) presents a demand function incorporating assortment decisions as well as backroom space for additional storage. Hübner and Schaal (2017b) integrate assortment planning and model stochastic demand. The approach of Irion et al. (2012) further details the demand function with cross-space elasticities. A new aspect was demonstrated by Hübner and Schaal (2017a) where replenishment costs for direct replenishment and from the backroom are specified. However, these state-of-the-art approaches in shelf-space literature still lack possibilities for integrating shelf segments and varying height levels. None of these models takes into account different height levels of segments as they can be found in practice. Total shelf space is described by a one-dimensional value and not differentiated by segments. Usually shelf space is represented by a single value for the width of a whole category (e.g., 10 meter in a single line). Consequently, solutions generated with these models can hardly be applied to a real shelf, as usually only the number of facings is considered and solutions must be split up among different shelf segments.

Literature related to product allocation with multiple segments. Some more sophisticated approaches already consider the fact that a shelf consists of different segments. Yang (2001) present a simple model that considers several vertical levels but on the other hand lacks important components such as a space-elastic demand function or cost function. A later approach of Hwang, Choi, and Lee (2005) provides a demand function that also incorporates in addition to several shelf segments, neighborhood relations of items. Hansen, Raut, and Swami (2010) create a model that integrates detailed location effects within their profit function. Further, Zhao, Zhou, and Wahab (2016) incorporate effects of spatial relationships between different items in addition to a space-elastic demand function. A stochastic demand function combined with location-dependent demand effects can be found in Hübner and Schaal (2017c). However, none of these papers factor in varying segment sizes and vertical levels. All these papers assume equally sized and identical segments across all vertical levels. For all of them, the shelf and segment sizes are given input parameters and not part of the decision problem.

Literature related to product allocation with multiple and differently sized segments. The following two papers are more related to our problem as they assume differently sized segments. Bai, Van Woensel, Kendall, and K. Burke (2013) provide a model where several shelf segments are available that can each be defined with an individual height. Yet, the different heights are not part of the decision problem but used as input parameter in their model. Further, they integrate in addition to a space-elastic demand function, location effects of different segments. Depending on the height of a shelf segment, they precalculate the number of items that can be stacked one above the other and then decide about the number of horizontal facings. On the other hand they neglect the fact that shelf segments can also have different depths and that this affects the resulting inventory for each item. Further, they only provide solutions for small data sets of up to 29 items generated with a multi-neighborhood heuristic. Düsterhöft et al. (2020) provide the first model to address the problem with multiple shelf racks and multiple-sized shelf segments. The authors present a model that, alongside the space-elastic demand and location effects, considers shelf segments with different dimensions regarding height, width and depth. However, in contrast to our work the authors assume the shelf dimensions as input parameters and shelf segment dimensioning is therefore not part of the optimization problem. This also means that the number of shelf segments is fixed in advanced. We integrate these decisions in our model and decide simultaneously on the product allocation and shelf segment dimensioning. The integration of shelf segment dimensioning significantly increases the model complexity of a product allocation model as each shelf segment setting offers different allocation possibilities. Düsterhöft et al. (2020) could be described as a static model (i.e., given dimensions), while we address a dynamic problem setting (i.e., varying dimensions). Enhancing product allocation with shelf segment dimensioning impacts the decision model and its variables, constraints and overall complexity to solve. This requires a tailored solution approach that is able to handle the new dimension of complexity. We discuss the given model complexity in more detail in Section 3.

2.4. Research gap

Retail practice shows that the dimensioning of shelf segments constitutes an important planning problem as shelf planners would struggle to find feasible allocation solutions when using current models from literature, and need to resort to trial-and-error approaches with limited optimization support for the dimensioning problem (Bianchi-Aguiar et al., 2017; Hübner & Kuhn, 2012). Further, our literature review shows that the integration of shelf dimensioning into product allocation problems has not yet been addressed in related literature. The review shows that common models reduce the product allocation problem to a one-dimensional problem with a single segment. Some papers are extended to multiple segments, but again assume only given shelf dimensions, reduce the three-dimensional problem to a two-dimensional problem by assuming some shelf parameters, do not explicitly model all relevant decision variables or are not capable of dealing with practically relevant problem sizes (see Table 1). Even though Bai et al. (2013) and Düsterhöft et al. (2020) extend the product allocation literature by modeling the shelf space with multi-dimensional segments, they do neither include a decision on number of shelf segments, nor the vertical level of segments and also not the height and depth of segments. These are given parameters in these models. It therefore becomes obvious that no available approach in literature yet covers the interdependent decision problem of shelf segment dimensioning and product allocation.

We further refer the reader to the publications of Hübner and Kuhn (2012), Kök et al. (2015) and Bianchi-Aguiar et al. (2021) for a more detailed review.

Table 1
Overview of product allocation literature related to shelf segment dimensions.

Literature	Demand effects			Item/shelf	Shelf	Shelf	Number of	Vertical	Height/Depth
	Space	Horizontal	Vertical	dimensions	racks	segments	segments	seg. level	of segments
Hansen and Heinsbroek (1979)	✓			1D	single	single	–	–	–
Corstjens and Doyle (1981)	✓			1D	single	single	–	–	–
Zufryden (1986)	✓			1D	single	single	–	–	–
Borin et al. (1994)	✓			1D	single	single	–	–	–
Urban (1998)	✓			1D	single	single	–	–	–
Irion et al. (2012)	✓			1D	single	single	–	–	–
Hübner and Schaal (2017a)	✓			1D	single	single	–	–	–
Yang (2001)	✓			1D	single	multiple	given	given	identical[a]
Hwang et al. (2005)	✓		✓	1D	single	multiple	given	given	identical[a]
Hansen et al. (2010)	✓	✓	✓	1D	single	multiple	given	given	identical[a]
Hübner and Schaal (2017c)	✓		✓	1D	single	multiple	given	given	identical[a]
Bai et al. (2013)	✓	✓	✓	2D	single	multiple	given	given	given
Düsterhöft et al. (2020)	✓	✓	✓	3D	multiple	multiple	given	given	given
This paper	✓	✓	✓	3D	multiple	multiple	decision	decision	decision

– means not applicable/not considered in model.
[a] Identical dimensions across all segments; segments given.

Table 2
Notation of the general model for product allocation and shelf segment dimensioning.

Indices	
I	Set of items i within the category, with $i \in I$
J	Set of shelf segments j, with $j \in J$
R	Set of shelf racks r available, with $r \in R$
Product-related parameters	
α_i	Basic demand of item i
$\beta_i (\gamma_i)$	Horizontal (vertical) space elasticity of item i
m_i	Margin of one unit of item i
ν_i	Costs of replenishing one unit of item i from backroom
$\overline{w}_i, \overline{h}_i, \overline{d}_i$	Width, height, depth of item i
$Q_i^{max} (Q_i^{min})$	Maximum (minimum) shelf inventory of item i
RSS_i	Minimum representation inventory and safety stock at the shelf of item i
Shelf-related parameters	
δ_r	Attractiveness factor of rack r
ϵ_j	Attractiveness factor of shelf segment level for the given level l_{jr}
b	Minimum grabbing gap, i.e., height between items of segment level l and next segment level $l+1$
g_{ijr}	Inventory per facing of item i at shelf segment j at rack r
$w_{jr}, h_{jr}, d_{jr}, l_{jr}$	Width, height, depth and level of shelf segment j on shelf rack r
$\tilde{w}_r, \tilde{h}_r$	Width and height of shelf rack r
Decision variables	
x_{ijr}^h	Number of horizontal facings of item i at shelf segment j at rack r; integer variable
x_{ijr}^v	Number of vertical facings of item i at shelf segment j at rack r; integer variable
y_{jr}	1 if on rack r segment j is active, else 0; binary variable
Auxiliary variables	
q_i^b	Backroom inventory (i.e., additional refill quantity) of item i; integer variable
q_i^s	Available shelf inventory of item i; integer variable
$\overline{q}_i^s$	Total shelf inventory of item i; integer variable
x_{ijr}	Total number of facings of item i at shelf segment j at rack r; integer variable

3. Model development

This section introduces the formal description of the **P**roduct **A**llocation **M**odel with **i**ntegrated **S**helf Segment **D**imensioning (PAMiSD) and discusses the complexity of this NP-hard Knapsack and non-linear optimization problem. The complete, non-linear mathematical model can be found in Appendix.

Notation. Table 2 summarizes the notation, including parameters and decision variables for the general production allocation and shelf segment-dimensioning model.

Decision variables and constraints. With the objective of maximizing total profit, retailers must assign a given set of items $i, i \in I$ to a total shelf space, where only the number of shelf racks $r, r \in R$ and the height $\tilde{h}_r$ and width $\tilde{w}_r$ of each rack are known. In order to find optimal shelf dimensions, retailers must consider different types of shelf segment $j, j \in J$. Each segment is characterized by its dimensions, i.e., its depth d_{jr}, height h_{jr} and level l_{jr} on a given rack $r, r \in R$. The actual segment dimensions depend on the choice or rather combination of segments on the corresponding rack and are not determined in advance. Consequently, the model optimizes two types of decision variables. First, the binary variable y_{jr} defines if a shelf segment $j, j \in J$ with specified shelf segment dimensions d_{jr}, h_{jr} and l_{jr} at rack $r, r \in R$ is chosen or not. The segment dimensions of the corresponding shelves have to adhere to the following relationships, with j_1 and j_2 as two consecutive segments from bottom to top: (1) $d_{j_1 r} \geq d_{j_2 r}$ and (2) $h_{j_1 r} \geq h_{j_2 r}$, which means the next higher segment j_2 is equal or less deep and high; (3) $l_{j_2 r} = l_{j_1 r} + h_{j_1 r}$, which means the level of the upper segment j_2 is the sum of the level and height of the lower segment j_1 on the considered rack $r, r \in R$. (4) The width w_{jr} of a segment j at rack r is determined by the width of rack $\tilde{w}_r$ (i.e., $w_{jr} = \tilde{w}_r$).

Second, the integer variable x_{ijr} defines the total number of facings of each item $i, i \in I$ at shelf segment $j, j \in J$ and rack $r, r \in R$. The total number of facings x_{ijr} is computed by the number of horizontal facings x^h_{ijr} times the number of vertical facings x^v_{ijr} at shelf segment j of rack r. Each item i can only be assigned to one segment j: If $x_{ijr} \geq 1$, then $x_{ikr} = 0 \ \forall i \in I, j, k \in J : k \neq j, r \in R$. As the assortment is given, each item needs to be allocated and cannot have zero facings: $\sum_{j \in J} \sum_{r \in R} x_{ijr} \geq 1, \forall i \in I$. A shelf segment j at rack r is selected if at least one item i is allocated. This is expressed by setting the binary variable $y_{jr} = 1$ if $x_{ijr} \geq 1$, and 0 otherwise. The available width (w_{jr}) and height (h_{jr}) of a segment j on rack r cannot be exceeded. In the width dimension it therefore needs to be ensured that $\sum_{i \in I} \overline{w}_i \cdot x^h_{ijr} \leq w_{jr}, \forall j \in J, r \in R$, with $\overline{w}_i$ as the item width. In the height dimension, it needs to be ensured that $\overline{h}_i \cdot x^v_{ijr} + b \leq h_{jr}$ for all $i, i \in I, \ j, j \in J$ and $r, r \in R$. This means that at most so many vertical facings x^v_{ijr} of an item with item height $\overline{h}_i$ can be assigned that fit into the segment height and respect an additional grabbing gap b. The total height $\tilde{h}_r$ of each rack r is limited by $\sum_{j \in J} h_{jr} \cdot y_{jr} \leq \tilde{h}_r, \forall r \in R$.

Two associated auxiliary variables are applied to define the total available *shelf inventory* q^s_i and the additional refill quantity from the *backroom* q^b_i. Both quantities are available to fully satisfy total demand λ_i, with $\lambda_i \leq q^s_i + q^b_i$. The total shelf inventory is computed by $\overline{q}^s_i = \sum_{j \in J} \sum_{r \in R} g_{ijr} \cdot x_{ijr}$. The stock per facing g_{ijr} of an item i at segment j and rack r is a parameter as retailers usually use the total segment depth and fill up accordingly. Hence, the parameter g_{ijr} depends on the item depth $\overline{d}_i$ and on the shelf segment depth d_{jr} at rack r, i.e., $g_{ijr} = \lfloor d_{jr} / \overline{d}_i \rfloor, \forall i \in I, j \in J, r \in R$. The available shelf inventory q^s_i is the total shelf inventory $\overline{q}^s_i$ minus the representation minimum and safety stock RSS_i. The parameter RSS_i is exogenously defined, for example by taking into account lead-time for warehouse replenishment, demand volatility or minimum representation quantities. The safety stock RSS_i is part of the shelf stock. This allows modeling of the remaining demand as deterministically known. Furthermore, out-of-stock situations cannot arise as if total demand λ_i exceeds available shelf inventory q^s_i, items are directly replenished from sufficiently available backroom inventory. This is expressed in the second auxiliary variable, the refill quantity from the backroom q^b_i. The quantity for this additional replenishment is calculated by $q^b_i = \max[\lceil \lambda_i - q^s_i \rceil; 0]$. Finally, retailers impose restrictions on the shelf inventory. Minimum and maximum shelf inventory levels are defined by $Q^{min}_i \leq q^s_i \leq Q^{max}_i, \forall i \in I$. The minimum shelf quantity Q^{min}_i factors in the minimum inventory reach and minimum representation quantity and Q^{max}_i the maximum inventory reach accordingly (e.g., for perishable products). These quantities also factor in minimum and maximum number of facings.

Objective function. The retailer pursues the objective of maximizing the total profit P through selecting the optimal number of facings x_{ijr} and shelf segments y_{jr} across all items, shelf segments and racks, represented by the respective vectors $\bar{x}$ and $\bar{y}$, with $\bar{x} = \{x_{111}, x_{112}, \ldots, x_{211}, x_{212}, \ldots, x_{|I||J||R|}\}$ and $\bar{y} = \{y_{11}, y_{12}, \ldots, y_{21}, y_{22}, \ldots, y_{|J||R|}\}$.

$$\max P(\bar{x}, \bar{y}) = \sum_{i \in I} p_i(x_{ijr}, y_{jr}) \tag{1}$$

To obtain the item's profit p_i, the cost of replenishment CR_{ijr} is deducted from the item's gross margin m_i.

$$p_i(x_{ijr}, y_{jr}) = m_i \cdot \lambda_i(x_{ijr}, y_{jr}) - \mathrm{CR}_{ijr}(x_{ijr}, y_{jr}) \tag{2}$$

The gross margin of an item is calculated as the product of its total demand λ_i and its unit margin m_i. The item unit margin m_i corresponds to sales price minus purchase cost and further costs per unit (e.g., listing costs, quantity-independent replenishment costs). The total demand $\lambda_i(x_{ijr}, y_{jr})$ of an item i is a composite function of the basic demand α_i, allocation- and segment-dependent demand. The basic demand α_i represents the retailer's forecast for an item that is independent of the facing, segment and rack position (cf. Bianchi-Aguiar et al., 2021; Hansen & Heinsbroek, 1979; Hübner & Kuhn, 2012). The forecast may be based on historical sales (i.e., average demand across multiple periods), but may also incorporate further marketing effects. The higher the visibility of an item, the higher its demand (Chandon et al., 2009; Cox, 1964; Curhan, 1972; Eisend, 2014). The visibility increases with the number of horizontal and vertical facings (x^h_{ijr}, x^v_{ijr}) and also depends on the item size. In accordance with prior research (cf. e.g., Hansen & Heinsbroek, 1979; Irion et al., 2012), the facing-dependent demand rate is a polynomial function of the number of horizontal facings x^h_{ijr}, visible frontal item width ($\overline{w}_i$) and the horizontal space-elasticity β_i (with $0 \leq \beta_i \leq 1$) as well as number of vertical facings x^v_{ijr}, visible frontal item height ($\overline{h}_i$), and vertical space-elasticity γ_i (with $0 \leq \gamma_i \leq 1$). The factor δ_r represents the attractiveness of rack r and ϵ_j the attractiveness of shelf segment j. As such, the model incorporates space-elastic demand as well as vertical and horizontal attractiveness of shelf segments. Eq. (3) summarizes the demand calculation applied.

$$\lambda_i(x_{ijr}, y_{jr}) = \alpha_i \cdot \sum_{j \in J} \sum_{r \in R} (\overline{w}_i \cdot x^h_{ijr})^{\beta_i} \cdot \sum_{j \in J} \sum_{r \in R} (\overline{h}_i \cdot x^v_{ijr})^{\gamma_i} \cdot \sum_{j \in J} \sum_{r \in R} \delta_r \cdot \epsilon_j \cdot y_{jr} \quad \forall i \in I \tag{3}$$

Costs of replenishment CR_{ijr} presented in Eq. (4) occur whenever the available shelf quantity q^s_i of an item i is not sufficient to cover the demand λ_i of an item i within the time horizon considered and thus additional replenishment with quantity from backroom q^b_i has to be performed at quantity-dependent refill costs ν_i.

$$CR_{ijr}(x_{ijr}, y_{jr}) = \nu_i \cdot q^b_i(x_{ijr}, y_{jr}) \quad \forall i \in I, j \in J, r \in R \tag{4}$$

Model complexity. The model presented constitutes a combinatorial optimization problem and belongs to the class of Knapsack problems. Knapsack problems are known to be NP-hard (see, e.g., Kellerer, Pferschy, & Pisinger, 2004), which underlines the given complexity. In our application, the model complexity is driven by two factors. First, as in classical allocation problems, we consider a large number of different product allocation combinations for a given shelf space. Eq. (5) indicates the possible combinations (Y) of placing K items ($K = |I|$) into a knapsack (= shelf) of the size S.

$$Y(N, S) = \binom{S-1}{K-1} \cdot K \cdot (K-1) \tag{5}$$

Second, we additionally deal with the combination of available shelf space S and the number and size of corresponding shelf segments. For each shelf rack, a different choice of shelf segments (i.e., number and size of segments) is possible, each resulting in a different setting and space availability. Assuming, that each shelf segment only shows discrete, predefined depth and height options, let $\underline{h}$ be the number of different shelf heights and $\underline{d}$ the number of different shelf depths. We can then determine the possible combinations of shelf segments Z for one shelf rack, as shown in Eq. (6).

$$Z = \sum_{j \in J} \underline{h}^j \cdot \sum_{j \in J} \underline{d}^j \tag{6}$$

Table 3
Three-step solution approach of the *PAMiSD*.

Stages	Approach	Implementation
(1) Preprocessing	Determination of problem parameters to exclude non-feasible values for each decision variable	Java
(2) Precalculation to overcome non-linear model	Transfer non-linear parts of the model into parameters that are fed into a BIP. Precalculate these parameters for all possible combinations within the individual bounds obtained from Stage 1 for each decision variable	Java
(3) Solving BIP	Input model parameters obtained from Stage 2 and solve BIP to obtain an optimal solution	CPLEX

The resulting large number of combinations for different shelf segments can be shown by looking at a simple example. Consider two shelf racks with up to 5 segments per rack ($|J| = 5$), and each shelf segment has $\underline{h} = 6$ different heights and $\underline{d} = 3$ different depths options. For two shelf racks this results in 19,386 different combinations of shelf layouts with regard to available space, number and size of segments. Altogether, the allocation combinations arising from Eq. (5) have to be considered for all Z segment/space combinations and for each single shelf rack. The complexity therefore increases significantly as the number of items, shelf space and available segment types increases. Furthermore, some parts of the objective function and constraints are non-linear, e.g., the demand model. Consequently, an efficient solution approach is required to solve the NP-hard and combinatorial complex multiple-choice knapsack problem that is based on a non-linear objective function.

4. Solution approach

In this section, we detail the solution approach proposed to solve PAMiSD to optimality. As stated above, PAMiSD constitutes a non-linear model. Our approach therefore consists of three different stages to address the non-linearity efficiently: (1) preprocessing, (2) precalculation, and (3) the solution of a Binary Integer Problem (BIP). The BIP is a reformulation of the PAMiSD to obtain a model that is solveable by a commercial solver. The different stages are summarized in Table 3.

The *first stage* is necessary to reduce the solution space and combinatorial options by determining model parameters needed as input for the BIP (e.g., limiting possible number of facings per item and segment heights). It is based on the idea of decreasing the solution space by transferring constraints of the Integer Non-linear Program (INLP) formulation of the PAMiSD into a preprocessing step. Doing so we reduce the possible values for the decision variables as we exclude non-feasible settings, but keep all feasible values for the decisions variables. The *second stage* helps to overcome the non-linearity induced by the non-linear demand function. This is done by precalculating demand, margin and replenishment values for the given set of integer facings obtained in Stage 1. Finally, the BIP reformulation of PAMiSD is solved in the *third stage*, leveraging the input parameters determined in Stage 1 & 2. Due to the presteps, the PAMiSD can be solved optimally as a BIP using CPLEX. As the calculated bounds on the decision variables reflect actual constraints of the PAMiSD and its INLP formulation, the solution obtained is a global optimum of PAMiSD. Table 4 summarizes the additional notation used.

Stage 1: preprocessing. In this section we exploit problem specifics (see Section 2.1) to reduce the combinatorial complexity and exclude non-feasible solutions upfront. In detail, we reduce the solution space by considering constraints to obtain tighter bounds for each decision variable. First, we define feasible heights of the segments. Second, we use the feasible heights to set limits for the stacking (i.e., the number of vertical facings) and stringing together of items. The parameters obtained are combined with minimum and maximum inventory reach to calculate feasible ranges for the total number of facings. Ultimately, ranges for the number and dimensions of levels for shelf segments are determined using constraints provided by retail practice (e.g., higher segments are less deep).

Each shelf segment is defined by its vertical level, height and depth. To specify the dimensions of a shelf segment we use $l, l \in L$, for the level, $h, h \in H$ for the height and $d, d \in D$ for the depth. First we define the set of possible segment heights $\hat{H}$. We define $\hat{H}$ during the preprocessing across all racks and not specifically by rack, as this set refers to the possible vertical distance between two segments in general, and is thus equal for all racks. We leverage the fact that retailers define minimum and maximum heights of segments for optical reasons. The parameters H^{min} and H^{max} represent these distances between two levels. Furthermore, heights are bounded to the cavities of a shelf rack, i.e., options where the bottom level of a segment can be hung. These cavities are represented by the vertical points VP of the shelf rack. The set for all potential heights H can therefore be reduced to $\hat{H}$, which only comprises $h^{max} = \lfloor (H^{max} - H^{min})/VP \rfloor + 1$ elements. For example, if we have a minimum segment height $H^{min} = 20$ centimeter and a maximum segment height $H^{max} = 50$ centimeter, the segment height is adjustable within 30 centimeter. Knowing that a segment height can only be placed every 5 centimeter due to the cavities of the shelf rack, i.e., $VP = 5$ centimeter, we only need to consider $h^{max} = 30$ centimeter/5 centimeter + 1 $= 7$ possible settings of a segment height. That also means that the height for each potential segment is defined by $H^{min} + (h \cdot VP), \forall h \in H$.

After reducing the set to $\hat{H}$, we can efficiently calculate the potential values for the vertical number of facings k_{ih} for each item i. The number of vertical facings is precalculated in our approach as retailers usually fill up segments to the maximum. However, some items are not stackable at all or only stackable to a certain limit. This is represented by the upper stacking limit k_i^{max}. Fig. 6 represents the pseudo code to calculate how many units can be stacked for each possible segment height h by respecting the minimum segment height H^{min}, grabbing distances b, vertical points VP, the item height $\overline{h}_i$ and the maximum stacking quantity k_i^{max}.

Secondly, we define the set of potential segment depth $\hat{D}$. Similarly as for the segment height, we leverage minimum and maximum depths of segments. The parameters D^{min} and D^{max} represent these distances. The segment depth can only be varied in given steps DP (like the vertical points VP for the height) and has a minimum and maximum depth D^{min} and D^{max}. Thus, the set for potential depths D can be reduced to the set $\hat{D}$ with only $d^{max} = \lfloor (D^{max} - D^{min})/DP \rfloor + 1$ different depths. The reduced set enables us to efficiently compute for each item i the number of units g_{id} that can be lined up one behind the other on a certain shelf depth d. Fig. 7 represents the pseudo code to calculate how many units g_{id} can be stacked behind each other for each possible segment depth d by respecting minimum segment depth D^{min}, depth points DP and the item depths $\overline{d}_i$.

The parameters obtained enable us to determine an upper bound on the number of facings f_i^{max} for each item i. We use g_{id} and k_{ih} as well as the maximum shelf inventory Q_i^{max} to calculate

Table 4
Additional notation.

Further indices	
D	Set of shelf depths d, with $d \in D$
H	Set of shelf heights, with $h \in H$
L	Set of shelf levels, with $l \in L$
N	Set of number of facings, with $n \in N$
Further parameters	
k_i^{max}	Maximum number of units of item i that can be stacked
$DL_d^{max}(DL_d^{min})$	A certain shelf depth must not be chosen above (below) this level
DP	Distance between two steps in the depth dimension of a segment
$D^{max}(D^{min})$	Maximum (minimum) depth of a segment
$H^{max}(H^{min})$	Maximum (minimum) distance between two segment levels
VP	Distance between two vertical points on which the bottom level of a segment can be hanged (i.e., interval between two adjacent potential shelf levels)
Variables	
x_{inrlhd}	1 if item i has n facings on rack r at level l with height h and depth d, else 0
y_{rlhd}	1 if segment with level l, height of h and depth d at rack r is active, else 0
Further values calculated	
λ_{inrlhd}	Total demand of item i with n facings at rack r, level l, height h and depth d
g_{id}	Number of units behind one another for a certain segment depth d
k_{ih}	Number of stacked units of item i for shelf height h (i.e., vertical number of facings)
CR_{inrlhd}	Costs of replenishment from backroom of item i with n facings at rack r with level l, height h and depth d
M_{inrlhd}	Total margin of item i with n facings at rack r, level l, height h and depth d

```
Input: Set of items I, set of possible segment heights Ĥ
for all items i ∈ I
        for all possible segment heights h ∈ Ĥ
                k_ih = min[k_i^max; ⌊(H^min − b + (h · VP))/h̄_i⌋]
        end for
end for
return k_ih ∀i ∈ I, h ∈ Ĥ
```

Fig. 6. Pseudo code for preprocessing of the stacking parameter k_{ih}.

```
Input: Set of items I, set of possible segment depths D̂
for all items i ∈ I
        for all possible segment depths d ∈ D̂
                g_id = ⌊(D^min + (d · DP))/d̄_i⌋
        end for
end for
return g_id ∀i ∈ I, d ∈ D̂
```

Fig. 7. Pseudo code for preprocessing of parameter g_{id} for stacking items behind each other.

the possible number of facings f_{irhd} for each item i at rack r for a given segment height h and depth d. Fig. 8 summarizes the associated computations.

First of all, if a combination leads to a shelf quantity of zero units ($g_{id} = 0$ or $k_{ih} = 0$), e.g., when it is not possible to allocate at least one unit to a segment due to a certain segment height, this combination is ignored within the process. We also check whether the resulting shelf quantity of one horizontal facing (computed by $k_{ih} \cdot g_{id}$) already exceeds Q_i^{max}. If so, one horizontal facing of this item on the shelf segment considered is only allowed at most as items cannot have zero facings (i.e., cannot be removed from the assortment). Otherwise, for each item i it is checked how many facings can be placed at most on a segment with height h and depth d of rack r, so that the possible number of facings f_{irhd} does not exceed Q_i^{max}. The possible number of facings is determined by $f_{irhd} = \lfloor Q_i^{max}/(k_{ih} \cdot g_{id}) \rfloor$. In a subsequent step, the feasibility of the values found for f_{irhd} is checked. This means that whenever the number of facings f_{irhd} is too high to allocate it to any segment of rack r due to the given segment and rack widths, the corresponding number of facings f_{irhd} is reduced accordingly to $f_{irhd} = \lfloor \tilde{w}_r/\overline{w}_i \rfloor$. This allows non-feasible solutions to be excluded from the precalculation. The maximum value of all f_{irhd} for item i across all possible heights and depths is saved in f_i^{max} as a global maximum for this item i. Moreover, for later iterations the maximum number of facings across all items is determined by $n^{max} = max[f_1^{max}, f_2^{max}, f_3^{max}, \dots, f_{|I|}^{max}]$. The set of possible facings is accordingly defined by $\hat{N} = \{1, \dots, n^{max}\}$.

Finally, also for later iterations, we limit the set of possible levels. Using the maximum height across all racks $\max_r h_r$ and the minimum height of a segment H^{min}, the maximum number of levels on a rack l^{max} can be defined by $l^{max} = \lfloor \max_r \tilde{h}_r / H^{min} \rfloor$. Consequently, the set of possible shelf levels for a shelf rack is given by $\hat{L}$ and contains l^{max} different levels. For each possible level $l, l \in \hat{L}$, it is further necessary to determine a specific depth d from the set of available depths $\hat{D}$. As the shelf depth decreases for higher levels and vice versa, some depth levels are no longer possible if a shelf segment has a certain level. DL_d^{min} represents the fact that a certain shelf depth d cannot be chosen below the level l, or DL_d^{max} above the level l. The connection between the set $\hat{D}$

```
Input: Set of items I, set of racks R, set of possible heights Ĥ, set of possible depths D̂
for all items i ∈ I
    for all racks r ∈ R
        for all possible heights h ∈ Ĥ
            for all possible depth d ∈ D̂
                if (k_ih · g_id > 0)
                    if (Q_i^max ≤ k_ih · g_id)
                        f_ihd = 1
                        if (f_irhd > f_i^max)
                            f_i^max = 1
                        end if
                    end if
                else
                    f_ihd = ⌈Q_i^max/(k_ih · g_id)⌉
                    if (f_irhd > f_i^max)
                        f_i^max = f_irhd
                    end if
                end if else
            end for
        end for
    end for
end for
return f_irhd and f_i^max ∀i ∈ I, r ∈ R, h ∈ Ĥ, d ∈ D̂
```

Fig. 8. Pseudo code for preprocessing the maximal possible number of facings f_i^{max}.

and the parameters DL_d^{min} and DL_d^{max} is considered within the decision for the shelf segment dimensions in y_{rlhd}. An exemplary set of possible shelf depths $\hat{D} = \{50, 40, 30\}$ with $d \in \hat{D}$ on a shelf rack where at most 7 levels l can be determined $\hat{L} = \{1, \ldots, 7\}$ is tied to lower and upper limits for each depth of $DL_d^{min} = \{1, 1, 3\}$ and $DL_d^{max} = \{2, 4, 7\}$. In this case the corresponding boundaries are $DL_1^{min} = 1$ and $DL_1^{max} = 2$ for the first shelf depth of 50 centimeter, indicating that the 50 centimeter-deep shelf segment is available from level 1 to at most level 2. Consequently the decision variable y_{rlh1} must be 0 for all levels $l \in \{DL_d^{max} + 1, \ldots, 7\}$. Equally this applies to the shelf depth of 40 centimeter, which is also available from level 1, but can be chosen up to level 4. For the shelf depth of 30 centimeter, the lower limit is $DL_3^{min} = 3$, indicating that this segment type is invalid for any shelf level below level 3 but is eligible up to level 7 defined by $DL_3^{max} = 7$.

Stage 2: precalculation. Based on the preprocessing and limiting the sets for $\hat{H}$, $\hat{L}$, $\hat{D}$ and $\hat{N}$ and the values obtained for f_i^{max}, g_{id} and k_{ih}, we start the precalculation of the non-linear parts of the model. We calculate values for the total item demand λ_{inrlhd} (cf. Eq. (7)), the resulting total item margin M_{inrlhd} (cf. Eq. (8)) and the costs of replenishment CR_{inrlhd} (cf. Eq. (9)) of each item $i, i \in I$, possible facings $n, n \in \hat{N}$, at rack $r, r \in R$, possible level $l, l \in \hat{L}$, possible height $h, h \in \hat{H}$ and possible depth $d, d \in \hat{D}$ as shown in the pseudo code of Fig. 9. The demand λ_{inrlhd} of an item in Eq. (7) is defined as in Eq. (3) by its basic demand α_i, the number of horizontal facings n, the number of vertical facings k_{ih} together with the horizontal and vertical space elasticities β_i and γ_i as well as the attractivity factors for the rack δ_r and for the segment level ϵ_l.

$$\lambda_{inrlhd} = \alpha_i \cdot (\overline{w}_i \cdot n)^{\beta_i} \cdot (\overline{h}_i \cdot k_{ih})^{\gamma_i} \cdot \delta_r \cdot \epsilon_l \tag{7}$$

The margin of an item is shown in Eq. (8) as the unit margin m_i of an item multiplied by the resulting demand λ_{inrlhd} of the allocated product.

$$M_{inrlhd} = m_i \cdot \lambda_{inrlhd} \tag{8}$$

Replenishment costs CR_{inrlhd} presented in Eq. (9) occur (as similarly defined in Eq. (4) whenever the available shelf quantity q_i^s of an item is not sufficient to cover the demand δ_{inrlhd}, and thus additional replenishments at a quantity of q_i^b have to be made at costs ν_i. This is computed by $q_i^b = \max[\lceil \delta_{inrlhd} - q_{inrlhd}^s \rceil; 0]$, whereas the available shelf inventory q_{inrlhd}^s is computed by $\overline{q}_{inrlhd}^s = n \cdot k_{ih} \cdot g_{id} - RSS_i$. The profit p_{inrlhd} per item is the delta between M_{inrlhd} and CR_{inrlhd}.

$$CR_{inrlhd} = \nu_i \cdot q_{inrlhd}^b \tag{9}$$

Stage 3: application of BIP. The model can now be formulated as BIP as the non-linear terms have been precalulated and flow, as parameters, into the objective function and constraints. The BIP can be solved using the CPLEX-Solver. We use two sets of binary decision variables, one for shelf dimensioning and the other for product allocation:

- Shelf segment dimension: The binary variable y_{rlhd} indicates whether the segment level l with a height h and a depth d is activated on rack r.
- Product allocation: The binary variable x_{inrlhd} indicates whether item i is allocated with n facings on rack r and segment level l, where units can be stacked and placed one behind the other according to the height h and depth d.

The objective function (10) maximizes the total profit P of all items of the considered category.

$$Max!P = \sum_{i \in I} \sum_{n \in \hat{N}} \sum_{r \in R} \sum_{l \in \hat{L}} \sum_{h \in \hat{H}} \sum_{d \in \hat{D}} p_{inrlhd} \cdot x_{inrlhd} \tag{10}$$

subject to

$$\sum_{i \in I} \sum_{n \in \hat{N}} x_{inrlhd} - \mathrm{M} \cdot y_{rlhd} \leq 0 \quad \forall r \in R, l \in \hat{L}, h \in \hat{H}, d \in \hat{D} \tag{11}$$

$$\sum_{n \in \hat{N}} \sum_{r \in R} \sum_{l \in \hat{L}} \sum_{h \in \hat{H}} \sum_{d \in \hat{D}} x_{inrlhd} = 1 \quad \forall i \in I \tag{12}$$

$$\sum_{h \in \hat{H}} \sum_{d \in \hat{D}} y_{rlhd} \leq 1 \quad \forall r \in R, l \in \hat{L} \tag{13}$$

$$\sum_{l \in \hat{L}} \sum_{h \in \hat{H}} \sum_{d \in \hat{D}} (H^{min} + h \cdot VP) \cdot y_{rlhd} = \tilde{h}_r + b \quad \forall r \in R \tag{14}$$

$$\sum_{i \in I} \sum_{n \in \hat{N}} \sum_{h \in \hat{H}} \sum_{d \in \hat{D}} n \cdot \overline{w}_i \cdot x_{inrlhd} \leq \tilde{w}_r \quad \forall r \in R, l \in \hat{L} \tag{15}$$

```
Input: Sets I, N̂, R, L̂, Ĥ and D̂
for all items i ∈ I
    for all possible facings n ∈ N̂
    if (n ≤ f_i^max)
        for all racks r ∈ R
        if (n · h_i ≤ h_r)
            for all possible levels l ∈ L̂
                for all possible heights h ∈ Ĥ
                    for all possible depths d ∈ D̂
                    if (n ≤ f_ihd)
                        λ_inrlhd = α_i · (w̄_i · n)^β_i · (h̄_i · k_ih)^γ_i · δ_r · ε_l
                        M_inrlhd = m_i · λ_inrlhd
                        CR_inrlhd = v_i · q^b_inrlhd
                        p_inrlhd = M_inrlhd − CR_inrlhd
                    end if
                    end for
                end for
            end for
        end if
        end for
    end if
    end for
end for
return p_inrlhd  ∀i ∈ I, n ∈ N̂, r ∈ R, l ∈ L̂, h ∈ Ĥ, d ∈ D̂
```

Fig. 9. Pseudo code for precalculation of non-linear model parts to obtain item profit p_{inrlhd}.

$$\sum_{n\in\hat{N}\setminus\{1\}}\sum_{r\in R}\sum_{l\in\hat{L}}\sum_{h\in\hat{H}}\sum_{d\in\hat{D}} n\cdot k_{ih}\cdot g_{id}\cdot x_{inrlhd} \le Q_i^{max} \ \forall i\in I \tag{16}$$

$$\sum_{n\in\hat{N}}\sum_{r\in R}\sum_{l\in\hat{L}}\sum_{h\in\hat{H}}\sum_{d\in\hat{D}} n\cdot k_{ih}\cdot g_{id}\cdot x_{inrlhd} \ge Q_i^{min} \ \forall i\in I \tag{17}$$

$$\sum_{d\in\hat{D}} h\cdot y_{rohd} \le \sum_{d\in\hat{D}} h\cdot y_{rlhd} \ \forall r\in R, l,o\in\hat{L}: o>l, h\in\hat{H} \tag{18}$$

$$\sum_{d\in\hat{D}} d\cdot y_{rohd} \le \sum_{d\in\hat{D}} d\cdot y_{rlhd} \ \forall r\in R, l,o\in\hat{L}: o>l, h\in\hat{H}; \tag{19}$$

$$\sum_{h\in H} y_{rlhd} = 0 \ \forall r\in R, d\in\hat{D}, l\in\{1,\dots,DL_d^{min}-1\} \tag{20}$$

$$\sum_{h\in H} y_{rlhd} = 0 \ \forall r\in R, d\in\hat{D}, l\in\{DL_d^{max}+1,\dots,L\} \tag{21}$$

We use Restriction (11) to ensure that if an item i is assigned to a shelf segment with n facings, the corresponding segment at rack r is active with the required dimensions l, h and d. This is formulated using the "BigM" method, where M represents a sufficiently large number (e.g., $M \ge |N|\cdot|I|$). Restriction (12) ensures that each item i is exactly assigned once on a segment of a certain rack. Assigning the same item to multiple segments is not permitted, but ultimately each item has to be assigned. Further, if a segment level l on rack r is activated, its dimensions (height and depth) are defined distinctly (Restriction (13)). Restriction (14) ensures that none of the segment levels activated on a shelf rack exceeds the total height of the rack, $\tilde{h}_r$. The minimum grabbing distance b is added here to the top level. Restriction (15) ensures that the facings of all items allocated to a certain segment do not exceed the width of the segment, or the equally sized rack width $\tilde{w}_r$. Further, Restrictions (16) and Restriction (17) ensure that the available shelf inventory q_i^s determined, lies within the upper and lower bounds for shelf inventory, Q_i^{min} and Q_i^{max}. Within Restriction (16) it is ensured that at least one facing of an item is always allowed even if the shelf inventory of one facing would already exceed the maximum inventory. Restriction (18) ensures that the height between the segments is constant or decreasing from the bottom to the top of a rack. The same logic is applied in Restriction (19) regarding the depths of the segments. Restrictions (20) and (21) ensure that the chosen segment depth of each level is in line with the lower and upper limits for each depth DL_d^{min} and DL_d^{max}.

5. Numerical results

This section provides numerical tests with simulated data as well as a case study with a major German retailer in order to analyze the general applicability to realistic problem sizes, its computational performance and to develop managerial insights. We use the IBM ILOG CPLEX Optimization Studio 12.6.2.0 on a Windows 8 64 Bit machine with 16 gigabyte RAM and an Intel(R) Core(TM) i5-6440HQ CPU with 2.60 gigahertz. The runtimes indicated in our tests refer to the total time for preprocessing, precalculation and solution of the BIP with CPLEX. Table 5 provides an overview of the numerical experiments.

In the first set of tests we use simulated data to analyze the performance of our approach. We leverage our data generation process based on insights from literature, retail practice and (if applicable) actual values received from our partner company. More precisely, we obtain replenishment costs $v_i = 0.22$, while mini-

Table 5
Overview of numerical experiments.

Section	Experiments and purpose	Analyzed parameter(s)/ scenario	Data set	# instances
5.1	Runtime analysis and efficiency	Number of items and shelf racks, profit margins	{Simulated data, informed by case study	520
5.2	Integrating shelf dimensioning and product allocation	Optimization with/without shelf dimensioning		100
5.3	Impact of shelf dimensioning and related demand	Shelf dimension parameters, attractiveness factors		120
5.4	Case study		Real-world data	10

Table 6
Impact of preprocessing on runtime for different problem sizes, averages of 20 instances per problem size.

Avg. runtime [sec.]	$\|I\| = 20$	$\|I\| = 30$	$\|I\| = 50$	$\|I\| = 100$	$\|I\| = 200$
– 3-Stage Approach	1.2	2.0	3.7	7.6	16.1
– Direct PAMiSD (w/o Stage 1)	8.9	20.1	60.8	> 3600	> 3600

mum and maximum inventory reach was limited to 3 and 30 time periods and a minimum representation quantity of 2 facings. The latter are applied to set Q_i^{min} and Q_i^{max}. Further, we use additional model parameters that are informed by our insights from work with the retailer: $w_i \in [9, 30]$, $h_i \in [10, 40]$, $d_i \in [5, 40]$, $m_i \in [0.1, 3.0]$ and $\alpha_i \in [0.2, 13]$. Space-elasticity parameters β_i and γ_i are set at 0.17 in line with the findings of a meta-analysis by Eisend (2014) on various related empirical studies. Finally, shelf-related global parameters are set as follows: $w_r = 200, b = 15, H^{min} = 40, H^{max} = 70, VP = 5$. The rack attractiveness factor δ_r was set up to 1.05 and the segment factor ϵ_l up to 1.10. The depths for segments at levels 1–3 were set at 57 centimeter, and 47 centimeter for all levels starting from level 2. These values are based on direct information from retail practice. If different values were used in the numerical studies, these are specified in the following.

5.1. Efficiency of the solution approach

Run time tests in comparison to benchmark approach. The first analysis assesses the efficiency of our approach. We compare the runtimes of our three-step approach to a direct solution of PAMiSD (denoted as *direct PAMiSD*), i.e., without the preprocessing (Stage 1). Stage 2 is always necessary as otherwise the model remains non-linear. The benchmark (direct PAMiSD) does not leverage on the preprocessing of parameter settings (i.e., predetermined number of facings per item, segment heights, depths, and levels) to reduce the solution space by excluding non-feasible values for the decision variables. Instead it incorporates the related constraints (see Section 3) with global limits using the original sets of potential heights H, depths D and levels L of shelf segments. These global limits are valid across all items (and not item individual), and constitute a reasonable choice for a feasible number of facings $|N|$. More precisely, the ranges are determined by dividing the widest segment by the smallest item, which then provides the physically highest number of horizontal facings possible. The ranges obtained are then used as input to Stage 2 and the precalculation of demands, margins and replenishment costs. In this way we can elaborate on the value of Stage 1 as both approaches provide optimal solutions. Please note that the comparison is based on the same setting, but with and without the preprocessing.

Table 6 shows that our three-stage approach can handle all problem sizes of practical relevance efficiently. With an increasing problem size $|I|$ the suggested solution approach with preprocessing shows significantly lower computation times. For the largest instances ($|I| = 200$), the average runtime with our approach is only 16 seconds, while the benchmark requires more than one hour on average to solve the corresponding problem.

Impact of available items and shelf racks. The runtime mainly depends on two data parameters: the number of available items $|I|$ and the number of shelf racks $|R|$. Hence we consider the relationship of I and R and its impact on runtime by subdividing the entire shelf width into multiple equally dimensioned racks. This allows the use of the same item set and the comparison of total profitability. Taking into account realistic shelf sizes, we do not allow shelf racks of $\tilde{w}_r \leq 60$ centimeter. The results are summarized in Table 7.

The results show that our approach is able to solve large data sets with up to 200 items within a reasonable time, especially when considering that a tactical planning problem is addressed. Naturally, the average runtime rises as I increases. Further, it is evident that for a given number of items $|I|$, a higher number of racks $|R|$ increases runtime. In particular, considering $|R| = 6$, a run time of 1 hour is exceeded for 15% of test instances with $|I| \geq 100$. However, in these cases the corresponding MIP GAPs after one hour are very small, ranging between 0.53 and 0.71% for $|I| = 100$ and between 0.54 and 1.09% for $|I| = 200$. Our three-step approach is able to reach an optimal solution for these instances in less than three hours. Further, the number of racks $|R|$ in this analysis can be considered as “break points” of the total shelf space. It divides the total shelf space, e.g., due to constructional reasons or when space is split into different aisles. However, retailer will always tend to reduce these break points to obtain a clean and steady optic of the resulting shelves. Furthermore, a higher number of $|R|$ racks does not have a positive effect on the objective value. Compared to the solution found with $|R| = 1$, the objective values remain more or less steady within a range of ±0.5%, whereas the runtime increases significantly.

5.2. Tests for integrating segment dimensioning into product allocation planning

The core contribution of this paper is the integration of shelf dimensioning into product allocation. In this subsection we accordingly emphasize the additional value of the integration and enhancement. Please note that this paper this is the first approach for the integrated problem. There are consequently no benchmark instances with respect to dimensioning of segments as other approaches use the segment dimensions as an input factor. To investigate the impact of integration, we compare PAMiSD with the model of Düsterhöft et al. (2020), which is the most related product allocation model with respect to shelf size considerations (see Section 2.3). This model does not optimize for shelf segment dimensions, but is based on given sizes of multi-dimensional shelf segments. As the choice of different shelf segments sizes is arbitrary when the corresponding dimensions are fixed and not part of the optimization, we consider multiple, identical shelf segments for the model of Düsterhöft et al. (2020). This approach is therefore in line with most product allocation models presented (e.g., Hansen et al. (2010) or Hwang et al. (2005)). Furthermore, we apply the identical demand model for the benchmark approach. Four equal shelf segments are considered for the model of Düsterhöft et al. (2020), each with a height of 50 centimeter. Further, all segments are of the same depth (57 centimeter), such that the number of items that can be placed one behind another is also fixed in advance.

Table 8 summarizes the results with varying problem sizes. Even though the results of the benchmark of Düsterhöft et al. (2020) also provide optimal allocations for the given setting, we show that integrating shelf segment dimensioning further improves the objective value by an average of about 3.3% across all test instances. The profit increases at least by 1.15% and up to 5.67%. Usually, the more items need to be allocated, the higher the profit potential. We apply in the following sections sensitivity analysis to investigate the value of integration.

Table 7
Impact of number of items I and number of racks R on runtime [in seconds] and objective value [in euros], average of 20 instances per problem size.

$\|I\|$		$\|R\| = 1$	$\|R\| = 2$	$\|R\| = 4$	$\|R\| = 6$
20	Shelf rack width w_r	1 · 160 centimeter	2 · 80 centimeter	–	–
	Avg. objective value	227.32	223.18		
	Avg. runtime	1.20	2.35		
30	Shelf rack width w_r	1 · 240 centimeter	2 · 120 centimeter	4 · 60 centimeter	–
	Avg. objective value	342.68	340.10	337.15	
	Avg. runtime	1.99	3.93	4.37	
50	Shelf rack width w_r	1 · 400 centimeter	2 · 200 centimeter	4 · 100 centimeter	6 · 67 centimeter
	Avg. objective value	599.57	604.50	601.12	597.98
	Avg. runtime	3.67	9.94	15.32	15.33
100	Shelf rack width w_r	1 · 800 centimeter	2 · 400 centimeter	4 · 200 centimeter	6 · 133 centimeter
	Avg. objective value	1195.74	1195.06	1197.02	1195.82
	Avg. runtime	7.59	17.55	97.46	820.39
200	Shelf rack width w_r	1 · 1,600 centimeter	2 · 800 centimeter	4 · 400 centimeter	6 · 267 centimeter
	Avg. objective value	2427.83	2427.24	2425.57	2424.84
	Avg. runtime	16.09	34.56	180.83	898.12

Table 8
Impact of integrating shelf segment dimensioning (PAMiSD vs. Düsterhöft et al. (2020)), 20 instances per problem size.

Profit delta of PAMiSD vs. Düsterhöft et al. (2020) for data sets with differing numbers of items									
$\|I\| = 20$	Δ in %	$\|I\| = 30$	Δ in %	$\|I\| = 50$	Δ in %	$\|I\| = 100$	Δ in %	$\|I\| = 200$	Δ in %
Avg.	+ 2.89	Avg.	+ 3.12	Avg.	+ 2.97	Avg.	+ 4.49	Avg.	+ 3.17
Min.	+ 1.15	Min.	+ 1.48	Min.	+ 1.67	Min.	+ 3.49	Min.	+ 2.54
Max.	+ 4.70	Max.	+ 5.31	Max.	+ 4.41	Max.	+ 5.67	Max.	+ 4.17

Table 9
Impact of shelf dimensions.

Parameter	Reduction	Base value	Extension
Rack width ($\tilde{w}_r$)	350 centimeter	400 centimeter	450 centimeter
Avg. objective value in %	99.9	100.0	101.4
Min. deviation from base value in %	0.00	–	0.00
Max. deviation from base value in %	−1.98	–	+3.01
Shelf depths ($d \in D$)	57 centimeter	47, 57 centimeter	47, 57, 67 centimeter
Avg. objective value in %	99.6	100.0	101.1
Min. deviation from base value in %	−0.33	–	0.00
Max. deviation from base value in %	−2.62	–	+2.20
Min. segment height (H^{min})[1]	20 centimeter	40 centimeter	50 centimeter
Avg. objective value in %	100.9	100.0	98.6
Min. deviation from base value in %	+0.04	–	−0.81
Max. deviation from base value in %	+2.17	–	−1.85

[1] Max. segment height (H^{max}): 70 centimeter.

5.3. Impact of shelf dimensions and shelf-related demand

In this section, we examine the impact of different parameters on the overall problem. This comprises the impact of available shelf dimensions (i.e., rack width, minimum segment height, shelf depth) and shelf attractiveness factors (δ_r and ϵ_l). We chose these parameters as the shelf dimensions directly impact the overall model complexity (i.e., degree of freedom set within our precalculations) and the optimization potential for shelf space planning. Additionally, the study of attractiveness factors for rack- and segment dependent demand highlight the importance of their incorporation within shelf space planning. We use 20 instances per parameter with $|I| = 50$ and $|R| = 1$. We focus in our analysis mainly on the impact on total profit, as the average run time of all test instances is below 6 seconds.

Impact of shelf dimensions. Table 9 summarizes the results for the analysis of available shelf dimensions. A reduction in total shelf space (i.e., reducing $\tilde{w}_r$) leads to a minor decrease of possible profits. On the other hand, if more shelf space is available, more products can be placed on the shelf (i.e., number of facings increases), which naturally leads to an increase in profits. Further, the rack width impacts needed computational times as less space makes the selections of products harder, while more space simplifies the space allocation as the competition for shelf space between products is mitigated. Increasing the number of available depths $|D|$ and lower minimum segment heights H^{min} leads to increasing profits. Once there are more shelf depth options or segment heights, the solution generated with a higher degree of freedom is at least as good as the solution with a lower degree of freedom. Provided that all other parameters remain untouched, it holds true that $P_{|D|=1} \leq P_{|D|=2} \leq P_{|D|=3}$ and respectively for small minimum segment heights. Knowing this, it is advisable for retailers to include different shelf depths and lower segment heights not only for optical reasons but also in the course of profit maximization.

Impact of rack- and segment-dependent demand. Another test is applied to investigate the impact of different rack- and segment-dependent demand factors. We consider the two spatial attrac-

Table 10
Average changes in solution structure with segment and rack dependent demand effects.

Demand effects	δ_r = off, ϵ_l = off	δ_r = on, ϵ_l = off	δ_r = off, ϵ_l = on	δ_r = on, ϵ_l = on
Different facings		8.8% [−3;1]	9.8% [−2;2]	11.0% [−3;3]
Different rack		75.4%	78.6%	78.8%
Different segment		59.6%	67.6%	65.8%
Impact on obj. value[1]	−2.2%	−0.9%	−0.8%	0.0%

[1] Ex-post evaluation of solution obtained with δ_r or/and ϵ_l in comparison to solution with δ_r =on and ϵ_l =on.

tivity parameters δ_r and ϵ_l within the demand function. In total, we compare four solutions. In the first set both effects are switched off. Afterwards we activate each attractivity factor while the other one remains unattended, and finally both effects are activated together. We compare the structure of the average solutions (i.e., number of facings, assignment to rack and segment) of each setting to the case where shelf-segment-dependent demand effects are not considered at all. For this analysis we use the data sets containing $|I| = 50$ items and $|R| = 4$ shelf racks. For the vertical level of the segment we assume a linear increase from the bottom to the top of 10% apportioned to the levels in between, with $\epsilon_l = \{1.000, 1.020, 1.040, 1.060, 1.080, 1.100\}$. The rack attractiveness factor follows the same intention, with a higher demand for items on the first rack of 5%, so that $\delta_r = \{1.050, 1.033, 1.017, 1.000\}$.

Table 10 summarizes the impact of varying rack- and segment-dependent demand effects. Modeling the effects has a notable impact on objective values. The profit decreases by up to 2.2% if demand effects are not included in the model, but exist in reality. Given the small magnitude of 10% and 5% of these effects, this is already significant. With respect to facing changes, the effect is moderate. When only δ_r is modeled, we see that the number of facings displayed to customers is changed in 8.8% of cases on average compared to the base case assuming no rack- and segment-dependent demand effects. Further, the number of facings was changed by −3 and +1 unit at most. The results are similar when instead only ϵ_l is considered. The average number of items with facings changed increases to 9.8% with ranges of ±2. When both effects are used within the model the number of changed items is again increases by up to 11.0%, while changes appeared ±3 units. The effect of rack and segment changes due to this effect is significant. The assignment of items to specific shelf racks changes between 75.4% and 78.8% when spatial demand effects are considered. There is a high percentage of items with changed rack assignment even where the rack-specific demand factor δ_r is not assumed. This can be explained by the varying attractiveness of shelf segments and the related switching of items from an unattractive segment of one rack to a more attractive segment on another rack. The number of items with a changed vertical level is 59.6% on average when only rack-dependent demand is considered, and 67.6% for segment-dependent demand. In the combined case 65.8% of items are assigned to another segment on average.

5.4. Case study: practical application of PAMiSD

This section presents a case study that is subject to a close cooperation with one of Europe's biggest grocery companies, which provided data and insights from their daily operations. In line with this, we use real data from the tea and tinned food category across five stores located in Eastern Europe. We had access to relevant shelf data (e.g., rack and segment sizes, minimum distances) and item data (e.g., margins, sizes, minimum inventory, replenishment costs). The item-specific parameters are subject to a non-disclosure agreement. We calculated the base demand α_i for each item with actual sales and number of facings. Demand factors are applied as in the tests above. The tinned food category consists of $|I| = 113$ different products. The retailer set $H^{min} = 40$ centimeter and $H^{max} = 70$ centimeter. Within tea we consider $|I| = 197$ items and $H^{min} = 30$ centimeter and $H^{max} = 70$ centimeter. The retailer allows a segment depth of 47 centimeter, 57 centimeter and 67 centimeter for both categories. In the stores, each category has an available shelf width of eight running meters, which comprises 6 racks with $w_r = 133$ centimeter each. Every shelf rack is $h_r = 220$ centimeter high and the grabbing gap $b = 15$ centimeter and $VP = 5$ centimeter are set in accordance with the actual shelf. While the given shelf space for each store and category was equal, the actual sales differed significantly across the stores.

Benckmarks. We use the status quo to benchmark it with PAMiSD. The status quo stems from the current planograms that are created with the latest version of a state-of-the-art commercial software for shelf planning. This means that we compare PAMiSD with a solution that is already promised to be a practicable and optimized solution by the retailer's software. Profit of the status quo contains the margin m_i multiplied by the sales minus the replenishment costs that are derived ex-post from the ratio of demand to shelf inventory. In order to further verify the use of integrated shelf space dimensioning, we also compare PAMiSD with the model of Düsterhöft et al. (2020) that operates with fixed shelf dimensions. For both categories, we tested several segment sizes for the application of Düsterhöft et al. (2020) and chose the setting with the highest profit to have a solid benchmark. For tinned food, the height of each shelf level was fixed at 55 centimeter and the depth at 57 centimeter, resulting in four equally sized segments for each rack. For the tea category, five equally sized segments per rack with a height of 44 centimeter and a depth of 57 centimeter build the benchmark.

Comparison of status quo to PAMiSD. Table 11 shows that PAMiSD increases the retailer's profit significantly by 5.03 and 5.74% on average. This is impressive as the status quo is based on the best possible planning result of the planner using the solution of the applied commercial software. In detail, the profit increase calculated with PAMiSD for tinned food is between 3.34 and 7.12%, whereas for tea the profit increases lie between 5.07 and 6.29%. The broader range in the tinned food category is explained by the sales potential and different status quo in the stores.

Comparison of status quo to Düsterhöft et al. (2020). No feasible solution could be determined with this approach for tinned food in two stores due to insufficient shelf space. In all other cases we see a moderate profit increase. Yet in each case the model of Düsterhöft et al. (2020) is outperformed by PAMiSD. Once more we can prove that the integrated approach results in significantly higher profits than the current approaches that tackle only the product allocation problem.

Improvement of planning processes and shelf layout. Fig. 10 shows as an example for Store 2 the shelf layout of the status quo in the tea segment, whereas Fig. 11 illustrates the new segments. In this case the number of levels required for proper product allocation remains at five, but the distances between the vertical levels are optimized with PAMiSD. Further, in the current shelf layout we

Table 11
Profit increase and runtime for case studies compared to status quo.

	Store 1	Store 2	Store 3	Store 4	Store 5	Average
Tinned food						
Profit Δ Düsterhöft et al. (2020), in %	–[a]	+0.96	+2.05	–[a]	+3.56	+2.19
Profit Δ PAMiSD, in %	+4.71	+3.34	+4.70	+5.29	+7.12	+5.03
Runtime PAMiSD [seconds]	37	16	18	24	38	26
Tea						
Profit Δ Düsterhöft et al. (2020), in %	+1.51	+2.27	+2.25	+0.98	+1.72	+1.75
Profit Δ PAMiSD, in %	+6.29	+6.06	+5.92	+5.36	+5.07	+5.74
Runtime PAMiSD [seconds]	82	56	69	99	90	79

[a] No feasible solution with Düsterhöft et al. (2020) possible.

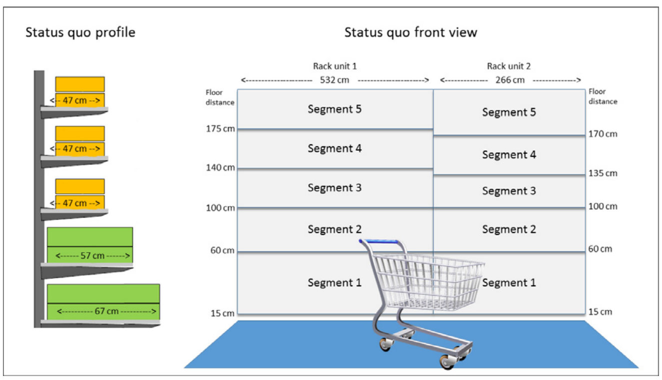

Fig. 10. Manually created shelf segments for the current product allocation with commercial software.

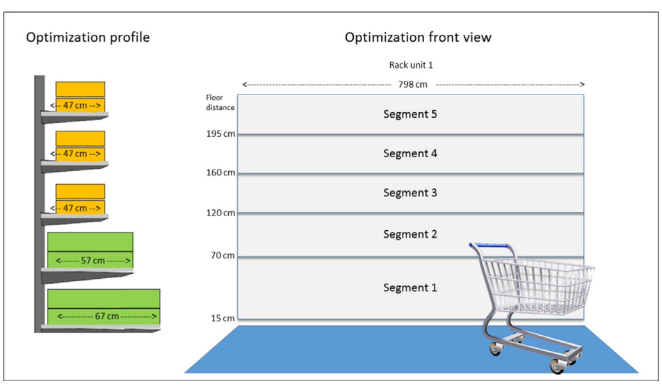

Fig. 11. Optimized shelf segments generated using PAMiSD.

find an interruption which gives the whole shelf an unsteady appearance. In contrast, PAMiSD provides an equal shelf layout for the whole category. Furthermore, despite the retailer using professional software for the current planning, the shelf dimensions are not defined by this commercial tool. This means that shelf planners have to plan the shelf layout manually and test several possible settings until they accomplish a satisfying solution.

5.5. Summary and managerial insights

In conclusion, the suggested approach is an improvement in several ways. First of all, shelf planners do not need to create the shelf dimensions manually and thus do not need several manual iterations to evaluate a plan in a try-and-error approach that is expected to be suboptimal. PAMiSD is fast to generate optimal solutions for practical purposes. Secondly, equal-looking shelves tend to generate higher profits. We show that a different segment height across racks tends to have a negative effect on the profit. The profit decreases when the total shelf of a category is split up into several racks (see Table 7). This is explained by the resulting very small racks and thus limited space for allocating facings of one product next to each other. Designing the entire shelf width as one rack for one category gives the resulting shelves a consistent shelf layout. As splitting up the entire shelf into multiple racks does not improve profit, it is sufficient to design and model only one shelf rack for the planning that depicts the total available shelf width. This strategy leads to two benefits: (i) retailers reach a steady and equal look over the different racks of a certain category and (ii) the performance is superior with respect to profitability and computation time. Furthermore, it is expected that equal-looking shelves make orientation much easier and positively impact customer satisfaction and sales. Thirdly, the profit magnitude is related to the total shelf space, number of segment depths that can be selected, and the magnitude of demand effects. A higher flexibility in choosing segment depths leads to increasing profits (see Table 9). Incorporating vertical and horizontal demand effects on the segments in essential. We show that a small magnitude of 5–10% of these effects already causes an increase in profit by more than 2%. It affects the number of facings and the assignment of items to different levels. Finally and most importantly, the new shelf segment dimensions in combination with product allocation increase the total profit of a category significantly (see Table 8). The integration leads to an increase of around 3.3% on average, with a minimum of 1.2% and a maximum of 5.7%. As margins in retail are often only 2–3%, integrating the product allocation and shelf dimensioning can become a significant contribution to the retailers profitability. With our case study, we could show that PAMiSD increases the retailer's profit significantly by more than 5% on average. This is impressive, as the status quo is based on the best possible planning result of the planner using the solution of the applied commercial software. As such, our result can also be used to evaluate current retail practice.

6. Conclusion

In this work, we presented a shelf space optimization approach that is extended with shelf segment dimensioning. As such, it enables more realistic planning for retailers and constitutes an appropriate decision support tool for practitioners. In our approach, we consider both the product allocation to shelves and the actual shelf layout by defining the number and dimensions of shelf segments. These two decisions are highly interrelated as given shelf dimensions serve as bounds for the product allocation, while the products considered limit decisions on shelf dimensions due to the given product characteristics. Determining the optimal number of shelf segments and defining individual segment dimensions was previously subject to the manual adjustments of a planner without any decision support. Also the literature falls short in this aspect as the segment dimensions are a parameter that is at least partially predetermined (see e.g., Bai et al. (2013) or Düsterhöft et al. (2020)). By integrating the segment decision into the shelf space optimization process, we provide practical and applicable results for retailers. The practicability of our approach is further ensured by considering actual customer behavior (i.e., rack-, segment- and facing-dependent demand function) and retail profitability (i.e., margins and impact on replenishment costs).

We obtain optimal solutions by applying a three-step process. In the first two steps, we set tighter bounds and eliminate non-linear parts of the model. This makes it possible to solve a BIP exactly and fast. The relevance and contribution of our model and solution approach is shown in numerical studies. Firstly, we show the time-efficiency of the solution approach for practical relevant problem sizes. We also provide a sensitivity analysis to identify important factors for the computation time. Secondly, we highlight the impact of integrating shelf dimensioning into the solution approach. In contrast to a shelf space optimization approach that does not consider segment dimensions, improvements of up to 5% can be achieved. Finally, we demonstrate the practical use of our approach in a case study with a major European retailer. Here we show that our approach is able to improve a given planning situation and corresponding profits by 3–7% across different stores and categories.

Future areas of research. Integrating shelf dimensioning into product allocation closes an important gap in literature and practice. However, there are still numerous possibilities for future research to improve shelf space planning. To begin with, retailers may imply certain aesthetic rules for the rack layout as part of the store layout and aisle network plan. For instance, it may only be possible to have standard heights and depths of segments within one category. Such a policy can be incorporated in our model by applying just one large shelf where each rack has the identical dimensions. However, it is not yet researched if standardized or varying racks have an additional demand impact. It would be interesting to test this within an empirically study and to incorporate the demand effects with a scenario analysis in our model. In our approach we already consider demand fluctuations by defining safety stocks for each product. However we do not explicitly cover stochastic demand. Besides stochastic demand, seasonal demand and demand effects caused by promotions or item pricing are a valuable path for further research in this area (see Flamand, Ghoniem, & Maddah, 2016). Another possible extension of our model approach is the consideration of detailed merchandising and item sequencing decisions. This means that a defined order for neighboring products has to be respected for the product allocation (Bianchi-Aguiar et al., 2017). In this context other possible extensions are the integration of assortment decision, and related effects for out-of-assortment or out-of-stock situations (see e.g., Honhon, Gaur, & Seshadri, 2010; Hübner & Schaal, 2017b; Kök & Fisher, 2007). An essential connecting factor between related planning steps would be the integration of inventory management and store delivery decisions (Holzapfel et al., 2016; Taube & Minner, 2018). Within our approach, we already consider upper and lower bounds for the shelf quantity of each item, which could then be reconciled and optimized in accordance with up-streamed logistics processes. A different aspect for future research is the available shelf space. In our approach the total shelf space of a category, more precisely the number of empty shelf racks, is an input parameter. But within a retail store the share of shelf space for each category is also a flexible factor within certain limits. An integrated approach that solves the product allocation problem for a category together with the question of how much shelf space should be assigned to this category at all could therefore further improve decision support models for retailers. A further extension might deal with optimizing shelf space across different stores and integrating local demand (see e.g., Corsten, Hopf, Kasper, & Thielen, 2018).

Acknowledgments

The authors are grateful to the retailer, i.e., the case company that we collaborated with on this project, for their support and many helpful contributions. A special thanks is due to the shelf planning department and their valuable recommendations, which significantly improved our research.

Appendix. Mathematical formulation of PAMiSD

This section provides the complete mathematical formulation of PAMiSD in addition to our formal problem description in Section 3. Using the notation introduced in Section 3, the PAMiSD can be formulated as Integer Non-Linear Program (INLP). It constitutes a non-linear model due to the given demand function and non-linear constraints.

$$\max P(\bar{x},\bar{y}) = \sum_{i\in I} m_i \cdot \lambda_i(x_{ijr},y_{jr}) - \mathrm{CR}_{ijr}(x_{ijr},y_{jr}) \quad \text{(A.1)}$$

Subject to

$$x_{ijr} = x^h_{ijr} \cdot x^v_{ijr} \forall i\in I, j\in J, r\in R \quad \text{(A.2)}$$

$$x_{ijr} \cdot (\sum_{\substack{k\in J\\ k\neq j}} \sum_{c\in R} x_{ikc}) = 0 \ \forall i\in I, j\in J, r\in R \quad \text{(A.3)}$$

$$\sum_{j\in J}\sum_{r\in R} x_{ijr} \geq 1 \ \forall i\in I \quad \text{(A.4)}$$

$$\sum_{i\in I} x_{ijr} - \mathrm{M}\cdot y_{jr} \leq 0 \ \forall j\in J, r\in R \quad \text{(A.5)}$$

$$\sum_{i\in I} \overline{w}_i \cdot x^h_{ijr} \leq w_{jr} \ \forall j\in J, r\in R \quad \text{(A.6)}$$

$$\overline{h}_i \cdot x^v_{ijr} + b \leq h_{jr} \ \forall i\in I, j\in J, r\in R \quad \text{(A.7)}$$

$$\sum_{j\in J} h_{jr}\cdot y_{jr} \leq \tilde{h}_r \ \forall r\in R \quad \text{(A.8)}$$

$$\lambda_i \leq q^s_i + q^b_i \ \forall i\in I \quad \text{(A.9)}$$

$$\overline{q}^s_i = \sum_{j\in J}\sum_{r\in R} g_{ijr}\cdot x_{ijr} \ \forall i\in I \quad \text{(A.10)}$$

$$q^s_i = \overline{q}^s_i - RSS_i \ \forall i\in I \quad \text{(A.11)}$$

$$q^b_i = \max[\lceil \lambda_i - q^s_i \rceil; 0] \ \forall i\in I \quad \text{(A.12)}$$

$$Q^{min}_i \leq q^s_i \leq Q^{max}_i \ \forall i\in I \quad \text{(A.13)}$$

$$x^h_{ijr}, x^v_{ijr}, x_{ijr} \in \mathbb{N} \ \forall i\in I, j\in J, r\in R \quad \text{(A.14)}$$

$$y_{jr} \in \{0,1\} \ \forall j\in J, r\in R \quad \text{(A.15)}$$

$$q^b_i, q^s_i, \overline{q}^s_i \in \mathbb{N}, \ \forall i\in I \quad \text{(A.16)}$$

The total number of facings x_{ijr} is defined by the number of horizontal (x^h_{ijr}) and vertical (x^v_{ijr}) facings, which is ensured by Constraint (A.2). Constraint (A.3) ensures that each item i can only be assigned to one segment j and one rack r. Further, each item needs to be allocated and cannot have zero facings (Constraint (A.4)). Constraint (A.5) activates a shelf segment j at rack r if at least one item i is assigned to it. Constraint (A.6) ensures that the width dimension of a shelf cannot be exceeded. Similarly, Constraint (A.7) ensures the adherence to height dimensions. The total height $\tilde{h}_r$ of each rack r limits the individual heights of the corresponding shelf segments (Constraint (A.8)). Constraint (A.9) ensures that total available shelf inventory q_i^s and the additional refill quantity from the backroom q_i^b are sufficient to fulfill total demand λ_i. The total shelf inventory is defined by Eq. (A.10), and the available shelf inventory q_i^s is given by Eq. (A.11). The quantity for additional replenishment is defined by Constraint (A.12). Constraint (A.13) ensures that the given minimum and maximum shelf inventory levels are respected. Finally, Constraints (A.14)–(A.16) define the variable domains.

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